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Continuity at zero of semi-groups on L_1 and differentiation of additive processes

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ABSTRACT. — We prove that the restriction of a semi-group of L_1-positive operators to its initially conservative part is strongly continuous at zero. We apply this result to prove the local ergodic theorems (l. e. t.) for a semi-group which need not be continuous at zero ([*2*] [*4*]) by using the l. e. t. for continuous semi-group ([*8*]).

We also simplify the proof of the n-parameter l. e. t. in L_1 and we improve the n-parameter l. e. t. in L_p $(1 < p < \infty)$.

RÉSUMÉ. — Nous montrons que la restriction d'un semi-groupe d'opérateurs positifs sur L_1 à sa partie conservative est fortement continue à l'origine. Nous appliquons ce résultat pour démontrer les théorèmes ergodiques locaux pour un semi-groupe éventuellement non continu à l'origine ([*2*] [*4*]) en utilisant le théorème ergodique local pour un semi-groupe continu [*8*].

Nous simplifions également la démonstration du théorème ergodique local à n paramètres dans L_1 et nous améliorons ce théorème dans L_p $(1 < p < \infty)$.

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1. INTRODUCTION AND NOTATIONS

Let $T = \{ T_t, t \in (\mathbb{R}^+ - \{ 0 \})^n \}$ be a strongly measurable and therefore strongly continuous semi-group of positive operators on $L_1(X, \mathscr{F}, \mu)$. We will assume that T is locally bounded (i. e. $\sup_{t \in (0,1]^n} \| T_t \|_1 < \infty$) and consequently there exists $\alpha > 0$ and $M > 0$ such that $\| T_t \|_1 \leqslant M e^{\alpha \Phi(t)}$ where $\Phi(t) = \sum_{i=1}^{n} t_i$ if $t = (t_1, \ldots, t_n)$. $T^* = \{ T_t^* \}$ is then a semi-group in L_∞ and is continuous in the w^*-topology.

Recall the definition of the initially conservative parts of T and T* (Akcoglu-Chacon [2], Akcoglu-Krengel [5], Akcoglu-Del Junco [3], M. Lin [9]):

Let $f \in L_1$, $g \in L_\infty$, $f > 0$ a. e. and $g > 0$ a. e., let $\beta > \alpha$. Put $f_0 = \int_0^\infty \ldots \int_0^\infty e^{-\beta\Phi(t)} T_t f \, dt$ and $g_0 = \int^\infty \ldots \int_0^\infty e^{-\beta\Phi(t)} T_t^* g dt.$

Then the initially conservative part of T (resp. of T*) is the set $C = \{ f_0 > 0 \}$ (resp. $C^* = \{ g_0 > 0 \}$), and the initially dissipative part of T (resp. of T*) is the set $D = X - C$ (resp. $D^* = X - C^*$).

The first local ergodic theorem (l. e. t.) for semi-group of operators is due to U. Krengel [8]: in dimension one, $\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_0^\varepsilon T_t f \, dt = T_0 f$ a. e. if T is a semi-group of positive contractions which is strongly continuous at zero.

This l. e. t. was generalized by Akcoglu-Chacon [2] as follows:

1.0 There exists L_1-semi-groups which do not admit continuous completion at 0.

1.1 $\lim_{\varepsilon \to 0^+} \varepsilon^{-1} \int_0^\varepsilon T_t f \, dt$ exists a. e. even if T is not continuous at 0.

1.2 The l. e. t. 1.1 implies that the restriction of T to $L_1(C)$ admits a continuous completion at 0.

The generalization of these results in dimension n is due to Akcoglu-Del Jungo [3].

In this paper we first show that these completion theorems do not depend on the l. e. t. and we generalize them to locally bounded semi-groups although the l. e. t. fails in this case (R. Sato [11], Akcoglu-Krengel [5]).

In section 3 we apply this result to prove the above l. e. t. 1.1 by using U. Krengel's l. e. t. By this way we will also obtain the decomposition of additive processes as the sum of an absolutely continuous process and a singular one, and we will see that singular processes converge to 0 without using a difficult localization property established by Akcoglu-Del Junco [*3*].

In section 4 we will show that R. T. Terrell's n-parameter l. e. t. for L_1-positive contractions is a consequence of his l. e. t. for contractions in L_1 and L_∞ [*12*].

