ANNALES DE L'I. H. P., SECTION B

T. K. CARNE

Brownian motion and stereographic projection

Annales de l'I. H. P., section B, tome 21, nº 2 (1985), p. 187-196
<http://www.numdam.org/item?id=AIHPB_1985__21_2_187_0>

Ann. Inst. Henri Poincaré,
Vol. 21, n° 2, 1985, p. 187-196. *Probabilités et Statistiques*

Brownian motion and stereographic projection

by

T. K. CARNE
King's College, Cambridge.

ABSTRACT. — Stereographic projection from R^N to S^N maps Brownian paths in R^N to the paths of Brownian motion on S^N conditioned to be at the centre of the projection at a negative exponential time.

Key-words: Stereographic projection; Conditioned Brownian motion; Conformal transformations.

RÉSUMÉ. — La projection stéréographique de R^N à S^N applique les trajectoires Browniennes de R^N sur les trajectoires Browniennes de S^N conditionnées par le fait d'être au centre de projection à un instant de loi exponentielle.

In this brief note we shall discuss how Brownian motion in R^N, for $N \geqslant 3$, can be interpreted as a Brownian bridge conditioned to go to the « ideal point at infinity ». This question was posed by Prof. L. Schwartz [*2*]. Prof. M. Yor [*3*] presents an alternative, more probabilistic, approach.

1. STEREOGRAPHIC PROJECTION

Consider the unit sphere S^N in R^{N+1} and the hyperplane

$$R^N = \{ y = (y_1, \ldots, y_{N+1}) : y_{N+1} = 0 \}.$$

Stereographic projection from the point $P = (0, \ldots, 0, 1)$ of S^N maps $y \in \mathbb{R}^N$

to the point $x \in S^N \backslash \{ P \}$ which lies on the straight line from P through y; see the diagram. This is a diffeomorphism between $S^N \backslash \{ P \}$ and R^N, so we regard P as being the point of S^N which corresponds to the « ideal point at infinity of R^N ».

PROPOSITION 1. — *Brownian motion on* R^N *is mapped by stereographic projection onto a time changed version of the Brownian motion on* S^N *together with a drift towards* P *at speed* $\frac{1}{2}(N-2)\tan\frac{1}{2}\theta$ *on the sphere.*

Proof. — Brownian motion on a Riemannian manifold with metric $g_{ab}dx_a dx_b$ has as its infinitesimal generator one half of the Laplacian, viz.

$$\frac{1}{2}\Delta = \frac{1}{2\sqrt{g}}\sum \frac{\partial}{\partial x_a}\left(\sqrt{g}\, g^{ab}\frac{\partial}{\partial x_b}\right)$$

where $g = \det(g_{ab})$ and $(g^{ab}) = (g_{ab})^{-1}$. On S^N take co-ordinates (θ, z) for $x \in S^N$ where $0 \leqslant \theta \leqslant \pi$ is the angle shown in the diagram and $z = y/\| y \| \in S^{N-1} = S^N \cap R^N$.

Then

$$\| dx \|^2 = | d\theta |^2 + \sin^2\theta \, . \, \| dz \|^2$$

so the Laplacian on S^N is

$$\Delta_{S^N} = \frac{1}{\sin^{N-1}\theta}\frac{\partial}{\partial\theta}\left(\sin^{N-1}\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\Delta_{S^{N-1}}\,.$$

Similarly, if we take co-ordinates (r, z) for $y \in \mathbb{R}^N$, where $r = \| y \|$, then

$$\| dy \|^2 = | dr |^2 + r^2 \| dz \|^2$$

so the usual Laplacian on $\mathbb{R}^N$ is

$$\Delta_{R^N} = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^{N-1}}\,.$$

The infinitesimal generator for the deterministic motion given by a drift towards P at speed $\frac{1}{2}(N-2)\tan\frac{1}{2}\theta$ is clearly

