

# ANNALES DE L'I. H. P., SECTION B

PETER IMKELLER

**Stochastic analysis and local times for  $(N, d)$ -  
Wiener process**

*Annales de l'I. H. P., section B*, tome 20, n° 1 (1984), p. 75-101

<[http://www.numdam.org/item?id=AIHPB\\_1984\\_\\_20\\_1\\_75\\_0](http://www.numdam.org/item?id=AIHPB_1984__20_1_75_0)>

© Gauthier-Villars, 1984, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques  
<http://www.numdam.org/>

## Stochastic analysis and local times for $(N, d)$ -Wiener process

by

**Peter IMKELLER**

Mathematisches Institut der Ludwig-Maximilians-Universität München,  
D-8000 München 2, Theresienstrasse 39, Federal Republic of Germany

---

**ABSTRACT.** — Let  $N, d \in \mathbb{N}, k \in \mathbb{N}_0^d$  be such that  $2|k| + d < 2N$ . It is well-known that in this case the  $(N, d)$ -Wiener process  $W$  has local times possessing (jointly in  $(t, x)$ , the time resp. space variables being  $t$  resp.  $x$ ) continuous derivatives in  $x$  of order  $k$ . In the framework of an appropriated stochastic calculus for  $(N, d)$ -Wiener process which generalizes Wong's and Zakai's calculus for the Wiener sheet, we derive Tanaka-like formulas for versions  $L^{(k)}$  of these derivatives. Using a method provided by the underlying calculus, we prove  $(t, x)$ -continuity for  $L^{(k)}$ : with the help of Burkholder's inequalities for the stochastic integral processes occurring in Tanaka's formula we establish Kolmogorov's continuity criterion. More generally, for  $\emptyset \neq V \subset \{1, \dots, N\}$ ,  $d \in \mathbb{N}, k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2|V|$ , the local time of  $(N, d)$ -Wiener process can be obtained by integrating the local times of the  $(|V|, d)$ -processes  $W_{(\cdot, t_{\bar{V}})}$  over  $t_{\bar{V}}$ . Using this observation, we get Tanaka-like formulas for the joint  $(t, x)$ -derivatives of the local time of  $W$  ( $k^{\text{th}}$  partial in  $x$ , w. r. to  $t_i, i \in \bar{V}$ , in  $t$ ) for which the above mentioned method yields continuity results, too.

**RÉSUMÉ.** — Soient  $N, d \in \mathbb{N}, k \in \mathbb{N}_0^d$  tels que  $2|k| + d < 2N$ . Il est bien connu qu'en ce cas le  $(N, d)$ -processus de Wiener possède un temps local ayant des dérivées en  $x$  d'ordre  $k$  continues (en  $(t, x)$ ,  $t$  étant la variable du temps,  $x$  celle de l'espace). Dans le cadre d'un calcul stochastique approprié pour le  $(N, d)$ -processus de Wiener qui généralise le calcul de Wong et Zakai pour le drap Brownien, on obtient des formules à la Tanaka pour

des versions  $L^{(k)}$  de ces dérivées. Utilisant une méthode provenant de ce calcul stochastique même, on démontre la continuité en  $(t, x)$  pour  $L^{(k)}$  : à l'aide des inégalités de Burkholder pour les processus intégraux stochastiques figurant dans la formule de Tanaka on vérifie le critère de Kolmogorov pour continuité. Plus généralement, étant donné  $\emptyset \neq V \subset \{1, \dots, N\}$ ,  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  tels que  $2|k| + d < 2|V|$ , le temps local du  $(N, d)$ -processus de Wiener s'obtient en intégrant les temps locaux des  $(|V|, d)$ -processus  $W_{(\cdot, t\bar{V})}$  en  $t\bar{V}$ . On utilise cette observation pour obtenir des formules à la Tanaka pour les dérivées en  $(t, x)$  du temps local de  $W$  (la  $k^{\text{ième}}$  en  $x$ , par rapport à  $t_i$ ,  $i \in \bar{V}$ , en  $t$ ) pour lesquelles la méthode déjà mentionnée fournit des résultats de continuité aussi.

---

## INTRODUCTION

It is well-known that  $N$ -parameter Wiener process with values in  $\mathbb{R}^d$  for  $d < 2N$  has local times whose  $k^{\text{th}}$  partial derivatives in the space variables exist and are jointly continuous (in space and time) up to  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2N$ . Roughly speaking, local times become smoother if  $N$  increases. The opposite is true if  $d$  increases. To prove this result, Ehm [8] has generalized Berman's [3] method of Fourier-analyzing occupation times (in fact, Ehm considered a large class of « Lévy-processes »): take the Fourier-transform of occupation time and study its integrability and differentiability properties. This yields very sharp results on moduli of continuity of local time (for this method, see also Tran [17] and Adler [1]).

But local times are quite generally accessible from stochastic analysis, too. This fact is well-known from one-parameter semimartingale theory (see Meyer [14], p. 361-371, Azema, Yor [2], Bichteler [5]). There are a few results for multi-parameter processes, too: Cairoli, Walsh [6] give a representation by a Tanaka-like formula for a local time of the Wiener sheet; Walsh [19] investigates smoothness properties of a local time of the Wiener sheet by means of Tanaka's formula. Local times for  $N$ -parameter « semimartingales » have been studied in [11]. See Geman, Horowitz [9] for a survey on local times.

This paper's aim is two-fold: firstly, to describe the local time of  $(N, d)$ -Wiener process with  $d < 2N$  and its (space-and-time) partial derivatives by Tanaka-like formulae in the framework of an appropriate stochastic calculus; secondly, to prove the smoothness properties of these functions

by means of the underlying stochastic calculus. Hereby, no attempt is made to cope the sharpness of the Fourier-analytic method's results.