Finally we will improve the n-parameter l. e. t. in L_p $(1 < p < \infty)$ as follows: $\lim (\varepsilon_1 \ldots \varepsilon_n)^{-1} \int_0^{\varepsilon_1} \ldots \int_0^{\varepsilon_n} T_t f\, dt = T_0 f$ a. e. as the ε_i tend to 0^+ *independently* (The previous version were stated as $\varepsilon_1 = \ldots = \varepsilon_n = \varepsilon \to 0^+$ [*6*] [*9*]).

2. THE CONTINUITY OF T AT ZERO

To obtain the following generalization of Akcoglu-Chacon-Del Junco theorem we will use a convexity argument and some ideas of R. Sato [*10*] [*11*] and M. Lin [*9*], and as we said the proof does not depend on the l. e. t.:

2.1 THEOREM. — Let T be as in section one, then the restriction of T to $L_1(C \cup D^*)$ admits a continuous completion at zero and thus on $\mathbb{R}_+^n$.

Proof. — The definitions of the initial parts easily imply the following properties.

$$T_t L_1(X) \subset L_1(C) \tag{2.2}$$

$$e^{-\beta\Phi(t)} T_t f_0 \leqslant f_0 \quad \text{a. e.} \tag{2.3}$$

$$T_t(1_{D^*} f) = 0 \quad \text{a. e. for any} \quad f \in L_1 . \tag{2.4}$$

(2.2) then implies that $L_1(C \cup D^*)$ is invariant under T since $T_t L_1(C \cup D^*) \subset L_1(C) \subset L_1(C \cup D^*)$.

Let S_t be the restriction of T_t to $L_1(C)$, let $\overline{f}_0$ (resp. $\overline{\mu}$) be the restriction of f_0 (resp. of μ) to the set C.

Putting $R_t = e^{-\beta\Phi(t)} S_t$, we see that $\{ R_t \}$ is a locally bounded semi-group of positive operators on $L_1(C, \overline{\mu})$ and since $f_0 > 0$ a. e. on C we may also put $m = \overline{f}_0 . \overline{\mu}$ and $H_t(h) = \frac{1}{\overline{f}_0} R_t(\overline{f}_0(h)$ for any $h \in L_1(C, m)$, so that $\{ H_t \}$ is a semi-group on $L_1(C, m)$.

Now the properties (2.2) and (2.3) imply that

$$\begin{aligned} R_t\overline{f}_0 &= e^{-\beta\Phi(t)}S_t\overline{f}_0 = e^{-\beta\Phi(t)}1_C T_t(1_C f_0) \\ &= e^{-\beta\Phi(t)}1_C T_t f_0 \leqslant 1_C f_0 = \overline{f}_0 \qquad \overline{\mu}\text{-a. e. on } C. \end{aligned}$$

We then obtain (2.5): $H_t(1) \leqslant 1$ m-a. e. on C and also that $\| H_t \|_{L_1(C,m)} = \| R_t \|_{L_1(C,\overline{\mu})} \leqslant K = \sup_{t\in(0,1]^n} \| T_t \|_{L_1(X,\mu)}$ for any $t \in (0, 1]^n$.

By (2.5), we see that $\{ H_t, t \in (\mathbb{R}_+ - \{ 0 \})^n \}$ is a locally bounded semi-group on $L_1(C, m)$ and $L_\infty(C, m)$ and is also strongly continuous on $L_1(C, m)$. Thus, a convexity argument shows that H_t is a semi-group in $L_p(C, m)$ for each p, $1 < p < \infty$. Since m is a finite measure, the strong-measurability of H_t on $L_1(C, m)$ implies the strong-measurability on $L_p(C, m)$ and hence the strong-continuity of H_t at every point $t \in (\mathbb{R}_+ - \{ 0 \})^n$. Then a lemma due to R. Sato (lemma 1, [*10*]) shows that there exists an operator H_0 on $L_p(C, m)$ such that H_t converges strongly to H_0 at t tends to 0.