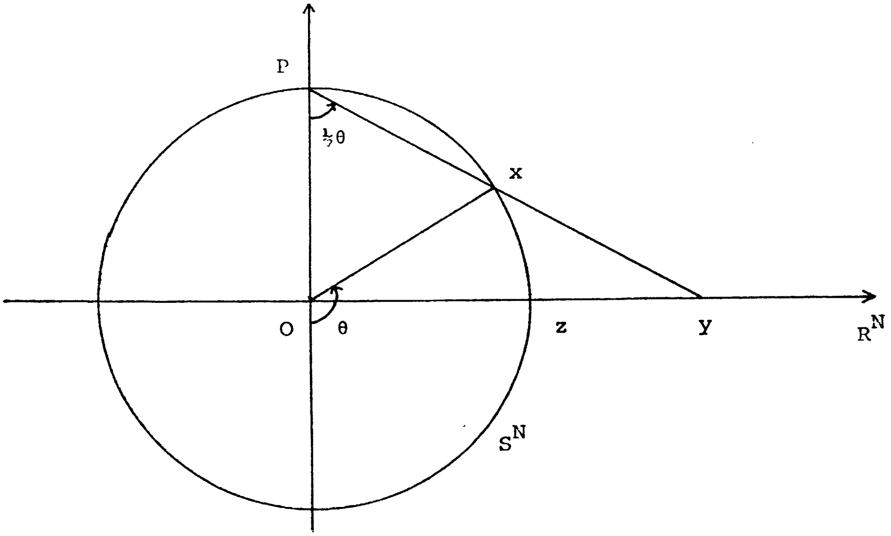
$$\frac{1}{2}(N-2)\tan\frac{1}{2}\theta \, . \, \frac{\partial}{\partial\theta}$$

Hence, to prove the proposition we need to show that, under stereographic projection $\frac{1}{2}\Delta_{R^N}$ corresponds to some strictly positive function times

$$\mathcal{G}_P = \frac{1}{2}\Delta_{S^N} + \frac{1}{2}(N-2)\tan\frac{1}{2}\theta\,.\frac{\partial}{\partial\theta}.$$

Under stereographic projection we have $r = \tan\frac{1}{2}\theta$ so

$$\begin{aligned}
\Delta_{S^N} &= \left(\frac{2r}{1+r^2}\right)^{1-N}\left(\frac{1+r^2}{2}\right)\frac{\partial}{\partial r}\left[\left(\frac{2r}{1+r^2}\right)^{N-1}\left(\frac{1+r^2}{2}\right)\frac{\partial}{\partial r}\right]+\left(\frac{1+r^2}{2r}\right)^2\Delta_{S^{N-1}}\\
&= \left(\frac{1+r^2}{2}\right)^2\left\{\left(\frac{2}{1+r^2}\right)^{2-N}\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left[\left(\frac{2}{1+r^2}\right)^{N-2}r^{N-1}\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\Delta_{S^{N-1}}\right\}\\
&= \left(\frac{1+r^2}{2}\right)^2\left\{\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left[r^{N-1}\frac{\partial}{\partial r}\right]-(N-2)\left(\frac{2r}{1+r^2}\right)\frac{\partial}{\partial r}+\frac{1}{r^2}\Delta_{S^{N-1}}\right\}\\
&= \left(\frac{1+r^2}{2}\right)^2\left\{\Delta_{R^N}-(N-2)\left(\frac{2r}{1+r^2}\right)\frac{\partial}{\partial r}\right\}.
\end{aligned}$$



Equivalently,

$$\begin{aligned}
\frac{1}{2}\Delta_{R^N} &= \left(\frac{2}{1+r^2}\right)^2\left\{\frac{1}{2}\Delta_{S^N}+\frac{1}{2}(N-2)r\left(\frac{1+r^2}{2}\right)\frac{\partial}{\partial r}\right\}\\
&= (1+\cos\theta)^2\left\{\frac{1}{2}\Delta_{S^N}+\frac{1}{2}(N-2)\tan\frac{1}{2}\theta\frac{\partial}{\partial\theta}\right\}.
\end{aligned}$$

This completes the proof. □

We now wish to obtain the random process with infinitesimal generator $\mathcal{G}_P$ by conditioning the standard Brownian motion $BM(S^N)$ on the sphere to be at P at an appropriate time. To do this we will follow the analysis of conditioning given by J. L. Doob [*1*, Chapter 10]. Note that we are seeking a time-homogeneous process, so that conditioning $BM(S^N)$ to be at P at a fixed time will not do. Furthermore, we cannot simply condition $BM(S^N)$ to hit P at some time since, to do so, we would require a positive harmonic function on $S^N \backslash \{ P \}$ with a singularity at P. No such function exists. However, we do obtain time homogeneous processes by conditioning $BM(S^N)$ to be at P at a random time T which is independent of $BM(S^N)$ and has a negative exponential distribution.

