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On Non-Euclidean Harmonic Measure

by

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ABSTRACT. — Let H_ε be the harmonic measure of the ε -sphere in a Riemannian manifold. We compute $\lim_{\varepsilon \downarrow 0} \frac{H_\varepsilon - H_{0+}}{\varepsilon^3}$ in terms of the gradient of the Gaussian curvature.

RÉSUMÉ. — Soit H_ε la mesure harmonique de la sphère du rayon ε d'une variété riemannienne de dimension deux. On exprime $\lim_{\varepsilon \downarrow 0} (H_\varepsilon - H_{0+})/\varepsilon^3$ en fonction de la dérivée de la courbure gaussienne.

1. INTRODUCTION

Let (M, g) be an n -dimensional Riemannian manifold with Brownian motion process $\{X_t, t \geq 0\}$. According to the Onsager-Machlup formula [3], the « most probable path » is that of a classical particle in a conservative force field whose potential energy is one-twelfth the scalar curvature. This suggests that the « most probable path » follows the negative gradient of the scalar curvature. The purpose of this note is to make this precise in terms of the harmonic measure of a small sphere. For technical reasons we restrict the discussion to surfaces. The technique follows the method used to study the mean exit time [1].

2. NOTATIONS AND DEFINITIONS

Let (M, g) be an n -dimensional Riemannian manifold. We use the following notations:

$\bar{M}_m$ is the tangent space at $m \in M$.

$B_m(\varepsilon)$ is the ball of radius ε in M with center at $m \in M$.

$\bar{B}_m(\varepsilon)$ is the ball of radius ε in $\bar{M}_m$ with center at $\dot{0} \in \bar{M}_m$.

$\exp_m$ is the exponential mapping (which is defined on all of $\bar{M}_m$ in case M is complete; otherwise it is a mapping) from $\bar{B}_m(\varepsilon)$ to $B_m(\varepsilon)$ for sufficiently small $\varepsilon > 0$.

Φ_ε is the mapping on functions defined by

$$(\Phi_\varepsilon f)(\exp_m \varepsilon x) = f(x);$$

Φ_ε maps from $C^\infty(\bar{B}_m(1))$ to $C^\infty(B_m(\varepsilon))$ for sufficiently small $\varepsilon > 0$.

Δ is the Laplace-Beltrami operator of the Riemannian manifold:

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x_j} \right) \quad \text{where} \quad g^{ij} = (g^{-1})^{ij}, \quad g = \det(g_{ij}).$$

The following result, which will be used repeatedly, was proved in [1].

PROPOSITION 2.0. — *There exist second order differential operators $(\Delta_{-2}, \Delta_0, \Delta_1, \dots)$ on $C^\infty(\bar{M}_m)$ such that for each $N \geq 0$ and each $f \in C^\infty(\bar{M}_m)$*

$$(2.1) \quad \Phi_\varepsilon^{-1} \Delta \Phi_\varepsilon f = \varepsilon^{-2} \Delta_{-2} f + \sum_{j=0}^N \varepsilon^j \Delta_j f + o(\varepsilon^{N+1}) \quad (\varepsilon \downarrow 0).$$

Δ_j maps polynomials of degree k to polynomials of degree $k + j$. In any normal coordinate chart $(x_1, \dots, x_n)$ we have

$$(2.2) \quad \Delta_{-2} f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

$$(2.3) \quad \Delta_0 f = (1/3) \sum_{i,a,j,b=1}^n R_{iajb} x_a x_b \frac{\partial^2 f}{\partial x_i \partial x_j} - (2/3) \sum_{i,a=1}^n \rho_{ia} x_a \frac{\partial f}{\partial x_i}$$

Here R_{iajb} is the Riemann tensor and $\rho_{ij} = \sum_{a=1}^n R_{iaja}$ is the Ricci tensor at $m \in M$.

Let (X_t, P_x) be the Brownian motion process with infinitesimal generator Δ . For each $m \in M$ let T_ε be the exit time from the geodesic ball $B_m(\varepsilon)$.