Next, given $f \in L_p(C, m)$, we have $H_0 f \in L_p(C, m)$ and thus $H_0 f \in L_1(C, m)$. Furthermore, if q is such that $\frac{1}{p} + \frac{1}{q} = 1$ then we have

$$\| H_0 f \|_1 \leqslant \| H_0 f - H_t f \|_1 + \| H_t f \|_1 \leqslant \| H_0 f - H_t f \|_{L_p(C,m)} m(C)^{q^{-1}} + K \| f \|_1$$

and the L_p-strong continuity of $\{ H_t \}$ at 0 implies that $\| H_0 f \|_1 \leqslant K \| f \|_1$. Thus, we can extend H_0 to $L_1(C, m)$ and $\{ H_t, t \in (R_+ - \{ 0 \})^n \cup \{ 0 \} \}$ is a strongly continuous semi-group on $L_1(C, m)$ by the following approximation argument:

Let $f \in L_1(C, m)$ and $\varepsilon > 0$. Let $g \in L_p(C, m)$ such that $\| g - f \|_{L_1(C,m)} < \varepsilon$. Then

$$\begin{aligned} \| H_t f - H_0 f \|_1 &\leqslant \| H_t f - H_t g \|_1 + \| H_t g - H_0 g \|_p m(C)^{q^{-1}} + \| H_0 g - H_0 f \|_1 \\ &\leqslant 2K\varepsilon + \| H_t g - H_0 g \|_p m(C)^{q^{-1}}. \end{aligned}$$

Therefore, $\limsup_{t\to 0} \| H_t f - H_0 f \|_1 \leqslant 2K\varepsilon$, and $\lim_{t\to 0} \| H_t f - H_0 f \|_1 = 0$.

Now if we put $R_0 f = \overline{f}_0 H_0\left(\frac{f}{\overline{f}_0}\right)$ for any $f \in L_1(C, \overline{\mu})$, then $\{ R_t, t \in (R_+ - \{ 0 \})^n \cup \{ 0 \} \}$ is a strongly continuous semi-group on $L_1(C, \overline{\mu})$, and since $e^{\beta\Phi(t)} \to 1$ as $t \to 0$, $T_{t|L_1(C,\overline{\mu})} = e^{\beta\Phi(t)} R_t$ admits a continuous completion at 0.

Finally $T_t(1_{D^*} f) = 0$ implies that $T_{|L_1(C\cup D^*)}$ also admits a continuous completion at 0 and thus is uniformly strongly continuous on $(\mathbb{R}_+ - \{0\})^n$,

(i. e. given $f \in L_1(C \cup D^*)$, $\varepsilon > 0$, there exists $\delta > 0$ such that $\| u - v \|_{\mathbb{R}^n} < \delta$ implies $\| T_u f - T_v f \|_{L_1(C \cup D^*)} < \varepsilon$). Hence $T_{|L_1(C \cup D^*)}$ can be completed continuously on $\mathbb{R}^n_+$.

The theorem is completely proved.

3. DIFFERENTIATION OF ADDITIVE PROCESSES

In this section we will assume that $n = 1$.

The result of Akcoglu-Krengel [4] deals with additive processes and it generalizes the Akcoglu-Chacon's l. e. t. We will prove it by using U. Krengel's l. e. t. The proof uses the technique introduced in [2] and developped in [4]; it depends on the filling scheme of Chacon-Ornstein but we avoid the construction of a « local » Brunel's function.

It is important to note here that a proof which only depends on Hopf's maximal lemma (as in U. Krengel's paper [8]) has been given by D. Feyel [7] at the same time that the present one. It can be adapted directly to additive processes without using the isomorphism between abelian processes and additive processes.

We recall that an additive process is a family $F = \{ F_t, t > 0 \}$ of L_1-functions satisfying the equation $F_{t+s} = F_t + T_t F_s$ $(t, s > 0)$. Examples of additive processes are $F_t = \int_0^t T_s f ds$ and $F_t = (T_t - I) f$. We will assume that F is locally bounded, i. e. $\sup_{0<t<1} \| t^{-i} F_t \|_1 < \infty$. A process is called absolutely continuous if $F_t = \int_0^t T_s f ds$ for some $f \in L_1(C)$, and a positive process is called singular if it does not dominate any nontrivial positive absolutely continuous process. Note that U. Krengel's l. e. t. and our completion theorem 2.1 only shows that $\lim_{t \to 0^+} t^{-1} \int_0^t T_s f ds$ exists for any $f \in L_1(C)$. Akcoglu-Chacon's l. e. t. goes beyond since the l. e. t. is also proved for any $f \in L_1(D)$.