PROPOSITION 2. — *Let* T *be a random time which is independent of* $BM(S^N)$ *and has a negative exponential distribution with parameter* $\lambda = N(N-2)/8$. *Then* $BM(S^N)$ *conditioned to be at* P *at time* T *has infinitesimal generator*

$$\mathcal{G}_P = \frac{1}{2}\Delta_{S^N} + \frac{1}{2}(N-2)\tan\frac{1}{2}\theta\frac{\partial}{\partial\theta}$$

on $S^N \backslash \{ P \}$. *Hence,* $BM(R^N)$ *is mapped by stereographic projection to a time-changed version of* $BM(S^N)$ *conditioned to be at* P *at the time* T.

Proof. — To condition $BM(S^N)$ to be at P at time T we need to find a positive function h on $S^N \backslash \{ P \}$ with a singularity at P and

$$\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)h = 0$$

Then the conditioned process will have the h-transform:

$$u \to h^{-1}\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)(h \, . \, u)$$

as its infinitesimal generator. Such a function h must be a multiple of the Green's function for $\frac{1}{2}\Delta_{S^N} - \lambda I$ with a pole at P and hence it must be a function of θ only. Thus we wish to solve

$$\frac{1}{2\sin^{N-1}\theta}\frac{\partial}{\partial\theta}\left[\sin^{N-1}\theta\frac{\partial h}{\partial\theta}\right] - \lambda h = 0\,.$$

When $\lambda = N(N-2)/8$ the required function h is given by $h = \left(\cos\frac{1}{2}\theta\right)^{-N+2}$

Consequently, the conditioned process has infinitesimal generator

$$\begin{aligned} u \to\ & h^{-1}\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)(h.u) \\ &= h^{-1}\left(\frac{1}{2}h\Delta_{S^N}u + \nabla h.\nabla u + \frac{1}{2}u\Delta_{S^N}h - \lambda u.h\right) \\ &= \frac{1}{2}\Delta_{S^N}u + h^{-1}\nabla h.\nabla u \\ &= \frac{1}{2}\Delta_{S^N}u + \frac{1}{2}(N-2)\tan\frac{1}{2}\theta\frac{\partial}{\partial\theta} \end{aligned}$$

where ∇ is the gradient for the Euclidean metric on S^N. This proves the first assertion and the second follows from Proposition 1.

(Note that the conditioning described above does correspond to the naïve idea of conditioning a process by its position at time T. For suppose that U is a subset of S^N with a smooth boundary. If (x_t) is the Brownian motion on S^N, then we may form a new process

$$\begin{aligned} x_t^* &= x_t \quad \text{for} \quad t < T \\ &= \partial \quad \text{for} \quad t \geqslant T \end{aligned}$$

which jumps to a coffin state ∂ at the random time T. If we condition (x_t^*) so that $x_{T-}^* \in U$ then we obtain the transition semigroup P_t given by

$$\begin{aligned} P_t f(x) &= E^x(f(x_t^*) \mid x_{T-}^* \in U) \\ &= E^x(f(x_t)1_{(t<T)} \mid x_T \in U) \\ &= \frac{E^x(f(x_t)1_{(t<T)}1_U(x_T))}{E^x(1_U(x_T))} \end{aligned}$$

Setting

$$h(x) = E^x(1_U(x_T))$$

we find that

$$\begin{aligned} P_t f(x) &= h(x)^{-1}E^x(f(x_t)1_{(t<T)}h(x_T)) \\ &= h(x)^{-1}\int_t^\infty E^x(f(x_t)h(x_s))\lambda e^{-\lambda s}ds \\ &= h(x)^{-1}e^{-\lambda t}E^x(f(x_t)h(x_t)) \end{aligned}$$

by using the Markov property of the Brownian motion. Thus the conditioned process is the h-transform of the Brownian motion for h the distributional solution of

$$\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right)h = 1_U .$$

We can now decompose this process into an average of the processes conditioned to be at a point $X \in U$ at the time T. See J. L. Doob [*1*] for further details.) $\square$