An « appropriate stochastic calculus » has been presented in a previous paper (see Imkeller [10]). As direct generalizations of Wong, Zakai's [20], the stochastic integrals necessary for a complete calculus are constructed in the following way: for each partition  $\mathcal{C}$  of  $\{1, \dots, N\}$  we take a function  $\phi: \mathcal{C} \rightarrow \{0, 1, \dots, d\}$  to note whether in T-direction the measure with respect to which we integrate is Lebesgue measure ( $\phi(T) = 0$ , i. e. «  $T \in \mathcal{C}^0$  ») or is the stochastic measure associated with  $W^j$  ( $\phi(T) = j$ , i. e. «  $T \in \mathcal{C}^j$  »),  $1 \leq j \leq d$ ,  $T \in \mathcal{C}$ . We obtain a set of integrals  $I^{(\mathcal{C}, \phi)}$ , such that for  $f \in C^{2N}(\mathbb{R}^d)$ , with  $D^{(\mathcal{C}, \phi)} f(W)$  square integrable w. r. t.  $P \times \lambda^N$  for all  $(\mathcal{C}, \phi)$ , we have the (« Ito's ») formula

$$f(W_t) - f(0) = \sum_{(\mathcal{C}, \phi)} \frac{1}{2^{|\mathcal{C}^0|}} I^{(\mathcal{C}, \phi)}([1_{\Omega \times ]0, t]} D^{(\mathcal{C}, \phi)} f(W)]^{\mathcal{C}}), \quad t \in [0, 1]^N.$$

Here  $D^{(\mathcal{C}, \phi)}$  is a differential operator obtained by applying  $|\mathcal{C}^0|$  times the Laplacian  $\mathbb{D}$  and  $|\mathcal{C}^j|$  times partial differentiation in direction  $j$ ,  $1 \leq j \leq d$ ; for any process  $Y$ ,  $Y^{\mathcal{C}}$  is the «  $\mathcal{C}$ -corner function » of  $Y$ :  $(s^T)_{T \in \mathcal{C}} \rightarrow Y(\sup_{T \in \mathcal{C}} s^T)$ . To derive a Tanaka-like formula, we take the term

of highest differentiation order

$$\frac{1}{2^N} \int_{]0, t]} \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du$$

in Ito's formula for formally describing a local time of  $W$  over  $]0, t]$  at  $x \in \mathbb{R}^d$  by

$$\int_{]0, t]} \delta_{W_u - x} \prod_{1 \leq i \leq N} u_i^{N-1} du,$$

$\delta_y$  being Dirac's  $\delta$ -distribution at  $y \in \mathbb{R}^d$ , which is « natural » for our calculus. Therefore, a representation of local time is obtained by generalizing Ito's formula to the solutions  $F^{N,d}(x, \cdot)$  of the partial differential equations  $\mathbb{D}^N F^{N,d}(x, \cdot) = \delta_{\cdot - x}$ ,  $x \in \mathbb{R}^d$ . This, however, requires allowing the integrals  $I^{(\mathcal{C}, \phi)}$  to be distribution-valued (for  $N = 1$ , see Ustunel [18]). It turns out that there is, yet, another possibility which requires starting with a modification of Ito's formula (but keeps the values of the representing integrals in  $\mathbb{R}$ ): by « partial stochastic integration » like in the classical Gauss' integral theorem we replace integrals over intervals by integrals

$I^{(\bar{\mathcal{C}}, \phi, t\bar{U})}$  of the processes  $W_{(\cdot, t\bar{U})}$ , i. e. integrals over « affine submanifolds » of  $[0, 1]^N$ ,  $U_i \subset \{1, \dots, N\}$ ,  $(\bar{\mathcal{C}}, \phi)$  being related to  $|U|$ -parameter space. This procedure essentially reduces the orders of occurring differential operators to at most  $N$ ; the sum in the resulting formula extends over  $(\bar{\mathcal{C}}, \phi) \in \Lambda$ , i. e. each  $T \in \bar{\mathcal{C}}^0$  has at least two elements:

$$\begin{aligned} f(W_t^v) - f(0) = & \sum_{(\bar{\mathcal{C}}, \phi) \in \Lambda, \bigcup_{T \in \bar{\mathcal{C}}} T = U} \frac{1}{2^{|\bar{\mathcal{C}}|}} \alpha_{(\bar{\mathcal{C}}, \phi)} I^{(\bar{\mathcal{C}}, \phi, t\bar{U})} ([1_{\Omega \times ]0, t[} D^{(\bar{\mathcal{C}}, \phi)} f(W_{(\cdot, t\bar{U})})]^{(\bar{\mathcal{C}})}) \\ & + \frac{1}{2^N} \int_{]0, t[} \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du, \quad t \in [0, 1]^N, \end{aligned}$$

with suitable constants  $\alpha_{(\bar{\mathcal{C}}, \phi)}$  (theorem 4 of [10]).

By showing that the corresponding integrals for  $D^{(\bar{\mathcal{C}}, \phi)} F^{N,d}(x, W)$  exist, we prove that this formula makes sense for  $F^{N,d}$ . Indeed, it even makes sense for  $D^{(k)} F^{N,d}$ , if  $k \in \mathbb{N}_0^d$  is such that  $2|k| + d < 2N$ , a fact which leads us directly to a Tanaka-like formula for the  $k^{\text{th}}$  partial derivative of local time:

$$\begin{aligned} M^{(k)}(\cdot, t, x) = & 2^N [D^{(k)} F^{N,d}(x, W_t) - D^{(k)} F^{N,d}(x, 0)] \\ & - \sum_{(\bar{\mathcal{C}}, \phi) \in \Lambda, \bigcup_{T \in \bar{\mathcal{C}}} T = U} \frac{1}{2^{|\bar{\mathcal{C}}|}} \alpha_{(\bar{\mathcal{C}}, \phi)} I^{(\bar{\mathcal{C}}, \phi, t\bar{U})} ([1_{\Omega \times ]0, t[} D^{(k)} D^{(\bar{\mathcal{C}}, \phi)} F^{N,d}(x, W_{(\cdot, t\bar{U})})]^{(\bar{\mathcal{C}})}). \end{aligned}$$

In theorem 1 we show that  $M^{(0)}$  is in fact a good candidate for local time, whereas  $M^{(k)}$  is the  $k^{\text{th}}$  distributional derivative of  $M^{(0)}$  in the space variables. The remainder of this paper is devoted to establishing the smoothness of  $M^{(k)}$  in space and time by means of the stochastic calculus presented in [10]. To do this, a method proposed by Walsh [19] is employed. In order to establish Kolmogorov's criterion for continuity of  $M^{(k)}$  in space and time, the moments of each one of the terms figuring in Tanaka's formula are estimated with the help of Burkholder's martingale inequalities for the « martingales »  $I^{(\bar{\mathcal{C}}, \phi, t\bar{U})}$ . The latter are developed in proposition 1, generalizing Metraux's [13] inequalities for discrete martingales and using ideas of Cairoli, Walsh [7] for the continuous parameter case. Thus, in theorem 2 we obtain functions  $L^{(k)}$  such that  $L^{(k)}(\cdot, s, t, \cdot)$  is a version of the usual  $k^{\text{th}}$  partial derivative of a local time of  $W$  over the interval  $]s, t[$  which is jointly continuous in  $(s, t, x)$  as long as  $s$  is not on  $\partial \mathbb{R}_+^N$ . Of course,  $L^{(k)}(\cdot, 0, \cdot, \cdot)$  is a version of  $M^{(k)}$ . If  $\emptyset \neq V \subset \{1, \dots, N\}$ ,  $d \in \mathbb{N}$  is such