Finally recall that there is no loss of generality in assuming that F is positive and that $R_0 = \text{strong-}\lim_{t \to 0^+} T_{t|L_1(C)} = I$ the identity operator ([4] [3]).

3.1 THEOREM (Akcoglu-Krengel [4]). — Let $T = \{ T_t, t > 0 \}$ be a semi-group of L_1-positive contractions, let F be a positive process then

$\lim_{t \to 0^+} t^{-1} F_t = \overline{F}$ exists a. e. and $\overline{F}$ is the greatest function f of $L_1^+(C)$ such that $\int_0^t T_s f ds \leqslant F_t$ for all $t > 0$. In particular if F is singular then $\overline{F} = 0$ a. e.

3.2 REMARK. — $F_t = \int_0^t T_s \overline{F} ds + G_t$ is then the decomposition of F given in ([3], 4.1).

Proof. — It is clear that $1_D F_t = 0$ a. e. So, it suffices to prove the convergence on C. Let $f \in L_1^+(Y)$ be such that $f < \limsup_{t \to 0^+} t^{-1} F_t$ a. e. on a non-null subset Y of C. Then, since $f = R_0 f = \lim_{t \to 0^+} t^{-1} \int_0^t T_s f ds$ a. e. by 2.1 and U. Krengel's l. e. t., we have $\sup_{0<t<t_0} \left(F_t - \int_0^t T_s f ds \right) > 0$ a. e. on Y for any $t_0 > 0$.

Then the proofs of lemmas (2.7) and (2.6) in [4] show that $\int_0^t T_s f ds \leqslant F_{t+t_0}$ a. e. for any $t_0 > 0$. Thus $\int_0^t T_s f ds \leq F_t$ for any $t > 0$. Dividing the two members by t and applying again 2.1 and U. Krengel's l. e. t. we then obtain $f \leqslant \liminf_{t \to 0^+} t^{-1} F_t$ a. e. Hence $\limsup_{t \to 0^+} t^{-1} F_t = \liminf_{t \to 0^+} t^{-1} F_t = \overline{F}$ a. e.

Furthermore since $\overline{F} - \frac{1}{k} < \limsup_{t \to 0^+} t^{-1} F_t$ a. e. on $E_k = \left\{ \overline{F} > \frac{1}{k} \right\}$, the above arguments also show that $\int_0^t T_s \left(1_{E_k} \left(\overline{F} - \frac{1}{k} \right) \right) ds \leqslant F_t$ for all $k \in \mathbb{N}^*$ and thus $\int_0^t T_s \overline{F} ds \leqslant F_t$ a. e. The proof is completed.

4. n-PARAMETERS LOCAL ERGODIC THEOREMS

4.1 An other application of the result of section 2 is the following interesting simplification of the proof of the n-parameter l. e. t. in L_1.

Recall that R. T. Terrell [12] has proved the n-parameter l. e. t. for contractions in L_1 and L_∞ by using a maximal lemma of Dunford-Schwartz and has given an other proof to obtain the l. e. t. for positive contractions on L_1, since the lemma fails in this case (see [12]). We prove this last result as follows:

If T is a semi-group of L_1-positive contractions, then the proof of 2.1 shows that $\{ H_t, t \in \mathbb{R}_+^n \}$ is a semi-group of contractions in $L_1(C, m)$ and

$L_\infty(C, m)$. Since $\frac{T_0 f}{f_0} \in L_1(C, m)$, the l. e. t. for $L_1 - L_\infty$ contractions applied to $\{H_t\}$ yields $\lim_{\varepsilon \to 0^+} \varepsilon^{-n} \int_0^\varepsilon \dots \int_0^\varepsilon e^{-\Phi(t)} T_t\left(\frac{T_0 f}{f_0} \bar{f_0}\right) dt = \frac{1}{\bar{f_0}} T_0\left(\frac{T_0 f}{f_0} \bar{f_0}\right)$ m-a. e. on C. It then suffices to observe that $e^{-\Phi(t)} \to 1$ and that $T_t T_0 = T_t$ to obtain $\lim_{\varepsilon \to 0^+} \varepsilon^{-n} \int_0^\varepsilon \dots \int_0^\varepsilon T_t(f) dt = T_0 f$ μ-a. e. on C and thus a. e. on X since $1_D T_t = 0$.