For each $Y \in S^N$ let $h(Y, .)$ be the Green's function of $\frac{1}{2}\Delta_{S^N} - \frac{N(N-2)}{8} I$ with a pole at Y. Then the Brownian motion conditioned to be at Y at the negative exponential time T has infinitesimal generator

$$u \to h(Y, x)^{-1}\left(\frac{1}{2}\Delta_{S^N} - \frac{N(N-2)}{8} I\right)(h(Y, x)u(x))$$

on $S^N \backslash \{ Y \}$. As in Proposition 2 we find that this is

$$u \to \frac{1}{2}\Delta_{S^N}u(x) - (N-2)\,\| x - Y \|^{-1}\nabla \| x - Y \| . \nabla u(x) .$$

Call this generator $\mathcal{G}_Y$.

COROLLARY. — *Let* $(x_t : 0 \leqslant t \leqslant S)$ *be the process with generator* $\mathcal{G}_P$ *which starts from* Y *at time* $t = 0$ *and stops at the time* S *when it first hits* P. *Then the time reversed process* $(\tilde{x}_\tau : 0 \leqslant \tau \leqslant S)$ *given by*

$$\tilde{x}_\tau = x_{S-\tau}$$

has infinitesimal generator $\mathcal{G}_Y$, *starts from* P *at* $\tau = 0$ *and stops at the time* S *when it first hits* Y.

Proof. — Since stereographic projection maps (x_t) onto Brownian motion in R^N it is clear that $(x_t : t > 0)$ almost surely never hits Y. Thus the reversed process certainly starts from P at $\tau = 0$ and stops at the time S when it first hits Y. It remains to find its infinitesimal generator.

Let $g(Y, .)$ be the Green's function for $\mathcal{G}_P$ with pole at Y, then, for any smooth function f which is compactly supported within $S^N \backslash \{ P, Y \}$, we have

$$E\int_0^S f(x_t)dt = \int g(x, Y)f(x)dV(x) = E\int_0^S f(\tilde{x}_\tau)d\tau$$

where dV is the N-dimensional Lebesgue measure on S^N.

Consequently, if we denote by $\mathcal{G}_P$, (P_t) the generator and transition semigroup for (x_t) and by $\tilde{\mathcal{G}}_P$, $(\tilde{P}_\tau)$ the corresponding operators for $(\tilde{x}_\tau)$, then we obtain

$$\begin{aligned}\int g(x, X) f(x) P_r k(x) dV(x) &= E \int_0^S f(x_t) P_r k(x_t) dt \\ &= E \int_0^S f(x_t) k(x_{t+r}) dt \\ &= E \int_0^S f(\tilde{x}_{\tau+r}) k(\tilde{x}_\tau) d\tau \\ &= \int g(x, Y) k(x) \tilde{P}_r f(x) dV(x).\end{aligned}$$

So

$$\tilde{P}_r k(x) = g(x, Y)^{-1} P_r^*(g(x, Y) k(x))$$

and

$$\tilde{\mathcal{G}}_P k(x) = g(x, Y)^{-1} \mathcal{G}_P^*(g(x, Y) k(x)).$$

Now recall that $\mathcal{G}_P = h(P, .)^{-1}\left(\frac{1}{2}\Delta - \lambda I\right) h(P, .)$ so

$$g(x, Y) = \frac{h(Y, x) h(P, x)}{h(P, Y)}$$

and consequently

$$\tilde{\mathcal{G}}_P k(x) = h(Y, x)^{-1}\left(\frac{1}{2}\Delta - \lambda I\right)^* (h(Y, x) k(x)).$$

Since the Laplacian is self-adjoint, this gives the desired result. □

2. CONFORMAL TRANSFORMATIONS

In this section we wish to set the results of § 1 in a more general context.

For any $\lambda > 0$ we can condition $BM(S^N)$ to be at P at the independent random time T which has negative exponential distribution with parameter λ. Indeed, to do so we need only find a positive function h of θ with

$$\left(\frac{1}{2}\Delta_{S^N} - \lambda I\right) h = 0 \quad \text{on} \quad S^N \setminus \{ P \}$$

and a singularity at P. If we make the change of variables $q = \frac{1}{2}(1 - \cos\theta)$ this becomes

$$q(1 - q)\frac{d^2h}{dq^2} + \frac{1}{2}N(1 - 2q)\frac{dh}{dq} - 2\lambda h = 0$$

for $0 \leqslant q < 1$. This is in the standard hypergeometric form and may be solved by a power series

$$h = \sum_{n=0}^{\infty} a_n q^n .$$

This series has radius of convergence 1 and each a_n is positive, so h is certainly positive on $0 \leqslant q < 1$. For $\lambda \neq \mathrm{N}(\mathrm{N}-2)/8$ this formula does not define an elementary function. Although the conditioned process may be studied as in the previous section, it does not correspond to a simple process on $\mathbb{R}^{\mathrm{N}}$.