that  $d < 2|V|$ , the local times of the  $(|V|, d)$ -processes  $W_{(\cdot, t_{\bar{V}})}$  can be integrated over  $t_{\bar{V}}$  such as to give a local time of  $W$ . This observation is used to treat the joint differentiability in  $(t, x)$  of local time. It first yields another Tanaka-like formula, defining, in a similar manner as above, functions  $M^{(k, \bar{V})}$  ( $k \in \mathbb{N}_0^d$  with  $2|k| + d < 2|V|$ ), which turn out to be good candidates for joint distributional derivatives of local time:  $D^{(k)}$  in space and w. r. to  $t_i, i \in \bar{V}$ , in time. Finally, the methods indicated above yield corresponding continuity results for  $M^{(k, \bar{V})}$  (theorem 3).

## 0. NOTATIONS, PRELIMINARIES AND DEFINITIONS

This article is based upon an application of the main theorem of the stochastic calculus developed in [10] to local times. Consequently, it largely depends not only on the results proved there. It is convenient to take the same notation, too. Therefore, the reader is referred to [10] for general notations concerning processes, filtrations, parameter space, etc. as well as for special notations necessary for a neater treatment of the technical aspects of the stochastic calculus used here. By  $W$  we always denote Wiener process with parameter space  $\mathbb{I} = [0, 1]^N$ , taking its values in  $\mathbb{R}^d, N, d \in \mathbb{N}$  (occasionally,  $W$  is called  $(N, d)$ -Wiener process). The symbol  $\hat{\mathbb{I}}_0^2$  is used for the set of all pairs  $(s, t) \in \mathbb{I}^2$  with  $0 < s \leq t$ .

The following concept of occupation time is natural for the representation of local time of  $W$  by means of a Tanaka-like formula: for  $J \in \mathcal{J}$ , a function  $v(\cdot, J, \cdot): \Omega \times \mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{R}$  is called « *occupation time of  $W$  over  $J$*  » if

$$v(\omega, J, B) = \int_J 1_B(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du, \quad \omega \in \Omega, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

(In terms of [10],  $v(\omega, J, B)$  measures the «  $\mu^{(\mathcal{S}, \psi)}$ -amount of time » spent by  $W(\omega, \cdot)$  in  $B$  during the time interval  $J$ , where  $\mathcal{S} = \{\{i\}: 1 \leq i \leq N\}$ ,  $\psi = 0$ ; cf. corollary 1 of theorem 3 in [10]).

A function  $L(\cdot, J, \cdot) \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  is called « *local time of  $W$  over  $J$*  », if for P-a. e.  $\omega \in \Omega$

$$(0.1) \quad \int_B L(\omega, J, x) dx = v(\omega, J, B), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Finally, a function  $L \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{I}) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  is called « *local time of  $W$*  », if for P-a. e.  $\omega \in \Omega$

$$(0.2) \quad \int_B L(\omega, t, x) dx = v(\omega, R_t, B), \quad t \in \mathbb{I}, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

# 1. TANAKA'S FORMULA FOR W

We now show how to generalize Ito's formula (theorem 4 of [10]) in order to obtain a representation of local time of W by stochastic integrals (Tanaka's formula). By formally differentiating occupation time over J, we conclude that local time over J at  $x \in \mathbb{R}^d$  should be given by the « integral »

$\int_J \delta_{W_u - x} \prod_{1 \leq i \leq N} u_i^{N-1} du$ , where  $\delta_y$  is Dirac's  $\delta$ -distribution at  $y \in \mathbb{R}^d$ . Let

us briefly recall Ito's formula (theorem 4 of [10]). For  $f \in C^{2N}(\mathbb{R}^d)$  such that  $D^{(\bar{c}, \phi)} f(W) \in L^2(\Omega \times \mathbb{I}, \mathcal{P}, P \times \lambda^N)$ ,  $(\bar{c}, \phi) \in \Psi_N$  and  $D^{(\bar{c}, \phi)} f(W_{(\cdot, t_{\bar{c}})})^{\bar{c}} \in L_{(\bar{c}, \phi)}^{l_{\bar{c}}}$ ,  $(\bar{c}, \phi) \in \Psi$ ,  $t_{\bar{c}} \in \mathbb{I}_{\bar{c}}$ , we have, putting

$$\alpha_{(\bar{c}, \phi)} := \prod_{T \in \bar{c}^0} (|T| - 1)(-1)^{|\bar{c}| - 1} \sum_{0 \leq i \leq |\bar{c}|} \sum_{i \leq k \leq |\bar{c}|} (-1)^{|\bar{c}| - i} \binom{k}{i} i^{|\bar{c}|}, \quad (\bar{c}, \phi) \in \Lambda,$$

$$(1.1) \quad \Delta_J f(W) = \sum_{(\bar{c}, \phi) \in \Lambda} \frac{1}{2^{|\bar{c}^0|}} \alpha_{(\bar{c}, \phi)} \Delta_{J_{\bar{c}}} I^{(\bar{c}, \phi, \cdot)}([1_{\Omega \times J_{\bar{c}}} D^{(\bar{c}, \phi)} f(W_{(\cdot, \cdot)})]^{\bar{c}}) \\ + \frac{1}{2^N} \int_J \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du, \quad J \in \mathcal{J}.$$

Moreover, for each product  $\rho$  of finite measures  $\rho_i$ ,  $1 \leq i \leq N$ , on  $\mathcal{B}(\mathbb{I})$ , the existence of (in  $(\omega, s, t)$ ) measurable versions of the integrals occurring in (1.1) can be assured, such that (1.1) is valid for  $\rho^2$ -a. e.  $(s, t) \in \hat{\mathbb{I}}^2$ , with  $J = ]s, t]$ . Comparing the last term of (1.1) to the above « integral », we find that local time should be given by an extension of Ito's formula to a family of functions  $F^{N,d}(x, \cdot): \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  which satisfies the partial differential equations

$$(1.2) \quad \mathbb{D}^N F^{N,d}(x, \cdot) = \delta_{x-\cdot}, \quad x \in \mathbb{R}^d.$$

