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Probabilités et Statistiques

Convergence of martingales on manifolds of negative curvature

by

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ABSTRACT. — Let M be a Riemannian manifold whose sectional curvatures are bounded above by a negative constant, and let the stochastic process (X_t) be a martingale with values in M. We give sufficient conditions for (X_t) to have a limit almost surely in the sphere at infinity. We apply the theorem when (X_t) is Brownian motion, proving as a corollary that if the sectional curvatures of M go to minus infinity sufficiently slowly with radial distance, then M admits non-constant bounded harmonic functions. In passing, we obtain an estimate on the passage times of Brownian motion in negatively curved manifolds.

Key-words: Martingale, Brownian motion, negative curvature, Riemannian manifold, stochastic development, escape rates, passage times, bounded harmonic functions.

RÉSUMÉ. — Soit M une variété riemannienne dont les courbures sectionnelles sont majorées par une constante négative, et soit (X_t) une martingale à valeurs dans M. Nous donnons des conditions suffisantes afin que (X_t) ait une direction asymptotique. Nous appliquons le théorème quand (X_t) est le mouvement brownien, et nous prouvons que, si les courbures sectionnelles de M tendent suffisamment lentement vers moins l'infini, alors M admet des fonctions harmoniques bornées et non-constantes. Nous obtenons également une inégalité concernant les temps de passage du mouvement brownien dans les variétés à courbure négative.

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§ 1. INTRODUCTION

Let M be a d-dimensional Riemannian manifold and let Γ denote its Levi-Civita connection. A stochastic process X with values in M is called a Γ-martingale when, roughly speaking, the image of (X_t) under every local Γ-convex function is a local submartingale whenever X_t is in the domain of the function: see Darling [2] for details and examples. A convenient method of constructing Γ-martingales is to start with a continuous local martingale on $\mathbb{R}^d$ and « print » it on to M by means of a certain stochastic moving frame: see Darling's thesis [1] or Meyer [10]. This generalizes the construction of Brownian motion using the orthonormal frame bundle, presented in Elworthy [5].

In a previous article [3], Darling gave a condition for the almost sure convergence of (X_t), without reference to the curvature of M. The idea was to construct the scalar quadratic variation of (X_t), denoted by $\langle X, X \rangle_t$, by setting

$$d \langle X, X \rangle_t = g_{ij}(X_t) d \langle X^i, X^j \rangle_t$$

where (g_{ij}) is the metric tensor and $(X_t^1, \ldots, X_1^d)$ is the local co-ordinate representation of (X_t). The result is that $X_\infty = \lim_t X_t$ exists almost surely in M, the one-point compactification of M, on the set where $\langle X, X \rangle_\infty < \infty$. Zheng [15] proved that on the set where X_∞ exists and lies in M, we have $\langle X, X \rangle_\infty < \infty$ almost surely. These results are put together in the expository paper of Meyer [11]. Emery [6] attempts to characterize the manifolds on which

$$\{ \lim_{t \to \infty} X_t \text{ exists in M} \} = \{ \langle X, X \rangle_\infty < \infty \} .$$

The present article addresses a different question. We restrict ourselves to manifolds M whose sectional curvatures are bounded above by a negative constant, and which are diffeomorphic to $\mathbb{R}^d$.

We wish to know: when does a Γ-martingale (X_t) converge to a limit in the sphere at infinity (considered as the set of directions of geodesic rays)? Theorem A says that a sufficient condition is that

(1) the process (X_t) has at least two commensurate 'random directions', and

(2) the process (X_t) does not escape too fast from unit balls. These assumptions are made precise in § 3.

In his article of 1975 [12], Prat shows that when the sectional curvatures of M are pinched between two negative constants, Brownian motion on M converges to a limit in the sphere at infinity. To demonstrate that Theorem A is not trivial, we apply it when X is Brownian motion and improve Prat's result, showing that the sectional curvatures may go to minus infinity like $-(\log r(x))^{2-2h}$, where $0 < h < 1$ and $r(x)$ is the distance of x from some fixed point M (Theorem B). As a corollary, we obtain an improvement on Sullivan's result on bounded harmonic functions [*14*]: if the sectional curvatures $K(x)$ of M satisfy

$$-k^2 \geqslant K(x) \geqslant -c^2(\log r(x))^{2-2h} \quad \text{as} \quad x \to \infty$$

then M admits non-constant bounded harmonic functions.

On the way to Theorem B, we obtain at the end of § 6 a new result about Brownian motion (X_t) when the sectional curvatures of M are pinched between two negative constants: namely, a lower bound on the probability that Brownian motion remains in the ball of radius a up till time t.

§ 2. MARTINGALES IN MANIFOLDS AND ITO'S FORMULA

The idea of martingales on a manifold with a linear connection seems to have originated with Bismut; for an elementary explanation, see Meyer [*10*]. For a more abstract version, see Darling's thesis [*1*]. We shall use the definitions and notations presented in Darling [*3*]. In particular, we shall *not* repeat the definitions of Stratonovitch integration of differential forms, horizontal lifting of a process to the orthonormal frame bundle, or stochastic development. A full account is given in Ikeda and Watanabe [*9*].

Let M be a smooth d-dimensional manifold.

DEFINITION 1. — A *semimartingale* on M will mean a process X with almost surely continuous paths on M, whose image under every C^2 function from M to R is a real-valued semimartingale.