In case $n = 2$ we introduce geodesic polar coordinates (r, θ) by $x_1 = r \cos \theta$, $x_2 = r \sin \theta$. The coordinate formulas for the first three operators become

$$(2.4) \quad \Delta_{-2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$(2.5) \quad \Delta_0 = -\frac{K_0}{3} \left(r \frac{\partial}{\partial r} - \frac{\partial^2}{\partial \theta^2} \right)$$

$$(2.6) \quad \Delta_1 = - (K_1 \cos \theta + K_2 \sin \theta) \left[\frac{r^2}{4} \frac{\partial}{\partial r} - \frac{r}{6} \frac{\partial^2}{\partial \theta^2} \right] \\ - (K_2 \cos \theta - K_1 \sin \theta) \left[\frac{r}{12} \frac{\partial}{\partial \theta} \right]$$

where K is the Gaussian curvature and $K_0 = K(m)$, $K_1 = \frac{\partial K}{\partial x_1}(m)$, $K_2 = \frac{\partial K}{\partial x_2}(m)$. This computation is carried out in the Appendix, Sec. 5.

3. STATEMENT OF RESULTS

To study the harmonic measure we let $f \in C^\infty(S_1)$, a smooth function on the circle. Extend f to $\mathbb{R}^2 \setminus \{0\}$ by making f constant along rays through the origin. Equivalently f can be thought of as a function on $\overline{M}_m \setminus \{0\}$. Then $\Phi_\varepsilon f$ is a smooth function on $M \setminus \{m\}$ which is constant along geodesics emanating from m . The harmonic measure operator is

$$H_\varepsilon f(x) = E_x \{ (\Phi_\varepsilon f)(X_{T_\varepsilon}) \} \quad \varepsilon > 0, \quad x \in B_m(\varepsilon).$$

THEOREM 3.1. — When $\varepsilon \downarrow 0$ we have

$$H_\varepsilon f(m) = \int_{-\pi}^{\pi} f(\theta) \left[1 - \frac{\varepsilon^3}{32} \langle \nabla K, u_\theta \rangle \right] \omega(d\theta) + O(\varepsilon^4)$$

where ω is normalized Lebesgue measure and $\langle \nabla K, u_\theta \rangle = K_1 \cos \theta + K_2 \sin \theta$, $-\pi \leq \theta \leq \pi$.

Proof. — We follow the perturbation method introduced in [1]. Let $u_i \in C^\infty(\overline{B}_m(1))$ ($i = 0, 2, 3$) be defined by

$$(3.1) \quad \Delta_{-2} u_0 = 0 \quad u_0|_{\partial \overline{B}_m(1)} = f$$

$$(3.2) \quad \Delta_{-2}u_2 + \Delta_0u_0 = 0 \quad u_2|_{\partial\overline{B}_m(1)} = 0$$

$$(3.3) \quad \Delta_{-2}u_3 + \Delta_1u_0 = 0 \quad u_3|_{\partial\overline{B}_m(1)} = 0$$

The proof depends on the following three lemmas:

$$\text{LEMMA 1. — } u_0(0) = \int_{-\pi}^{\pi} f(\theta)\omega(d\theta).$$

$$\text{LEMMA 2. — } u_2(0) = 0.$$

$$\text{LEMMA 3. — } u_3(0) = -\frac{1}{32} \int_{-\pi}^{\pi} (K_1 \cos \theta + K_2 \sin \theta) f(\theta)\omega(d\theta).$$

Proof of Lemma 1. — u_0 is the solution of the classical Dirichlet problem in the unit disc. thus

$$u_0 = \int_{-\pi}^{\pi} \frac{(1-r^2)}{(1+r^2-2r\cos(\theta-\phi))} f(\phi)\omega(d\phi).$$

$$\text{In particular } u_0(0) = \int_{-\pi}^{\pi} f(\phi)\omega(d\phi). \quad \square$$

Proof of Lemma 2. — Equivalently we write u_0 as a Fourier series

$$u_0 = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta).$$

Thus

$$\Delta_0 u_0 = -\frac{K_0}{3} \sum_{n=1}^{\infty} (n^2 + n) r^n (A_n \cos n\theta + B_n \sin n\theta).$$

This is a sum of homogeneous polynomials $\sum_1^{\infty} h_n$. By Lemma 6.3 of [1] we have

$$u_2(0) = \sum_1^{\infty} \frac{1}{(n+2)^2} \int_{-\pi}^{\pi} h_n(\theta)\omega(d\theta). \text{ But from the orthogonality relations for}$$

the functions $(\cos n\theta, \sin n\theta)$, all of these integrals are zero, hence $u_2(0) = 0$. $\square$

(Note: The above depends on the fact that $\Delta_0 u_0$ is a harmonic function which vanishes at the origin. This may also be seen from the relation

$$\Delta_{-2}\Delta_0 - \Delta_0\Delta_{-2} = -\frac{2K_0}{3}\Delta_{-2} \text{ applied to } u_0.)$$