It is well-known (cf. Schwartz [16], p. 44-47) that

$$F^{N,d}: \mathbb{R}^{2d} \rightarrow \mathbb{R} \cup \{\infty\}, (x, y) \rightarrow$$

$$\begin{cases} \left( \prod_{1 \leq j \leq N-1} 2j \prod_{1 \leq i \leq N} (2j-d)\gamma_d \right)^{-1} |y-x|^{2N-d}, & \text{if } d \text{ is odd,} \\ \left( \prod_{1 \leq j \leq N-1} 2j \prod_{\substack{1 \leq i \leq N \\ 2j \neq d}} (2j-d)\gamma_d \right)^{-1} |y-x|^{2N-d} \log |y-x|, & \text{if } d \text{ is even,} \end{cases}$$

is a solution of (1.2),  $\gamma_d$  being the measure of the surface of the  $d$ -dimensional unit sphere ( $\gamma_1 := 2$ ). We will now establish that in case  $d < 2N$ , an extension of Ito's formula to  $F^{N,d}$  exists. But it turns out, that we can do better: if  $k \in \mathbb{N}_0^d$  is such that  $2|k| + d < 2N$ , we can even show that Ito's formula makes sense for  $D^{(k)}F^{N,d}$ . We thus obtain not only Tanaka's formula for  $W$  in case  $d < 2N$ , but a candidate for the  $k^{\text{th}}$  partial derivative (in the space variables) of the local time of  $W$ , if  $2|k| + d < 2N$ . Considering the terms of (1.1), our task can be put in the following words: establish that

$$D^{(k)}D^{(\bar{c}, \phi)}F^{N,d}(x, W_{(\cdot, t_{\bar{c}})})^{\bar{c}} \in L_{(\bar{c}, \phi)}^{l_{\bar{c}}}$$

for  $(\bar{c}, \phi) \in \Lambda$ ,  $t_{\bar{c}} \in \mathbb{I}_{\bar{c}}$ ,  $x \in \mathbb{R}^d$ ,  $|k| < [(2N - d)/2]$ .

For doing so, we have to estimate the partial derivatives of  $F^{N,d}(x, \cdot)$ . Use induction on the order  $|q|$  of the differential operator and observe that for each  $\delta > 0$  the function  $u \rightarrow u^\delta \log u$  is bounded on  $[1, \infty[$  to conclude that for  $q \in \mathbb{N}_0^d$ ,  $\delta > 0$  there is a constant  $c \in \mathbb{R}$  such that

$$(1.3) \quad |D^{(q)}F^{N,d}(x, y)| \leq c[|x - y|^{2N-d-|q|+\delta} + |x - y|^{2N-d-|q|-\delta}].$$

Consequently,

(1.4) for  $(\bar{c}, \phi) \in \Lambda$  with order  $m$ ,  $k \in \mathbb{N}_0^d$ ,  $\delta > 0$  there exists  $c \in \mathbb{R}$  such that

$$|D^{(k)}D^{(\bar{c}, \phi)}F^{N,d}(x, y)| \leq c[|y - x|^{2N-d-m-|k|+\delta} + |y - x|^{2N-d-m-|k|-\delta}].$$

With the help of (1.4) we can prove

LEMMA 1. — Let  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2N$ ,  $(\bar{c}, \phi) \in \Lambda$  with order  $m$  be given. Then  $(t_{\bar{c}}, x) \rightarrow \|D^{(k)}D^{(\bar{c}, \phi)}F^{N,d}(x, W_{(\cdot, t_{\bar{c}})})^{\bar{c}}\|_{(\bar{c}, \phi)}^{l_{\bar{c}}}$  is locally bounded on  $\mathbb{I}_{\bar{c}} \times \mathbb{R}^d$ .

*Proof.* — Since  $(\bar{c}, \phi) \in \Lambda$ , we have  $m \leq |\bar{c}| \leq N$  and thus

$$2N - d - m - |k| < -d/2 \vee (1/2 - m).$$

Taking (1.4) into account, it is enough to show for  $l > -d/2 \vee (1/2 - m)$

$$(t_{\bar{c}}, x) \rightarrow \| [|x - W_{(\cdot, t_{\bar{c}})}|^l]^{\bar{c}} \|_{(\bar{c}, \phi)}^{l_{\bar{c}}} \text{ is locally bounded on } \mathbb{I}_{\bar{c}} \times \mathbb{R}^d.$$

In case  $l \geq 0$  this is a simple consequence of the integrability of  $|W_1|^l$ . In case  $l < 0$  the fact that  $x \rightarrow E(|\xi - x|^l)$  has its global maximum at  $x = 0$  for any Gaussian unit vector  $\xi$  and scaling imply

$$\begin{aligned} \| [|x - W_{(\cdot, t_{\bar{c}})}|^l]^{\bar{c}} \|_{(\bar{c}, \phi)}^{l_{\bar{c}}} &\leq \| [|W_{(\cdot, t_{\bar{c}})}|^l]^{\bar{c}} \|_{(\bar{c}, \phi)}^{l_{\bar{c}}} \\ &= \prod_{i \in \bar{c}} t_i^{l/2 + m/2} \| [|W_{(\cdot, \frac{1}{t_{\bar{c}}})}|^l]^{\bar{c}} \|_{(\bar{c}, \phi)}^{l_{\bar{c}}}. \end{aligned}$$

Since  $l > 1/2 - m$  we are left with the assertion

$$(1.5) \quad \| [ |W|^l ]^{\mathcal{C}} \|_{(\mathcal{C}, \phi)} < \infty, \quad \text{if } (\mathcal{C}, \phi) \in \Lambda_N.$$