Suppose M has a linear connection Γ on the tangent bundle (for example, the Levi-Civita connection if M is Riemannian). Let ∇ denote covariant differentiation in the cotangent bundle, and for x in M and C^2 functions f on M, define as usual

$$\nabla df(x)(V, W) = (\nabla_V df(x), W), \ V, W \in T_xM$$

(some authors call this Hess $f(x)(V, W)$. In local co-ordinates,

$$\nabla df(x)_{ij} = \nabla df(x)(D_i, D_j) = D_{ij}f(x) - \Gamma^k_{ij}(x)D_k f(x)$$

where the (Γ^k_{ij}) are the Christoffel symbols.

For any semimartingale X on M,

$$\int_0^t \nabla df(X_s)(dX_s, dX_s) \tag{2.1}$$

can be defined by expressing (X_t) in terms of local co-ordinate processes $(X^1_t, \ldots, X^n_t)$ and setting the integrand equal to:

$$\nabla df(X_s)_{ij} d\langle X^i, X^j\rangle_s \tag{2.2}$$

using the angle brackets process of the continuous semimartingales X^i and X^j, whenever X_s is in the domain of the local co-ordinate system.

DEFINITION 2. — (The phrasing is due to Emery [*6*]).

A semimartingale X with values in M is called a Γ-*martingale* if, for all C^2 functions f on M,

$$f(X_t) - 1/2 \int_0^t \nabla df(X_s)(dX_s, dX_s) \tag{2.3}$$

is a real-valued local martingale.

Now assume that M is Riemannian, and Γ is the Levi-Civita connection. Experience in calculating with semimartingales on manifolds shows that it is inefficient to refer constantly to local co-ordinate processes as in expressions (2.1), (2.2) and (2.3). To avoid this, we introduce a process U with values in the orthonormal frame bundle 0(M), called a *horizontal lift* of X to 0(M) through Γ, such that U_t is an isometry from $\mathbb{R}^d$ to the tangent space to M at X_t, with Riemannian inner product. From U we construct a process Z in $\mathbb{R}^d$, called the *stochastic development* of X into $\mathbb{R}^d$. The precise definitions, which are not needed here, are available in Darling [*1*] [*3*]. The crucial relationship of X, U and Z is the Ito formula. For the sake of clarity, take an orthonormal basis $e_1, \ldots, e_d$ for $\mathbb{R}^d$, and write $Z_t = (Z^1_t, \ldots, Z^d_t)$ with respect to this basis. Then $U_s(e_i)$ is a tangent vector at X_s, for each i. Suppose f is a C^2 function from M to $\mathbb{R}$. Then the differential of f is the 1-form

$$df(x): T_xM \to \mathbb{R}$$

and so the composite process

$$df(X_s) \circ U_s(e_i)$$

is real-valued. Likewise

$$\nabla df(X_s)(U_s(e_i), U_s(e_j))$$

is real-valued. We may now state the

Ito Formula.

For C^2 functions f from M to $\mathbb{R}$,

$$f(X_t) - f(X_0) = \int_0^t (df(X_s) \circ U_s(e_i)) dZ_s^i + 1/2 \int_0^t \nabla df(X_s)(U_s(e_i), U_s(e_j)) d\langle Z^i, Z^j \rangle_s \qquad (2.4)$$

In fact the use of a basis is not necessary, and we can abbreviate this to:

$$\int_0^t (df(X_s) \circ U_s) dZ_s + 1/2 \int_0^t \nabla df(X_s)(U dZ \otimes U dZ)_s .$$

The virtues of (2.4) are as follows. First, $(U_s(e_1), \ldots, U_s(e_d))$ forms an orthonormal basis of the tangent space X_s, which simplifies many calculations. More importantly, we have:

PROPOSITION 1. — For X to be a Γ-martingale on M, it is necessary and sufficient that any stochastic development Z of X into $\mathbb{R}^d$ is a continuous local martingale; in this case (2.4) is exactly the Doob-Meyer decomposition of the semimartingale $f(X_t) - f(X_0)$.

Proof. — See Darling [*1*], or Meyer [*10*].

Since we refer to Brownian motion on Riemannian manifolds, let us state the following fact:

PROPOSITION 2. — The following there assertions are equivalent:

i) (X_t) is a Markov process on M whose generator is half the Laplacian Δ.

ii) For all C^2 functions f with compact support on M,

$$C_t^f \equiv f(X_t) - f(X_0) - 1/2 \int_0^t \Delta f(X_s) ds$$

is a real-valued martingale on M.

iii) Each stochastic development Z of X into $\mathbb{R}^d$ is a Brownian motion in $\mathbb{R}^d$.

Proof. — The equivalence of (*i*) and (*ii*) follows from the approach to diffusions presented in Stroock and Varadhan [*13*]. Assume (*iii*); then

$$d\langle Z^i, Z^j\rangle_s = \delta^{ij}ds$$

and the left side of (2.4) becomes

$$\int_0^t (df(X_s)\circ U_s)dZ_s + 1/2\sum_i\int_0^t \nabla df(X_s)(U_s(e_i), U_s(e_i))ds.$$

Since U_s is an isometry and df is a bounded 1-form the first integrand is bounded; hence the first integral is a martingale. As for the second, since $(U_s(e_i), \ldots, U_s(e_d))$ is an orthonormal basis of the tangent space at X_s, the integrand is $\Delta f(X_s)ds$. Hence

$$f(X_t) - f(X_0) - 1/2\int_0^t \Delta f(X_s)ds = \text{martingale}.$$

This proves (*ii*). The implication (*ii*) $\Rightarrow$ (*iii*) is easily proved in local coordinates, by the methods presented in Ikeda and Watanabe [*9*].