The key property of stereographic projection is that it is *conformal* so it alters the metric at any point only by a scale factor. We can develop the arguments above for any such conformal transformation.

PROPOSITION 3. — *Let* M *be an* N-*manifold* ($\mathrm{N} \geqslant 3$) *with a Riemannian metric* g_{ab} *and a conformally equivalent metric*

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \quad \textit{with} \quad \Omega > 0 .$$

Let R *and* $\tilde{\mathrm{R}}$ *be the scalar curvature for* g *and* $\tilde{g}$ *respectively. Then the Brownian motion relative to* $\tilde{g}$ *can be obtained, up to a time change, by conditioning the Brownian motion relative to* g *according to its behaviour at a negative exponential time if, and only if,* $\mathrm{R} - \Omega^2\tilde{\mathrm{R}}$ *is constant on* M.

Proof. — In terms of the infinitesimal generators $\frac{1}{2}\Delta$ and $\frac{1}{2}\tilde{\Delta}$ for the Brownian motions, the Proposition states that there exists $\lambda > 0$ and strictly positive functions h and c on M with

$$\frac{1}{2}\tilde{\Delta}u = c^2 h^{-1}\left(\frac{1}{2}\Delta - \lambda \mathrm{I}\right)(h\,.\,u) \tag{1}$$

if, and only if, $\mathrm{R} - \Omega^2\tilde{\mathrm{R}}$ is constant. (If we consider the second degree terms of (1) we see that the condition can only be satisfied if g and $\tilde{g}$ are conformal. So there was no loss of generality in restricting ourselves to this case.)

The proof is simply a standard calculation of the scalar curvature for conformal metrics. We shall use the usual index notation for vectors and tensors on M. Let ∇_a, $\tilde{\nabla}_a$ be the covariant derivatives relative to g and $\tilde{g}$.

Then a straightforward but tedious calculation yields the formulae:

$$\begin{aligned}
\tilde{\nabla}_a v_b &= \nabla_a v_b - \Omega^{-1}(v_a\nabla_b\Omega + v_b\nabla_a\Omega - g_{ab}g^{cd}v_c\nabla_d\Omega)\\
\tilde{\Delta} u &= \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b u = \tilde{g}^{ab}\tilde{\nabla}_a(\nabla_b u)\\
&= \Omega^2(\Delta u + (\mathrm{N}-2)\Omega^{-1}g^{ab}\nabla_a\Omega\nabla_b u)\\
\Omega^2\tilde{\mathrm{R}} &= \mathrm{R} - 2(\mathrm{N}-1)\Omega^{-1}\Delta\Omega - (\mathrm{N}-1)(\mathrm{N}-4)\Omega^{-2}g^{ab}\nabla_a\Omega\nabla_b\Omega\,.
\end{aligned}$$

Thus, for (1) to hold, we must have $c = \Omega$ and

$$h^{-1}\left(\frac{1}{2}\Delta - \lambda \mathrm{I}\right)(h\,.\,u) = \frac{1}{2}\Delta u + \frac{1}{2}(\mathrm{N}-2)\Omega^{-1}g^{ab}\nabla_a\Omega\nabla_b u\,.$$

Now

$$\Delta(h\,.\,u) = g^{ab}\nabla_a\nabla_b(h\,.\,u) = h\Delta u + 2g^{ab}\nabla_a h\nabla_b u + u\Delta h$$

so we obtain the two conditions:

$$h^{-1}\nabla_a h = \frac{1}{2}(\mathrm{N}-2)\Omega^{-1}\nabla_a\Omega$$

and

$$\left(\frac{1}{2}\Delta - \lambda \mathrm{I}\right)h = 0\,.$$

The first of these is satisfied if, and only if, $h = \mathrm{K}\,.\,\Omega^{\frac{1}{2}(\mathrm{N}-2)}$ for some constant K. In this case, the second condition becomes