Proof of Lemma 3. — We have

$$\left[\frac{r^2}{4} \frac{\partial}{\partial r} - \frac{r}{6} \frac{\partial^2}{\partial \theta^2} \right] u_0 = \sum_1^{\infty} r^{n+1} \left(\frac{n}{4} + \frac{n^2}{6} \right) (A_n \cos n\theta + B_n \sin n\theta)$$

$$\frac{r}{12} \frac{\partial u_0}{\partial \theta} = \sum_1^{\infty} r^{n+1} \left(\frac{n}{12} \right) (-A_n \sin n\theta + B_n \cos n\theta),$$

Referring to the formula (2.6), we see that

$$\Delta_1 u_0 = - \sum_1^{\infty} \left(\frac{n}{4} + \frac{n^2}{6} \right) r^{n+1} (K_1 \cos \theta + K_2 \sin \theta) (A_n \cos n\theta + B_n \sin n\theta)$$

$$- \sum_1^{\infty} \left(\frac{n}{12} \right) r^{n+1} (K_2 \cos \theta - K_1 \sin \theta) (-A_n \sin n\theta + B_n \cos n\theta)$$

a sum of homogeneous polynomials $\sum_2^{\infty} h_k$. By Lemma 6.3 of [1] we have

$$u_3(0) = \sum_2^{\infty} \frac{1}{(k+2)^2} \int_{-\pi}^{\pi} h_k(\theta) \omega(d\theta). \text{ But from the orthogonality relations}$$

of Fourier series all of these integrals are zero except for $k = 2$, i. e. $n = 1$. This gives

$$u_3(0) = - \left(\frac{1}{4} + \frac{1}{6} \right) \frac{1}{4^2} (K_1 A_1 + K_2 B_2)/2$$

$$- \left(\frac{1}{12} \right) \frac{1}{4^2} (K_2 B_2 + K_1 A_1)/2$$

$$= - \frac{1}{32} (K_1 A_1 + K_2 B_2)/2.$$

On the other hand $f(\theta) = A_0 + \sum_1^{\infty} (A_n \cos n\theta + B_n \sin n\theta)$ and thus

$$\int_{-\pi}^{\pi} f(\theta) (K_1 \cos \theta + K_2 \sin \theta) \omega(d\theta) = \frac{1}{2} (K_1 A_1 + K_2 B_2).$$

We have proved that $u_3(0) = - \frac{1}{32} \int_{-\pi}^{\pi} f(\theta) (K_1 \cos \theta + K_2 \sin \theta) \omega(d\theta)$ as required. $\square$

To complete the proof of the theorem we appeal to Proposition 2.0 with $f = u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3$; thus

$$\begin{aligned}\Phi_\varepsilon^{-1} \Delta \Phi_\varepsilon(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) &= \varepsilon^{-2} \Delta_{-2}(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &\quad + \Delta_0(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &\quad + \varepsilon \Delta_1(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3) + O(\varepsilon^2) \\ &= O(\varepsilon^2),\end{aligned}$$

where we have used the defining relations (3.1)-(3.3). Hence $\Delta U = O(\varepsilon^2)$ where $U = \Phi_\varepsilon(u_0 + \varepsilon^2 u_2 + \varepsilon^3 u_3)$. Noting that $U = \Phi_\varepsilon f$ on $\partial B_m(\varepsilon)$ we have that for any $x \in B_m(\varepsilon)$ by Dynkin's formula

$$\begin{aligned}H_\varepsilon f(x) &= E_x U(X_{T_\varepsilon}) \\ &= U(x) + E_x \int_0^{T_\varepsilon} \Delta U(X_s) ds \\ &= U(x) + O(\varepsilon^4).\end{aligned}$$

Setting $x = m$ we have $U(m) = u_0(0) + \varepsilon^2 u_2(0) + \varepsilon^3 u_3(0)$ from which the result follows. $\square$

4. COMPARISON WITH THE NON-STOCHASTIC MEAN VALUE

The mean-value operator of a Riemannian manifold is defined by

$$M_\varepsilon f(m) = \frac{\int_{S_\varepsilon} (\Phi_\varepsilon f) d\sigma_\varepsilon}{\int_{S_\varepsilon} 1 d\sigma_\varepsilon}$$

where σ_ε is the surface measure on the ε -sphere S_ε and f is a smooth function on the unit sphere. We have the following result.