DÉFINITION 3. — A semimartingale X on a Riemannian manifold M will be called a *Brownian motion* if the assertions of Proposition 2 hold.

§ 3. CONVERGENCE THEOREM

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a filtered probability space satisfying the usual conditions. All processes will be $(\mathscr{F}_t)$ adapted. Suppose X is a Γ-martingale on a d-dimensional Riemannian manifold (M, g), where Γ is the Levi-Civita connection. Let U be a horizontal lift of X to 0(M) through Γ, and be the corresponding stochastic development of X into $\mathbb{R}^d$.

Our theorem will depend on two main assumptions, (A1) and (A2), about the process X, plus a condition that the sectional curvatures of M are bounded above by a negative constant. The first assumption ensures that X has at least two « random directions » at all times. The second asks that X does not escape too fast from unit balls.

(A1) **Multidirectional assumption.**

This assumption is stated in terms of the stochastic development Z of X into $\mathbb{R}^d$. Actually Z is not unique, because different choices of initial frame U_0

give different processes Z. However if the following assumptions hold for *any* Z, they will hold for *all* stochastic developments Z.

First we make two mild technical assumptions:

$$\mathbb{E}\,|\,Z_t\,|^2 < \infty \quad \text{for all} \quad t \tag{3.1}$$

$$d\langle Z^i, Z^j\rangle_t = A^{ij}(t)dt \tag{3.2}$$

where $\langle Z^i, Z^j\rangle_t$ is the angle-brackets process of Z^i and Z^j, and $(A^{ij}(t))$ is some symmetric $d \times d$ matrix-valued process.

Let $\lambda_1(t) \geqslant \lambda_2(t) \geqslant \ldots \geqslant \lambda_d(t) \geqslant 0$ be the eigenvalues of $A^{ij}(t)$. The crucial assumption is that for some constant c_1 and c_2

$$0 < c_1 < \lambda_1(t) < c_2\lambda_2(t) \text{ a. s.,} \quad \text{for all } t \tag{3.3}$$

Among other things, this ensures that (X_t) moves in at least two 'random directions'. Observe that when (X_t) is Brownian motion on M, then (Z_t) is a Brownian motion in $\mathbb{R}^d$, and all the λ_i are identically 1.

(A2) **Moderated escape rates.**

We need to have a lower estimate for the rate at which (X_t) can escape from unit balls in M. We suppose henceforward that for some $k > 0$, the sectional curvatures of M are bounded *above* by $-k^2 < 0$. Define $c_3 = k(3 - 2\sqrt{2})/c_2$, and define ε_0 to be the value of ε which maximizes

$$h(\varepsilon) = \frac{1}{2}\varepsilon e^{-\varepsilon}(c_3 - \varepsilon)c_1\,; \quad 0 \leqslant \varepsilon \leqslant c_3 \wedge k\,.$$

Define

$$c_4 = c_2 h(\varepsilon_0) \tag{3.4}$$

Let $\delta = 1/c_4$, and define bounded stopping-times as follows:

$$u(0) = 0$$

$$u(n) = \inf\{\, t > u(n-1):\ d(X_{u(n-1)}, X_t) \geqslant 1 \,\} \wedge (u(n-1) + \delta)\,. \tag{3.5}$$

Note that necessarily $u(n) \leqslant n\delta$. For $n = 0, 1, 2, \ldots$, define $H(n)$ to be the event that (X_t) leaves the unit ball centered on $X_{u(n-1)}$ sometime before $(u(n-1) + \delta)$; more formally,

$$\begin{aligned} H(n) &= \{\, u(n) < u(n-1) + \delta \,\} \\ &= \{\, d(X_{u(n-1)}, X_t) \geqslant 1, \text{ some } u(n-1) < t < u(n-1) + \delta \,\} \end{aligned} \tag{3.6}$$

The assumption is that there exist real numbers $(a(n))$, $n = 1, 2, \ldots$ such that

$$P(H(n) \mid \mathcal{F}_{u(n-1)}) \leqslant a(n)$$

with

$$\sum_{N=1}^{\infty}\left(\prod_{n=1}^{N} a(n)\right) < \infty \tag{3.7}$$

Intuitively speaking, this says that the process (X_t) is not allowed to escape from unit balls too fast.

The theorem below refers to connected manifolds M whose sectional curvatures are bounded above by a negative constant. The Cartan-Hadamard theorem states that for such M, the exponential map at any point x of M provides a diffeomorphism $\exp_x$ from $\mathbb{R}^d$ to M where $d = \dim(M)$. Let us take polar co-ordinates in $\mathbb{R}^d$, so that we now have a diffeomorphism

$$\exp_x : (0, \infty) \times S^{d-1} \to M/\{x\}.$$

The *geometric compactification* of M is the space $M \cup S^{d-1}$ obtained by identifying M with $[0, \infty) \times S^{d-1}$, and adjoining $\{\infty\} \times S^{d-1}$. We say that a sequence $(r(n), \theta(n))$ in M converges to (∞, θ) in $M \cup S^{d-1}$ if $r(n)$ tends to infinity and $\theta(n)$ tends to θ in S^{d-1}. We speak of $\{\infty\} \times S^{d-1}$ as the *sphere at infinity*.