Let  $\beta_{2l} := E(|W_1|^{2l})$ . We have

$$\begin{aligned} \| [ |W|^l ]^{\mathcal{C}} \|_{(\mathcal{C}, \phi)}^2 &= E \left( \int_{\mathbb{R}^1} \left( \int_{\mathbb{R}^0} [ |W|^l ]^{\mathcal{C}}(., \mathcal{J}) d\mathcal{J}_{\mathcal{C}^0} \right)^2 d\mathcal{J}_{\mathcal{C}^1} \right) \\ &\leq \int_{\mathbb{R}^1} \int_{\mathbb{R}^0} \int_{\mathbb{R}^0} [E([ |W|^{2l} ]^{\mathcal{C}}(., \mathcal{J}_{\mathcal{C}^1}, \mathcal{U}_{\mathcal{C}^0}))]^{1/2} \quad (\text{Hölder}) \\ &\quad [E([ |W|^{2l} ]^{\mathcal{C}}(., \mathcal{J}_{\mathcal{C}^1}, \mathcal{V}_{\mathcal{C}^0}))]^{1/2} d\mathcal{U}_{\mathcal{C}^0} d\mathcal{V}_{\mathcal{C}^0} d\mathcal{J}_{\mathcal{C}^1} \\ &= \beta_{2l} \int_{\mathbb{R}^1} \int_{\mathbb{R}^0} \int_{\mathbb{R}^0} 1_{\mathbb{R}}(\mathcal{J}_{\mathcal{C}^1}, \mathcal{U}_{\mathcal{C}^0}) 1_{\mathbb{R}}(\mathcal{J}_{\mathcal{C}^1}, \mathcal{V}_{\mathcal{C}^0}) \quad (\text{definition of } [\cdot]^{\mathcal{C}}) \\ &\quad \prod_{T \in \mathcal{C}^1} \prod_{i \in T} (s_i^T)^l \prod_{T \in \mathcal{C}^0} \prod_{i \in T} (u_i^T v_i^T)^{1/2} d\mathcal{U}_{\mathcal{C}^0} d\mathcal{V}_{\mathcal{C}^0} d\mathcal{J}_{\mathcal{C}^1} \\ &= \beta_{2l} c_1 \left( \int_0^1 r^{m+l-1} dr \right)^N \end{aligned}$$

with a suitable constant  $c_1 \in \mathbb{R}$ . Since  $l > -d/2$ ,  $\beta_{2l}$  is finite and since  $l > 1/2 - m$ ,  $r \rightarrow r^{m+l-1}$  is integrable over  $[0, 1]$ . This gives (1.5).  $\square$

*Remark.* — The assertion of lemma 1 is not necessarily true for  $(\mathcal{C}, \phi) \in \Psi \setminus \Lambda$ . This shows that the formula of theorem 3 of [10] cannot be generalized in the same way as (1.1) to give a representation of local time.

By what has been said above we are motivated and by lemma 1 we are allowed to give

**DEFINITION 1.** — Let  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  be such that  $2|k| + d < 2N$ . For  $x \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$  let, setting  $J = ]s, t]$ ,

$$\begin{aligned} M^{(k)}(., s, t, x) &:= 2^N [\Delta_J D^{(k)} F^{N,d}(x, W) \\ &\quad - \sum_{(\mathcal{C}, \phi) \in \Lambda} \frac{1}{2^{|\mathcal{C}^0|}} \alpha_{(\mathcal{C}, \phi)} \Delta_{J_{\mathcal{C}}} I^{(\mathcal{C}, \phi, .)}([1_{\Omega \times J_{\mathcal{C}}} D^{(k)} D^{(\mathcal{C}, \phi)} F^{N,d}(x, W_{(.,.)})]^{\mathcal{C}})] . \end{aligned}$$

We will show now that  $M^{(k)}$  is, in fact, a good candidate for the  $k^{\text{th}}$  partial derivative of local time of  $W$ . For this purpose, we need a measurable version of  $M^{(k)}$  and some knowledge about the exchangeability of  $\langle I^{(\mathcal{C}, \phi, t)} \rangle$  and  $\langle dx \rangle$ .

**LEMMA 2.** — Let  $(\mathcal{C}, \phi) \in \Lambda$ ,  $\rho$  on  $\mathcal{B}(\mathbb{I})$  be a product of finite measures  $\rho_i$ ,

$1 \leq i \leq N$ . Further, suppose that a function  $g: \mathbb{R}^{2d} \rightarrow \mathbb{R} \cup \{\infty\}$  satisfies

- i)  $(t_{\bar{e}}, x) \rightarrow \|g(x, W_{(\cdot, t_{\bar{e}})})^{\bar{e}}\|_{(\bar{e}, \phi)}^{t_{\bar{e}}}$  is locally bounded on  $\mathbb{I}_{\bar{e}} \times \mathbb{R}^d$ ,
- ii)  $x \rightarrow g(x, y)$  is continuous on  $\mathbb{R}^d \setminus \{y\}$ ,  $y \in \mathbb{R}^d$ .

Then there exists  $G \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  such that

- iii)  $G(\cdot, s, t, x) = \Delta_{J_{\bar{e}}} I^{(\bar{e}, \phi, \cdot)}([1_{\Omega \times J_{\bar{e}}} g(x, W_{(\cdot, t_{\bar{e}})})]^{\bar{e}})$  for  $\rho^2 \times \lambda^d - a. e.$   
 $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$ , putting  $J = ]s, t]$ ,

- iv)  $G(\omega, s, t, \cdot)$  is locally square integrable w. r. t.  $\lambda^d$ , for all  $(\omega, s, t) \in \Omega \times \hat{\mathbb{I}}^2$ ,

$$v) \int_{\mathbb{R}^d} G(\cdot, s, t, x) h(x) dx = \Delta_{J_{\bar{e}}} I^{(\bar{e}, \phi, \cdot)} \left( \left[ 1_{\Omega \times J_{\bar{e}}} \int_{\mathbb{R}^d} h(x) g(x, W_{(\cdot, t_{\bar{e}})}) dx \right]^{\bar{e}} \right)$$

for  $\rho^2 - a. e. (s, t) \in \hat{\mathbb{I}}^2$ , putting  $J = ]s, t]$ , all  $h \in C_c^0(\mathbb{R}^d)$ .

*Proof.* — Since  $\rho$  is a product measure and iv) and v) are « local » properties, it is enough to show, that for  $m \in \mathbb{Z}^d$  there exists

$$G_m \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{I}) \times \mathcal{B}([m-1, m]), \mathcal{B}(\mathbb{R}))$$

such that

- iii')  $G_m(\cdot, t, x) = I^{(\bar{e}, \phi, t_{\bar{e}})}([1_{\Omega \times (\mathbb{R}_{t_{\bar{e}}})} g(x, W_{(\cdot, t_{\bar{e}})})]^{\bar{e}})$  for  $\rho \times \lambda^d - a. e.$   
 $(t, x) \in \mathbb{I} \times ]m-1, m]$ ,

- iv')  $G_m(\omega, t, \cdot)$  is square integrable w. r. t.  $\lambda^d$  for all  $(\omega, t) \in \Omega \times \mathbb{I}$ ,

$$v') \int_{]m-1, m]} h(x) G_m(\cdot, t, x) dx = I^{(\bar{e}, \phi, t_{\bar{e}})}([1_{\Omega \times (\mathbb{R}_{t_{\bar{e}}})} \int_{]m-1, m]} h(x) g(x, W_{(\cdot, t_{\bar{e}})}) dx]^{\bar{e}})$$

for  $\rho - a. e. t \in \mathbb{I}$ , all  $h \in C_b^0([m-1, m])$ .