THEOREM 4.1. — *When $\varepsilon \downarrow 0$*

$$M_\varepsilon f(m) = \int_{-\pi}^{\pi} f(\theta) \left[1 - \frac{\varepsilon^3}{12} \langle \nabla K, u_\theta \rangle \right] \omega(d\theta) + O(\varepsilon^4).$$

Proof. — From the method of Gray and Willmore [2], we have

$$M_\varepsilon f = \frac{I_\varepsilon f}{I_\varepsilon 1}$$

where

$$I_\varepsilon f = \int_{-\pi}^{\pi} f(\theta) G(\varepsilon, \theta) d\theta.$$

Using the expansion $G(\varepsilon, \theta) = \varepsilon - \frac{\varepsilon^3}{6} K_0 - \frac{\varepsilon^4}{12} (K_1 \cos \theta + K_2 \sin \theta) + O(\varepsilon^5)$ we have

$$I_\varepsilon f = \left(\varepsilon - \frac{\varepsilon^3}{6} K_0 \right) \int_{-\pi}^{\pi} f(\theta) d\theta - \frac{\varepsilon^4}{12} \int_{-\pi}^{\pi} f(\theta) (K_1 \cos \theta + K_2 \sin \theta) d\theta + O(\varepsilon^5).$$

In particular

$$I_\varepsilon 1 = 2\pi \left(\varepsilon - \frac{\varepsilon^3}{6} K_0 \right) + O(\varepsilon^5).$$

Thus

$$\frac{I_\varepsilon f}{I_\varepsilon 1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left[1 - \frac{\varepsilon^3}{12} (K_1 \cos \theta + K_2 \sin \theta) \right] d\theta + O(\varepsilon^4)$$

which was to be proved.

COROLLARY. — If $H_\varepsilon f(m) = M_\varepsilon f(m)$ for every $m \in M$, $\varepsilon > 0$, $f \in C^\infty(S^1)$, then (M, g) has constant curvature.

Proof. — We have $\lim_{\varepsilon \downarrow 0} \frac{M_\varepsilon f - H_\varepsilon f}{\varepsilon^3} = \frac{-5}{96} \int_{-\pi}^{\pi} \langle \nabla K, u_\theta \rangle f(\theta) \omega(d\theta).$

If this is zero for all f , then $K_1 = K_2 = 0$, i. e. $\nabla K(m) = 0$, and hence K is constant.

5. APPENDIX. COMPUTATION OF Δ_{-2} , Δ_0 , Δ_1

In geodesic polar coordinates we have

$$(5.1) \quad \begin{aligned} (g_{ij}) &= \begin{pmatrix} 1 & 0 \\ 0 & G^2 \end{pmatrix} \quad G = G(r, \theta) \\ \Delta &= \frac{1}{G} \frac{\partial}{\partial r} \left(G \frac{\partial}{\partial r} \right) + \frac{1}{G} \frac{\partial}{\partial \theta} \left(\frac{1}{G} \frac{\partial}{\partial \theta} \right) \\ &= \frac{\partial^2}{\partial r^2} + \frac{G_r}{G} \frac{\partial}{\partial r} + \frac{1}{G^2} \frac{\partial^2}{\partial \theta^2} - \frac{G_\theta}{G^3} \frac{\partial}{\partial \theta} \end{aligned}$$

The Jacobi equation connecting the metric with the curvature is

$$(5.2) \quad G_{rr} + KG = 0, \quad G(0^+, \theta) = 0, \quad G_r(0^+, \theta) = 1.$$

In a neighborhood of m , we can write

$$(5.3) \quad K = K_0 + r(K_1 \cos \theta + K_2 \sin \theta) + O(r^2) \quad (r \downarrow 0).$$

This yields

$$G = r - \frac{r^3}{6} K_0 - \frac{r^4}{12} (K_1 \cos \theta + K_2 \sin \theta) + O(r^5) \quad (r \downarrow 0).$$

Performing the indicated operations, we have

$$\frac{G_r}{G} = \frac{1}{r} - \frac{r}{3} K_0 - \frac{r^2}{4} (K_1 \cos \theta + K_2 \sin \theta) + O(r^3)$$

$$\frac{1}{G^2} = \frac{1}{r^2} + \frac{K_0}{3} + \frac{r}{6} (K_1 \cos \theta + K_2 \sin \theta) + O(r^2)$$

$$\frac{G_\theta}{G^3} = \frac{r}{12} (K_1 \sin \theta - K_2 \cos \theta) + O(r^2).$$

Substituting in (5.1) and equating coefficients of r yields (2.4)-(2.6).

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