THEOREM A. — Let (M, g) be a simply-connected Riemannian manifold with sectional curvatures bounded above by $-k^2 < 0$. Let (X_t) be a Γ-martingale on M. Suppose that the multidirectional assumption (A1), and the escape rate assumption (A2) are in force. Then we have

i) $r(X_t) \to \infty$ as $t \to \infty$.

ii) $X_\infty = \lim_{t\to\infty} X_t$ exists almost surely in $M \cup S^{d-1}$, and X_∞ lies in S^{d-1} almost surely.

§ 4. PROOF OF THE CONVERGENCE THEOREM

Step 1.

Let $(u(n))$ be the stopping-times defined in § 3.5. The Ito formula (2.4) may be applied to $e^{-\varepsilon r(x)}$ for $\varepsilon > 0$, since (X_t) has zero probability of hitting x_0: thus

$$e^{-\varepsilon r(X_{u(n)})} - e^{-\varepsilon r(X_{u(n-1)})} = \int_{u(n-1)}^{u(n)} (-\varepsilon e^{-\varepsilon r(X_s)} dr(X_s) \circ U_s) dZ_s - \frac{1}{2}\int_{u(n-1)}^{u(n)} e^{-\varepsilon r(X_s)} \varepsilon(\nabla dr(X_s) - \varepsilon dr \otimes dr(X_s))(U dZ \otimes U dZ)_s \tag{4.1}$$

Since (X_t) is a Γ-martingale, it follows from Proposition 1 that (Z_t) is a local martingale; by (A1), Z_t is in L^2 and so the first integral, whose integrand is bounded, is a martingale. The second integral can be written, using (A1), as

$$\frac{1}{2}\int_{u(n-1)}^{u(n)} e^{-\varepsilon r(X_s)}\varepsilon\left\{\sum_{i,j}(\nabla dr(X_s)(U_s(e_i), U_s(e_j)) - \varepsilon dr(U_s(e_i))dr(U_s(e_j)))A^{ij}(s)\right\}ds.$$

The right choice of orthonormal basis (e_i) diagonalizes $(A^{ij}(s))$, so that

$$\{\ \dots\ \} = \sum_{i=1}^{d}(\nabla dr(X_s)(U_s(e_i), U_s(e_i)) - \varepsilon(dr(U_s(e_i)))^2)\lambda_i(s) \tag{4.2}$$

Step 2. Hessian comparison theorem.

We shall first perform a small algebraic calculation. Let $(e_1, \dots, e_d)$ denote an orthonormal basis of $\mathbb{R}^d$, and let

$$v = (a_1e_1 + \dots + a_ne_n), \ \sum_{i=1}^{d} a_i^2 = 1$$

be a unit vector; define

$$\hat{e}_i = e_i(1 - v.e_i) = (1 - a_i)e_i$$
$$f(a_1, \dots, a_n) = |\hat{e}_1|^2 + |\hat{e}_2|^2 = (1 - a_1)^2 + (1 - a_2)^2 .$$

LEMMA. — $\min\{f(a_1, \dots, a_n) : a_1^2 + \dots + a_n^2 = 1\} = 3 - 2\sqrt{2}.$

Proof. — The minimum is attained when $a_1 = a_2 = 2^{-1/2}$.

We return to the integral obtained in step 1. A (random) tangent vector such as $U_s(e_i)$ can be decomposed into a radial part and an orthogonal part; denote the latter by $U_s(e_i)^\perp$. Greene and Wu (7, p. 21] show that:

$$\nabla dr(U_s(e_i), U_s(e_i)) = \nabla dr(U_s(e_i)^\perp, U_s(e_i)^\perp) \geqslant k\,\|U_s(e_i)^\perp\|^2$$

by the Hessian comparison theorem; we have used the fact that the Hessian of the distance function in M is greater (in a sense described precisely by Greene and Wu [7]) than the Hessian of the distance function $r_k(.)$ in a manifold of constant sectional curvature $-k^2$, and

$$\nabla dr_k(u, u) = k \coth(kr_k)\,\|u^\perp\|^2 \geqslant k\,\|u^\perp\|^2 .$$

Continuing from the end of step 1,

$$\sum_{i=1}^{d} \nabla dr(X_s)(U_s(e_i), U_s(e_i))\lambda_i(s) \geqslant \lambda_2(s) \sum_{i=1}^{2} \nabla dr(X_s)(U_s(e_i)^\perp, U_s(e_i)^\perp)$$

$$\geqslant \lambda_2(s)k \sum_{i=1}^{2} \| U_s(e_i)^\perp \|^2 \geqslant \lambda_2(s)k(3-2\sqrt{2})$$

by the lemma, using the fact that $U_s(e_i)$ is of unit length. Using (A1) this is

$$\geqslant \lambda_1(s)k(3 - 2\sqrt{2})/c_2$$

or in the notation of (A2).

$$\geqslant c_3\lambda_1(s)\,.$$

On the other hand,

$$\sum_{i=1}^{d} (dr(U_s(e_i)))^2\lambda_i(s) \leqslant \lambda_1(s),$$

Since

$$\sum_{i=1}^{d} (dr(U_s(e_i)))^2 = \| \operatorname{grad} r \|^2 = 1\,.$$

The conclusion about formula (4.2) is that

$$\{\ \dots\ \} \geqslant (c_3 - \varepsilon)\lambda_1(s) > (c_3 - \varepsilon)c_1 > 0 \tag{4.3}$$

by (A1), assuming $0 < \varepsilon < c_3$.