For simplicity, take  $m = \underline{1}$ . Put  $\Lambda = ]0, \underline{1}] \times \mathbb{I}$ ,  $\mathcal{G} = \mathcal{B}([0, \underline{1}]) \times \mathcal{B}(\mathbb{I})$ ,  $\nu := \lambda^d|_{\mathcal{B}([0, \underline{1}])} \times \rho$  (instead of  $]0, 1]$  resp.  $\mathcal{B}([0, 1])$  resp.  $\lambda|_{\mathcal{B}([0, 1])}$ ) in the proof of lemma 2 of [11]. To make this proof work, we further must replace  $\|\cdot\|_q$  by  $\|\cdot\|_{(\bar{e}, \phi)}$  and resort to « lemma 5 and its corollary » of [10] instead of « lemma 1 » of [11].  $\square$

**THEOREM 1.** — Let  $d < 2N$ ,  $\rho$  on  $\mathcal{B}(\mathbb{I})$  be a product of finite measures  $\rho_i$ ,  $1 \leq i \leq N$ . Then for each  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2N$  there exists  $K^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  which satisfies

- i)  $K^{(k)}(\cdot, s, t, x) = M^{(k)}(\cdot, s, t, x)$  for  $\rho^2 \times \lambda^2 - a. e. (s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$ ,
- ii)  $K^{(k)}(\omega, s, t, \cdot)$  is locally square integrable w. r. t.  $\lambda^d$  for all  $(\omega, s, t) \in \Omega \times \hat{\mathbb{I}}^2$ ,

$$iii) \int_{\mathbb{R}^d} h(x) K^{(k)}(\cdot, s, t, x) dx = (-1)^{|k|} \int_{\mathbb{R}^d} D^{(k)} h(x) K^{(0)}(\cdot, s, t, x) dx$$

for  $\rho^2 - a. e. (s, t) \in \hat{\mathbb{I}}^2$ , all  $h \in C_c^\infty(\mathbb{R}^d)$ .

Moreover,  $K^{(0)}(\cdot, s, t, \cdot)$  is a local time of  $W$  over  $]s, t]$  for  $\rho^2 - a. e. (s, t) \in \hat{\mathbb{I}}^2$ .  $W$  has a local time.

*Proof.* — Let  $k \in \mathbb{N}_0^d$  satisfy  $2|k| + d < 2N$ . For  $(\mathcal{C}, \phi) \in \Lambda$  set

$$g_{(\mathcal{C}, \phi)}^k := D^{(k)} D^{(\mathcal{C}, \phi)} F^{N, d}$$

According to lemma 1,  $g_{(\mathcal{C}, \phi)}^k$  fulfils *i*) and *ii*) of lemma 2. Therefore we can choose  $G_{(\mathcal{C}, \phi)}^k$  such that *iii*)-*v*) of lemma 2 are valid for the pair  $(g_{(\mathcal{C}, \phi)}^k, G_{(\mathcal{C}, \phi)}^k)$ . Define

$$K^{(k)}(\cdot, s, t, x) := 2^N \left[ \Delta_{[s, t]} D^{(k)} F^{N, d}(x, W) - \sum_{(\mathcal{C}, \phi) \in \Lambda} \frac{1}{2^{|\mathcal{C}_0|}} \alpha_{(\mathcal{C}, \phi)} G_{(\mathcal{C}, \phi)}^k(\cdot, s, t, x) \right],$$

$$(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d.$$

Of course,  $K^{(k)} \in \mathcal{H}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ . *i*) is a consequence of lemma 2, *iii*) and definition 1 ; *ii*) follows from lemma 2, *iv*) ; lemma 2, *v*) and the equality

$$\begin{aligned} & \int_{\mathbb{R}^d} h(x) D^{(k)} D^{(\mathcal{C}, \phi)} F^{N, d}(x, W) dx \\ &= (-1)^{|k|} \int_{\mathbb{R}^d} D^{(k)} h(x) D^{(\mathcal{C}, \phi)} F^{N, d}(x, W) dx, \quad h \in C_c^\infty(\mathbb{R}^d), (\mathcal{C}, \phi) \in \Lambda \end{aligned}$$

together impli *iii*).

Now fix  $t \in \mathbb{I}$  and take  $\rho := \bigtimes_{1 \leq i \leq N} (\varepsilon_{\{0\}} + \varepsilon_{\{t_i\}})$ ,  $\varepsilon_s$  being the point mass in  $s \in \mathbb{I}$ .

For  $h \in C_c^\infty(\mathbb{R}^d)$  set  $f := \int_{\mathbb{R}^d} h(x) F^{N, d}(x, \cdot) dx$ . We have  $f \in C^\infty(\mathbb{R}^d)$ ,

$$D^{(q)} f = (-1)^{|q|} \int_{\mathbb{R}^d} D^{(q)} h(x) F^{N, d}(x, \cdot) dx, \quad q \in \mathbb{N}_0^d,$$

and particularly

$$\mathbb{D}^N f = h.$$

By (1.4) and since  $W$  possesses moments of all orders, the hypotheses of theorem 4 of [10] are fulfilled. Therefore,

$$\begin{aligned} (1.6) \quad & \int_{\mathbb{R}^d} K^{(0)}(\cdot, 0, t, x) h(x) dx = 2^N [\Delta_{R_t} f(W) \\ & - \sum_{(\mathcal{C}, \phi) \in \Lambda} \frac{1}{2^{|\mathcal{C}_0|}} \alpha_{(\mathcal{C}, \phi)} I^{(\mathcal{C}, \phi, t_{\bar{\mathbb{I}}})}([1_{\Omega \times (R_t)_{\bar{\mathbb{I}}}} D^{(\mathcal{C}, \phi)} f(W_{(\cdot, t_{\bar{\mathbb{I}}}})])^{(\mathcal{C})}] \\ & = \int_{R_t} \mathbb{D}^N f(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du = \int_{R_t} h(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du. \end{aligned}$$

It is clear how (1.6) has to be generalized so as to give (0.1) for  $J = R_t$ .