4.2. **The l. e. t in** $L_p (1 < p < \infty)$.

In ([*6*], Remark 1) we have noted that the n-parameter l. e. t. also holds in L_p $(1 < p < \infty)$ and that the proof was just like the L_1-proof. In [*9*] M. Lin has proved the differentiation of additive process in L_p. In fact we will see that additive processes in L_p are represented as indefinite integrals so that the two previous results are equivalent. We improve them as follows:

4.3 THEOREM. — Let $T = \{T_t, t \in \mathbb{R}^n_+\}$ be a strongly continuous semi-group of positive contractions in L_p $(1 < p < \infty)$ and let F be a locally bounded additive process with respect to T. Then

i) there exists a function f in L_p such that $F_I = \int_I T_t f\, dt$ for any interval of $\mathbb{R}^n_+$

ii) $\lim (\varepsilon_1 \dots \varepsilon_n)^{-1} F_{[(0,\dots,0),(\varepsilon_1,\dots\varepsilon_n)]} = T_0 f$ a. e. as the ε_i tend to 0^+ *independently*.

Remarks. — *i*) generalizes the one-dimensional theorem of Akcoglu-Krengel ([*5*], theorem 10) and it also holds for locally bounded semi-groups in any reflexive space.

ii) fails for $p = 1$ (see [*12*]) and was proved in the particular case $\varepsilon_1 = \dots = \varepsilon_n = \varepsilon \to 0^+$ ([*6*] [*9*]).

Sketch of proof:

i) Since F is locally bounded there exists an $f \in L_p$ and a sequence $x_i \to 0^+$ such that $w\text{-}\lim_{x_i \to 0^+} x_i^{-n} F_{[(0,\dots,0),(x_i,\dots,x_i)]} = f$. Then using lemma 3.2 in [*3*] it can be shown that $\left| \left\langle F_I - \int_I T_t f\, dt, g \right\rangle \right| \leqslant \varepsilon$ for any $g \in L_q$ and any $\varepsilon > 0$.

ii) Denoting $M_{(\varepsilon_1,\dots,\varepsilon_n)} h = (\varepsilon_1 \dots \varepsilon_n)^{-1} \int_0^{\varepsilon_1} \dots \int_0^{\varepsilon_n} T_t h\, dt$ for any $h \in L_p$, we have

$$M_{(\varepsilon_1,\dots,\varepsilon_n)} f = (\varepsilon_1 \dots \varepsilon_n)^{-1} F_{[(0,\dots,0),(\varepsilon_1,\dots\varepsilon_n)]} \qquad \text{where } f \text{ is given by } i).$$

M. A. Akcoglu's estimates [1] applied to continuous semi-groups then yields $\| \sup_{\varepsilon_i > 0} | M_{(\varepsilon_1, \dots \varepsilon_n)} h | \|_p \leqslant (p/p - 1)^n \| h \|_p$. Thus the set

$$H = \{ h \in L_p / \lim_{\varepsilon_i \to 0^+} M_{(\varepsilon_1, \dots, \varepsilon_n)} h \text{ exists a. e.} \}$$

is strongly closed. Since $M_{(x_1, \dots, x_n)} f \in H$ for any $x_i > 0$,

$$\text{strong-}\lim_{x_i \to 0^+} M_{(x_1, \dots, x_n)} f = T_0 f \in H$$

that is $f \in H$. For more details see the proofs in [6].

4.4 In all L_1-l. e. t. we may replace the hypothesis contractions by T locally bounded and $\| R_0 \| \leqslant 1$ where $R_0 = \text{strong-}\lim_{t \to 0} T_{t|L_1(C)}$. Indeed M. Lin has proved the convergence on $C^* \cup D$. So it suffices to show that $D^* \subset D$:

For this, let $f > 0$ a. e., $f \in L_1$. Since $T_s f \in L_1(C)$, we have

$$\| T_s f \| = \lim_{t \to 0} \| T_{t+s} f \| = \lim_{t \to 0} \| T_t (1_{C^*} T_s f) \| = \| R_0 (1_{C^*} T_s f) \|$$

$\leqslant \| 1_{C^*} T_s f \|$. This implies that $1_{D^*} T_s f = 0$ and $1_{D^*} f_0 = 0$, thus $D^* \subset D$ (see the definitions in section 1).

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