Step 3. Applying (A2).

Continuing from (4.1) for $0 < \varepsilon < c_3$

$$\mathbb{E}[e^{-\varepsilon r(X_{u(n)})} - e^{-\varepsilon r(X_{u(n-1)})} \mid \mathscr{F}_{u(n-1)}] \leqslant -\frac{1}{2}\varepsilon_1(c_3 - \varepsilon)\mathbb{E}\left[\int_{u(n-1)}^{u(n)} e^{-\varepsilon r(X_s)}ds \mid \mathscr{F}_{u(n-1)}\right]$$

using (4.3) and the fact that the first integral on the right side of (4.1) is a martingale. By the definition of $u(n)$ in (3.5):

$$r(X_s) \leqslant r(X_{u(n-1)}) + 1, \qquad u(n-1) \leqslant s < u(n)$$

and hence

$$-\, e^{-\varepsilon r(X_s)} \leqslant -\, e^{-\varepsilon}e^{-\varepsilon r(X_{u(n-1)})}, \qquad u(n-1) \leqslant s < u(n)\,.$$

If the constant c_4 is as in formula (3.4) we have:

$$\mathbb{E}[e^{-\varepsilon r(X_{u(n)})} - e^{-\varepsilon r(X_{u(n-1)})} \mid \mathcal{F}_{u(n-1)}] \leqslant -c_4 e^{-\varepsilon r(X_{u(n-1)})} \mathbb{E}[u(n)-u(n-1) \mid \mathcal{F}_{u(n-1)}]$$

where $\varepsilon = \varepsilon_0$ as in (3.4). (Note that $0 < \varepsilon_0 < c_3$). Hence

$$\mathbb{E}[e^{-\varepsilon r(X_{u(n)})} \mid \mathcal{F}_{u(n-1)}] \leqslant \{1 - c_4 \mathrm{E}[u(n)-u(n-1) \mid \mathcal{F}_{u(n-1)}]\} e^{-\varepsilon r(X_{u(n-1)})} \tag{4.4}$$

Refer back to the definition of $\mathrm{H}(n)$ in (3.6). Evidently

$$u(n) - u(n-1) \geqslant \delta 1_{\mathrm{H}(n)^c}.$$

By definition of δ, $c_4\delta = 1$; therefore

$$\begin{aligned} 1 - c_4 \mathbb{E}[u(n)-u(n-1) \mid \mathcal{F}_{u(n-1)}] &\leqslant 1 - c_4 \delta \mathbb{E}[1_{\mathrm{H}(n)^c} \mid \mathcal{F}_{u(n-1)}] \\ &= \mathrm{E}[1_{\mathrm{H}(n)} \mid \mathcal{F}_{u(n-1)}] \leqslant a(n). \end{aligned}$$

Therefore (4.4) says that

$$\mathbb{E}[e^{-\varepsilon r(X_{u(n)})} \mid \mathcal{F}_{u(n-1)}] \leqslant a(n) \mathbb{E}[e^{-\varepsilon r(X_{u(n-1)})} \mid \mathcal{F}_{u(n-1)}] = a(n) e^{-\varepsilon r(X_{u(n-1)})}.$$

By induction,

$$\mathrm{E}[e^{-\varepsilon r(X_{u(n)})}] \leqslant \prod_{j=1}^{n} a(j) e^{-\varepsilon r(X_0)}.$$

Therefore by assumption (A_2), equation (3.7),

$$\sum_{n=0}^{\infty} \mathbb{E}[e^{-\varepsilon r(X_{u(n)})}] < \infty \tag{4.5}$$

This proves that $r(X_{u(n)})$ goes to infinity almost surely as n tends to infinity. Therefore $r(X_t)$ tends to infinity as t tends to infinity, proving (i).

Step 4. Angular convergence.

For points x and y in M, let $\theta(x, y)$ denote the angle in the tangent space at m between the geodesics from m to x and from m to y respectively. Since we are assuming $\varepsilon \leqslant k$, geometric calculation shows that for some $c_5 > 0$,

$$\theta(x, y) \leqslant c_5 e^{-\varepsilon r(x)}, \qquad \text{provided} \quad d(x, y) < 1$$

(see Prat [*12*]). Consequently, if $u(n) \leqslant t \leqslant u(n+1)$,

$$\theta(X_{u(n)}, X_t) \leqslant c_5 e^{-\varepsilon r(X_{u(n)})},$$

and

$$\theta(X_{u(n)}, X_{u(m)}) \leqslant c_5 \sum_{j=n}^{m-1} e^{-\varepsilon r(X_{u(j)})}$$

which tends to zero as n, m tend to infinity, by (4.5). These two assertions prove that almost surely,

$$\theta(X_s, X_t) \to 0 \quad \text{as} \quad s, t \to \infty .$$

Together with the fact that $r(X_t)$ tends to infinity as t tends to infinity, this shows that there exists a random variable X_∞ with values in S^{d-1}, representing a limiting direction for (X_t), in the sense that

$\theta(X_t, X_\infty) \to 0$ as $t \to \infty$, almost surely.