To obtain (0.2) with a suitable  $L \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{I}) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ , first define  $L(\cdot, t, \cdot)$  by  $K^{(0)}(\cdot, 0, t, \cdot)$  for rational  $t \in \mathbb{I}$ . Then use monotonicity in  $t$  of the occupation time of  $W$  over  $R_t$ .

For arbitrary  $\rho$ , the argument which proves that the corresponding  $K^{(0)}(\cdot, s, t, \cdot)$  is a local time over  $]s, t]$  for  $\rho^2 - a. e. (s, t) \in \hat{\mathbb{I}}^2$ , is contained in the one which was given.  $\square$

Theorem 1 particularly says that  $K^{(k)}(\omega, s, t, \cdot)$  is the  $k^{\text{th}}$  distributional derivative of a local time of  $W(\omega, \cdot)$  over  $]s, t]$  for  $P \times \lambda^2 - a. e. (\omega, s, t) \in \Omega \times \hat{\mathbb{I}}^2$ . Our next aim is to improve this statement by means of the stochastic calculus we dispose of: we will show that there exists a version  $L^{(k)}$  of  $K^{(k)}$  which is continuous in  $(s, t, x)$ .  $L^{(k)}$  proves to be the « classical »  $k^{\text{th}}$  partial derivative of a local time of  $W$ . Hereby, the following technique will be used (cf. Walsh [19]): Kolmogorov's well-known continuity criterion for stochastic processes is verified for each term of  $M^{(k)}$  separately. This makes it necessary for example to estimate the moments of

$$I^{(\mathcal{C}, \phi, t_{\bar{\mathbb{I}}})}([1_{\Omega \times (R_t)_{\bar{\mathbb{I}}}} D^{(k)}(\mathcal{C}, \phi) [F^{N,d}(x, W_{(\cdot, t_{\bar{\mathbb{I}}})}) - F^{N,d}(y, W_{(\cdot, t_{\bar{\mathbb{I}}})})])^{\mathcal{C}}), \\ t \in \mathbb{I}, \quad x, y \in \mathbb{R}^d, \quad (\mathcal{C}, \phi) \in \Lambda.$$

Since for  $(\mathcal{C}, \phi) \in \Psi$  the integral process in the  $\mathcal{C}$ -variables of  $I^{(\mathcal{C}, \phi)}$  can be seen to be a  $\mathcal{C}^1$ -martingale (cf. remark after lemma 4 of [10]), this is a job for Burkholder's martingale inequalities for  $I^{(\mathcal{C}, \phi)}$ .

## 2. BURKHOLDER'S INEQUALITIES FOR $I^{(\mathcal{C}, \phi)}$

Burkholder's inequalities for martingales with a discrete parameter set are well-known (cf. Metraux [13] and Merzbach [12], p. 43). They imply (2.1) for  $1 < p < \infty$  there are constants  $A_p, B_p > 0$  such that for all martingales  $M$  and all partitions  $(J^k: 1 \leq k \leq r)$  of  $\mathbb{I}$  in  $\mathcal{J}$

$$A_p E \left( \left[ \sum_{1 \leq k \leq r} (\Delta_{J^k} M)^2 \right]^{p/2} \right) \leq E(|M_1|^p) \leq B_p E \left( \left[ \sum_{1 \leq k \leq r} (\Delta_{J^k} M)^2 \right]^{p/2} \right).$$

In particular, (2.1) can be applied to the  $\mathcal{C}^1$ -parameter martingales

$$I_{0, (\cdot, \cdot)_{\bar{\mathbb{I}}}}^{(\mathcal{C}, \phi)}(Y_0), \quad Y_0 \in \mathcal{C}_{\mathcal{C}}, \quad (\mathcal{C}, \phi) \in \Psi \quad (\text{cf. lemma 4 of [10]})$$

to yield inequalities which, however, still depend on the partition chosen. Now suppose that a sequence of partitions of  $\mathbb{I}$  in  $\mathcal{J}$  is given, whose mesh goes to zero. If we can establish convergence of the corresponding qua-

dratic sums on the left and right sides of (2.1) to a suitable limit, we obtain inequalities depending only on this limit (« quadratic variation »). The following lemma shows that this can be done for  $Y_0 \in \mathcal{E}_{\mathcal{C}}$ ,  $(\mathcal{C}, \phi) \in \Psi$ . Finally, an appeal to density of  $\mathcal{E}_{\mathcal{C}}$  in  $L_{(\mathcal{C}, \phi)}$  will yield Burkholder's inequalities for all  $Y \in L_{(\mathcal{C}, \phi)}$ .

LEMMA 3. — For  $\emptyset \neq U \in \Pi_N$  let  $(\mathcal{C}, \phi) \in \Psi_U$ ,  $\mathcal{C}^1 = \mathcal{C}$ . Suppose that  $Y_0 \in \mathcal{E}_{\mathcal{C}}$  has an  $\mathbb{I}_{\mathcal{C}}$ -representation  $Y_0 = \sum_{\substack{1 \leq k \leq q \\ 1 \leq k^T \leq q}} \alpha_k \prod_{T \in \mathcal{C}} 1_{K^k T}$ , where  $K^k = ]u^k, v^k]$ ,  $1 \leq k \leq q$ . For  $n \in \mathbb{N}$  let  $(J^{j,n}: 1 \leq j \leq r(n))$  be the partition which is generated by  $\{u^k, v^k: 1 \leq k \leq q\} \cup \left\{\frac{i}{n}: 0 \leq i \leq n\right\}$ ,  $(J_U^{j,n}: 1 \leq j_U \leq r(n)_U)$  the partition of  $\mathbb{I}_U$  defined by the projections of  $J^{j,n}$  on  $\mathbb{I}_U$ ,  $1 \leq j \leq r(n)$ . Then

$$(L^2-) \lim_{n \rightarrow \infty} \sum_{1 \leq j_U \leq r(n)_U} (\Delta_{J_U^{j,n}} I_{0,(\cdot, \cdot, \cdot, \cdot)}^{(\mathcal{C}, \phi)}(Y_0))^2 = \int_{\mathcal{C}} Y_0^2(\cdot, \cdot) d\mathcal{C}.$$

*Proof.* — Taking  $\bar{U} = \emptyset$ , we can avoid some unessential technicalities. Further, omitting  $n$  as an index will cause no confusion, as it is kept fix during the following arguments. Note first that by linearity

$$\Delta_{J^j} I_{0,(\cdot, \cdot)}^{(\mathcal{C}, \phi)}(Y_0) = I_{0,(\cdot, \cdot)}^{(\mathcal{C}, \phi)}(Y_0 [1_{\Omega \times J^j}]^{\mathcal{C}}) = \sum_{1 \leq k^T \leq q} \alpha_k \prod_{T \in \mathcal{C}} \Delta_{K^k T \cap (J^j)^T} W^{\phi(T)}.$$