$$\begin{aligned}
0 &= \left(\frac{1}{2}\Delta - \lambda \mathrm{I}\right)(\Omega^{\frac{1}{2}(\mathrm{N}-2)})\\
&= \frac{1}{4}(\mathrm{N}-2)\Omega^{\frac{1}{2}\mathrm{N}-2}\Delta\Omega + \frac{1}{8}(\mathrm{N}-2)(\mathrm{N}-4)\Omega^{\frac{1}{2}\mathrm{N}-3}g^{ab}\nabla_a\Omega\nabla_b\Omega - \lambda\Omega^{\frac{1}{2}\mathrm{N}-1}.
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \lambda &= \frac{1}{4}(\mathrm{N}-2)\Omega^{-2}\Delta\Omega + \frac{1}{8}(\mathrm{N}-2)(\mathrm{N}-4)\Omega^{-2}g^{ab}\nabla_a\Omega\nabla_b\Omega\\
&= \frac{\mathrm{N}-2}{8(\mathrm{N}-1)}\,.\,(\mathrm{R} - \Omega^2\tilde{\mathrm{R}})\,. \qquad \square
\end{aligned}$$

If we take g to be the Euclidean metric on S^N and $\tilde{g}$ the metric on S^N which corresponds under stereographic projection to the Euclidean metric on R^N, then

$$\Omega = \frac{1}{1+\cos\theta}, \qquad \mathrm{R} = \mathrm{N}(\mathrm{N}-1), \qquad \tilde{\mathrm{R}} = 0$$

and we recover Proposition 2. The above formula may also be usefully applied to conformal mappings from S^N to itself.

PROPOSITION 4. — *Let* $(x_t : 0 \leqslant t < S)$ *be the process on* $S^N \setminus \{ P \}$ *with infinitesimal generator* $\mathcal{G}_P$ *and let* $T : S^N \to S^N$ *be a conformal automorphism of* S^N. *Then* $(Tx_t : 0 \leqslant t < S)$ *is a time-changed version of the process on* $S^N \setminus \{ TP \}$ *with infinitesimal generator* $\mathcal{G}_{TP}$.

Proof. — Recall that the group of conformal automorphisms of S^N is generated by the inversions in spheres orthogonal to S^N. We could prove the result by direct calculation, as in § 1, of the effect of such an inversion. However, it is simpler to argue indirectly.

Let $U : R^N \to S^N$ be stereographic projection with centre P and let $V : R^N \to S^N$ be the stereographic projection with centre TP from the N-dimensional subspace of R^{N+1} orthogonal to TP. Both of these maps are conformal, so the composite

$$Q = V^{-1}TU : R^N \to R^N$$

is conformal. Since $N \geqslant 3$, the only such conformal maps are the Euclidean similarities of R^N. These similarities obviously preserve Brownian motion on R^N to within alteration of the time scale by a constant factor. Now Proposition 1 shows that, to within a time change, U maps $BM(R^N)$ to the process with generator $\mathcal{G}_P$ and V maps $BM(R^N)$ to the process with generator $\mathcal{G}_{TP}$. Therefore, $T = VQU^{-1}$ does indeed transform the process with generator $\mathcal{G}_P$ to a time-changed version of the process with generator $\mathcal{G}_{TP}$. □

If we combine Proposition 4 with the earlier Corollary, we see that time-reversal of the process starting at Y with generator $\mathcal{G}_P$ corresponds to the image of the process under any inversion which maps S^N onto itself and interchanges Y and P. This should be compared with the results of M. Yor [*3*].

REFERENCES

[*1*] J. L. DOOB, *Classical Potential Theory and its Probabilistic Counterpart*, 1984, Springer-Verlag, Berlin.

[*2*] L. SCHWARTZ, Le mouvement Brownien sur $\mathbb{R}^N$, en tant que semi-martingale dans S_N. *Annales de l'I. H. P. Probabilités et Statistiques*, t. **21**, n° 1, 1985, p. 15-25.

[*3*] M. YOR, A propos de l'inverse du mouvement Brownien dans $\mathbb{R}^n$ ($n \geqq 3$). *Annales de l'I. H. P. Probabilités et Statistiques*, t. **21**, n° 1, 1985, p. 27-38.

(Manuscrit reçu le 7 décembre 1984).