§ 5. SOME ESTIMATES ON BROWNIAN MOTION IN R^m

We shall begin by deriving anew a classical estimate on m-dimensional Brownian motion. First we need some notation and a Lemma on 1-dimensional Brownian motion.

Notation. — If (Y_t) is a stochastic process taking non-negative real values, then define

$$Y_t^* = \sup_{0 \leqslant s \leqslant t} Y_s .$$

LEMMA 1. — Let (B_t) denote 1-dimensional Brownian motion, started at zero. Then for all $c > 0$

$$P(|B_t|^* < c) \geqslant (32t/\pi c^2)^{0.5}(1 - \exp(-c^2/2t)) \tag{5.1}$$

Proof. — Define

$$M_c = \inf\{ t : B_t = c \} .$$

Then

$$\begin{aligned} P(|B_t|^* \geqslant c) &= P(\{M_c \leqslant t\} \cup \{M_{-c} \leqslant t\}) \\ &= P(M_c \leqslant t) + P(M_{-c} \leqslant t) - P(\{M_c \leqslant t\} \cap \{M_{-c} \leqslant t\}) < 2P(M_c \leqslant t) \end{aligned}$$

using the symmetry of the distribution of Brownian motion about zero. The reflection principle (Ito and McKean [*16*, p. 26]) shows that

$$P(M_c \leqslant t) = 2 \int_c^\infty (2\pi t)^{-0.5} \exp(-u^2/2t) du .$$

Hence

$$P(|B_t|^* < c) \geqslant 1 - 4\int_c^\infty (2\pi t)^{-0.5} \exp(-u^2/2t)du$$

$$= \int_{-c}^{c} 2(2\pi t)^{-0.5} \exp(-u^2/2t)du \tag{5.2}$$

When u is in the interval $[0, c]$, $u^2 \leqslant uc$; therefore

$$\int_{-c}^{c} \exp(-u^2/2t)du \geqslant 2\int_0^c \exp(-uc/2t)du = \frac{4t}{c}(1-\exp(-c^2/2t)) \tag{5.3}$$

The combination of (5.2) and (5.3) give the result.

LEMMA 2. — Let m be a positive integer, and let $(W_t) = (W_t^1, \ldots, W_t^m)$ denote Brownian motion in $\mathbb{R}^m$. Then

$$P(|W_t|^* < a) \geqslant (32t/\pi)^{m/2}a^{-m}e^{m\ln(m)/2}(1 - \exp(-a^2/2tm))^m \tag{5.4}$$

Proof.

$$\{|W_t|^* < a\} \supseteq \bigcap_{j=1}^{m} \{|W_t^j|^* < a/\sqrt{m}\},$$

because for every sample path in the set on the right,

$$|W_t|^2 = \sum_{j=1}^{m} |W_t^j|^2 \leqslant m(a/\sqrt{m})^2 = a^2.$$

By the independence of the co-ordinate processes,

$$P(|W_t|^* < a) \geqslant \prod_{j=1}^{m} P(|W_t^j|^* < a/\sqrt{m}).$$

By the previous Lemma, with $c = a/\sqrt{m}$, this is

$$\geqslant \{(32tm/\pi a^2)^{0.5}(1 - \exp(-a^2/2tm))\}^m$$
$$\geqslant (32t/\pi)^{m/2}a^{-m}e^{m\ln(m)/2}(1 - \exp(-a^2/2tm))^m$$

§ 6. ESTIMATES ON THE RADIAL PROCESS OF BROWNIAN MOTION ON M

PROPOSITION 2. — Suppose M is a d-dimensional Riemannian manifold with sectional curvatures $K(x)$ satisfying

$$-q^2 \leqslant K(x) \leqslant -k^2 < 0, \qquad x \text{ in } M$$

for constants k and q. Let (X_t) be a Brownian motion on M and let

$$R_t = d(X_0, X_t), \qquad R_t^* = \sup\{R_s : 0 \leqslant s \leqslant t\}.$$

Then

$$P(R_t^* < a) \geqslant \left(\frac{32t}{\pi}\right)^{m/2} a^{-m} e^{m \ln(m)/2}\left(1 - \exp\left(\frac{-a^2}{2tm}\right)\right)^m \tag{6.1}$$

provided $m \geqslant d + (d-1)aq$.

In order to prove the proposition above, let M′ be a d-dimensional manifold of constant sectional curvature $-q^2$. Take and R^d-valued Brownian motion $(\overline{W}_t)$ and points x'_0 in M′, x_0 in M, and solve the canonical stochastic dynamical systems of M′ and M to obtain Brownian motions (X'_t) and (X_t) respectively, both driven by $(\overline{W}_t)$, with $X'_0 = x'_0$ and $X_0 = x_0$. Define real-valued processes

$$R'_t = d(x'_0, X'_t), \qquad R_t = d(x_0, X_t) \tag{6.2}$$

Let $r(y) = d(x_0, y)$ for y in M, and define

$$b(x) = \frac{1}{2}(d-1)q \coth(qx), \qquad x \text{ in } \mathbb{R}. \tag{6.3}$$

By appling Ito's formula to the distance function $r(.)$ and to its counterpart in M′, Prat [12] shows that there exist a real-valued Brownian motions (B_t) and (B'_t) on the same probability space such that

$$R_t = B_t + \int_0^t \frac{1}{2}\Delta_M r(X_s)ds \tag{6.4}$$

and

$$R'_t = B'_t + \int_0^t b(R'_s)ds \tag{6.5}$$

In actual fact, the function $b(.)$ is half the Laplacian of the distance function in the manifold of constant curvature $-q^2$. Now define two more real-valued processes to be the unique strong solutions, started at zero, to the stochastic integral equations:

$$S_t = B'_t + \int_0^t \frac{1}{2}(d-1)\left(q + \frac{1}{S_u}\right)du \tag{6.6}$$

$$Y_t^{(m)} = B'_t + \int_0^t \frac{1}{2}\left(\frac{m-1}{Y_s^{(m)}}\right)ds \tag{6.7}$$

where m is some positive integer $\geqslant 2$. It is well known—see Ikeda and Watanabe [9, p. 223]—that the law of $(Y_t^{(m)})$ is that of the distance of Brownian motion in $\mathbb{R}^m$ from the origin, known as the Bessel process of order m.

LEMMA 3. — *i)* $R'_t \leqslant S_t$ a.s., and $P(R_t < a) \geqslant P(R'_t < a)$ for all $a \geqslant 0$, $t \geqslant 0$. Assume that $m \geqslant d + (d-1)aq$, for some $a > 0$. Then

ii) On the set $\{ Y_t^{(m)*} \leqslant a \}$, $R'_s \leqslant Y_s^{(m)}$ a.s., for all $0 \leqslant s \leqslant t$.

iii) $P(R_t^* < a) \geqslant P(|W_t|^* < a)$

where (W_t) is Brownian motion in $\mathbb{R}^m$.

Remark. — The comparison of (R_t) and (R'_t) is due to Debiard, Gaveau and Mazet [4]. The idea of using (S_t) and $(Y_t^{(m)})$ was suggested to the author by S. R. S. Varadhan (Courant Institute).

Proof. — The Laplacian comparison theorem, found for example in Greene and Wu [7], shows that if $b(.)$ is the function defined in (6.3),

$$b(r(y)) \geqslant \frac{1}{2}\Delta_M r(y), \quad y \ \text{ in } \ M.$$

It is a simple calculation to verify that

$$\frac{1}{2}(d-1)\left(q + \frac{1}{x}\right) = \frac{1}{2}(d-1)q\left(1 + \frac{1}{qx}\right) \geqslant b(x).$$

Part (*i*) of the Lemma now follows from the comparison theorem in Ikeda and Watanabe [9, p. 352]. The same theorem is used to prove part (*ii*), noting that

$$(m-1) \geqslant (d-1)(aq+1),$$

and

$$0 < y \leqslant a \Rightarrow (m-1) \geqslant (d-1)(yq+1) \Rightarrow \frac{m-1}{y} \geqslant \left(d-1\right)\left(q + \frac{1}{y}\right)$$

which proves $S_u \leqslant Y_u^{(m)}$ on $0 \leqslant u \leqslant t$, on the set $\{ Y_t^{(m)*} \leqslant a \}$. Combined with the result of (*i*), this gives part (*ii*).

As for part (*iii*), let

$$M_a^{(m)} = \inf \{ t : Y_t^{(m)} = a \}, \qquad N_a = \inf \{ t : R'_t = a \}.$$

Part (*ii*) implies that

$$s < M_a^{(m)} \Rightarrow R'_s \leqslant Y_s^{(m)} < a \Rightarrow s < N_a.$$

Hence

$$P(M_a^{(m)} > s) \leqslant P(N_a > s)$$

or equivalently

$$P(R_s^* < a) \geqslant P(R_s'^* < a) \geqslant P(Y_s^{(m)*} < a) = P(|W_s|^* < a).$$

From (*iii*) and Lemma 2 of § 5, Proposition 2 now follows.

§ 7. BROWNIAN MOTION ON A MANIFOLD WITH NEGATIVE CURVATURE GOING TO $-\infty$

Ichihara [*8*] has proved that for Riemannian manifolds M whose Ricci curvature at distance r along any minimal geodesic is bounded below by $-ar^2 - b$ (here a and b are some positive constants) the Brownian motion (X_t) on M is non-explosive, meaning that

$$P(X_t \in M) = 1, \qquad 0 \leqslant t < \infty.$$

It is well-known, and easy to prove, that if the sectional curvatures of M are bounded above by a negative constant, then the distance process

$$R_t = d(X_0, X_t)$$

tends to infinity almost surely as t tends to infinity. An open problem is to characterize those manifolds M of negative curvature on which Brownian motion has an angular limit; in other words, those for which X_∞ ($= \lim_{t\to\infty} X_t$) exists in S^{d-1}. (See section § 3 for explanation.). A sufficient condition, given by Prat [*12*], is that the sectional curvatures of M are pinched between two negative constants. The following extension of Prat's result is included as an example of the application of Theorem A of § 3.

THEOREM B. — Suppose M is a d-dimensional Riemannian manifold containing some compact c_0 such that the sectional curvatures $K(x)$ at points x outside c_0 satisfy

$$-c^2 (\ln (r(x)))^{2-2h} \leqslant K(x) < -k^2 < 0 \tag{7.1}$$

for some $0 < h < 1$ and some constant c; here $r(x)$ is the distance of x from some fixed point x_0 in c_0. Then Brownian motion (X_t) on M has a limit almost surely on the sphere at infinity. In other words

$$X_\infty = \lim_{t\to\infty} X_t \text{ exists almost surely in } S^{d-1} .$$

COROLLARY. — A manifold with curvatures satisfying (7.1) admits non-constant bounded harmonic functions.

Proof of the Corollary. — Let h be any bounded, continuous function on S^{d-1}. Define for x in M

$$f(x) = \mathbb{E}_x[h(X_\infty)] ,$$

where $\mathbb{E}_x[\dots]$ means that $X_0 = x$. Then f is a bounded harmonic function on M.

Proof of Theorem B. — For the sake of simplicity, let the Brownian motion (X_t) begin at $X_0 = x_0$, and define $r(x) = d(x_0, x)$ (the distance from x_0 to x in M). Notice that

$$u(n-1) \leqslant t \leqslant u(n) \Rightarrow r(X_t) \leqslant \sum_{j=0}^{n-2} d(X_{u(j)}, X_{u(j+1)}) + d(X_{u(n-1)}, X_{u(n)}) .$$

From the definition of the stopping times $(u(n))$, it follows that $r(X_t) \leqslant n$ when $u(n-1) \leqslant t \leqslant u(n)$. Define for each n a real-valued process

$$R_s^n = d(X_{u(n-1)}, X_{u(n-1)+s}), \quad \text{stopped at } s = u(n) - u(n-1) .$$

On the time interval $[0, u(n)]$, the Brownian motion (X_t) lives inside the ball of radius n in M; therefore it experiences sectional curvatures K(.) bounded by

$$- q(n)^2 \leqslant K(x) \leqslant - k^2$$

where

$$q(n) = c\,(\ln(n))^{1-h} \tag{7.2}$$

Therefore the Corollary at the end of the previous section applies to the process (R_s^n), for each n. Recall the definition of H(n) in § 3, assumption (A2), equation (3.6):

$$\begin{aligned} 1 - P(H(n)) &= P(u(n) = u(n-1) + \delta) = P(R_\delta^{n*} < 1) \\ &\geqslant \left(\frac{32\delta}{\pi}\right)^{m/2} e^{m\ln(m)/2}(1 - \exp(-1/2\delta m))^m \end{aligned} \tag{7.3}$$

where $m = m(n) = dq(n)$. Let $\gamma = (32\delta/\pi)^{1/2}$ and let $\lambda = \exp(-1/2\delta)$. It can be proved by calculus that for $0 < h < 1$

$$1 - \lambda^h \geqslant h(1 - \lambda).$$

Taking $h = 1/m$, it follows that

$$(1 - \lambda^{1/m})^m \geqslant e^{-m\ln(m) - m\ln(b)}, \qquad b = 1/(1 - \lambda).$$

Putting this into (7.3) gives:

$$1 - \mathrm{P}(\mathrm{H}(n)) \geqslant \exp(-m\ln(m)/2 + m\ln(\gamma) - m\ln(b)) \qquad (7.4)$$

The fact that Brownian motion has independent increments implies that

$$\mathrm{P}(\mathrm{H}(n) \mid \mathscr{F}_{u(n-1)}) = \mathrm{P}(\mathrm{H}(n)).$$

Consider assumption (A2), equation (3.7) again; notice that

$$\prod_{n=1}^{\mathrm{N}}\left(1 - \frac{2}{n+3}\right) = \prod_{n=1}^{\mathrm{N}}\left(\frac{n+1}{n+3}\right) = \frac{2.3}{(\mathrm{N}+2)(\mathrm{N}+3)} \leqslant \frac{6}{\mathrm{N}^2}.$$

Hence if $a(n) = 1 - 2/(n+3)$, assumption (A2) holds. Hence it suffices to show that

$$\mathrm{P}(\mathrm{H}(n)) \leqslant 1 - \frac{2}{n+3},$$

or equivalently,

$$1 - \mathrm{P}(\mathrm{H}(n)) \geqslant \frac{2}{n+3}.$$

Using (7.4), it will be enough to verify that

$$m\ln(m)/2 - m(\ln(\gamma) - \ln(b)) \leqslant -\ln 2 - \ln(n+3) \qquad (7.5)$$

for all sufficiently large n. Now

$$m = dq(n) = cd(\ln(n))^{1-h}$$

So

$$m\ln(m) = 0(\ln(n))^{1-h}\ln(\ln(n)).$$

Hence

$$\frac{m\ln(m)}{2\ln(n+3)} = 0(\ln(\ln(n))/\ln(n))^h.$$

Moreover

$$\frac{m}{\ln(n+3)} = 0((\ln(n))^{-h}).$$

It follows that

$$\frac{m \ln (m)/2 + m (\ln (\gamma) - \ln (b))}{\ln (n + 3)} \to 0 \quad \text{as} \quad n \to \infty$$

and this verifies (7.5). The proof is complete.

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