Therefore, putting

$$Z_j^{\mathcal{S}} := \sum_{1 \leq k^T \leq q, T \in \mathcal{S}} \left( \sum_{1 \leq k^T \leq q, T \in \mathcal{S}} \alpha_k \prod_{T \in \mathcal{S}} \Delta_{K^k T \cap (J^j)^T} W^{\phi(T)} \right)^2 \prod_{T \in \mathcal{S}} 1_{K^k T \cap (J^j)^T},$$

$$1 \leq j \leq r, \text{ and } Z^{\mathcal{S}} := \sum_{1 \leq j \leq r} Z_j^{\mathcal{S}}, \quad \mathcal{S} \subset \mathcal{C},$$

the triangle inequality implies that it is enough to show

(2.2)

$$\left\| \int_{\mathbb{I}_{\mathcal{S}}} Z^{\mathcal{S}}(\cdot, \cdot) d\mathcal{S} - \int_{\mathbb{I}_{\mathcal{S} \cup \{S\}}} Z^{\mathcal{S} \setminus \{S\}}(\cdot, \cdot) d\mathcal{U} \right\|_2 \rightarrow 0 \quad (n \rightarrow \infty) \text{ for } S \in \mathcal{S} \subset \mathcal{C}.$$

Let  $S \in \mathcal{S} \subset \mathcal{C}$ . Since  $W$  has independent, centered increments, we have

$$(2.3) \quad E \left( \prod_{k=i,j} \left[ \int_{\mathbb{I}_{\mathcal{S}}} Z_k^{\mathcal{S}}(\cdot, \cdot) d\mathcal{S} - \int_{\mathbb{I}_{\mathcal{S} \cup \{S\}}} Z_k^{\mathcal{S} \setminus \{S\}}(\cdot, \cdot) d\mathcal{U} \right] \right) = 0, \quad \text{if } i_S \neq j_S.$$

As in addition for  $J \in \mathcal{J}$ ,  $1 \leq i \leq d$ ,  $(\Delta_J W^i)^4$  has variance  $c\lambda^N(J)^2$  with a suitable constant  $c$  independent of  $J$ , we get

$$\begin{aligned}
 (2.4) \quad & E \left( \left[ \int_{\mathbb{J}^{\mathcal{J}}} Z_j^{\mathcal{J}}(\cdot, \mathcal{J}) d\mathcal{J} - \int_{\mathbb{J}^{\mathcal{J} \cup \{S\}}} Z_j^{\mathcal{J} \setminus \{S\}}(\cdot, u) du \right]^2 \right) \\
 & \leq \prod_{T \in \mathcal{T}} \lambda^{|T|} (J_T^j) \int_{\mathbb{J}^{\mathcal{J}}} E \left( \left[ Z_j^{\mathcal{J}}(\cdot, \mathcal{J}) - \int_{\mathbb{J}^{\mathcal{J} \setminus \{S\}}} Z_j^{\mathcal{J} \setminus \{S\}}(\cdot, u^S) du^S \right]^2 \right) d\mathcal{J} \\
 & \leq c \prod_{T \in \mathcal{T} \cup \{S\}} \lambda^{|T|} (J_T^j) \int_{\mathbb{J}^{\mathcal{J} \cup \{S\}}} E([Z_j^{\mathcal{J} \setminus \{S\}}(\cdot, \mathcal{J})]^4) d\mathcal{J} \quad (\text{Jensen's inequality}) \\
 & \leq c \prod_{T \in \mathcal{T}} \lambda^{|T|} (J_T^j) \prod_{1 \leq i \leq N} q_i^{4|\mathcal{J} \setminus \{S\}|} \int_{\Omega \times \mathbb{J}} [1_{\Omega \times J^j}]^{\mathcal{C}} |Y_0|^4 d(P \times \lambda^N).
 \end{aligned}$$

Combining (2.3) and (2.4) yields

$$\begin{aligned}
 & \left\| \int_{\mathbb{J}^{\mathcal{J}}} Z^{\mathcal{J}}(\cdot, \mathcal{J}) d\mathcal{J} - \int_{\mathbb{J}^{\mathcal{J} \cup \{S\}}} Z^{\mathcal{J} \setminus \{S\}}(\cdot, u) du \right\|_2^2 \\
 & = \sum_{1s \leq js \leq rs} E \left( \left[ \sum_{1\bar{s} \leq j\bar{s} \leq r\bar{s}} \int_{\mathbb{J}^{\mathcal{J}}} Z_j^{\mathcal{J}}(\cdot, \mathcal{J}) d\mathcal{J} - \int_{\mathbb{J}^{\mathcal{J} \cup \{S\}}} Z_j^{\mathcal{J} \setminus \{S\}}(\cdot, u) du \right]^2 \right) \quad ((2.3)) \\
 & \leq \left( \prod_{i \in \mathcal{S}} r_i \right) c \left( \frac{1}{n} \right)^N \prod_{1 \leq i \leq N} q_i^{4|\mathcal{J} \setminus \{S\}|} \int_{\Omega \times \mathbb{J}} |Y_0|^4 d(P \times \lambda^N). \quad (\text{Cauchy-Schwartz,} \\
 & \quad (2.4))
 \end{aligned}$$

By choice of  $(J^j: 1 \leq j \leq r)$ ,  $r_i \leq n + 1 + q_i$  for  $1 \leq i \leq N$ . This implies (2.2).  $\square$

**PROPOSITION 1.** — For  $1 < p < \infty$  there exist real constants  $A_p, B_p > 0$ , such that for  $(\mathcal{C}, \phi) \in \Psi$ ,  $Y \in L_{(\mathcal{C}, \phi)}$

$$\begin{aligned}
 A_p E \left( \left[ \int_{\mathbb{J}^{\mathcal{C}^1}} \left( \int_{\mathbb{J}^{\mathcal{C}^0}} Y(\cdot, \mathcal{J}) d\mathcal{J}_{\mathcal{C}^0} \right)^2 d\mathcal{J}_{\mathcal{C}^1} \right]^{p/2} \right) & \leq E(|I^{(\mathcal{C}, \phi)}(Y)|^p) \\
 & \leq B_p E \left( \left[ \int_{\mathbb{J}^{\mathcal{C}^1}} \left( \int_{\mathbb{J}^{\mathcal{C}^0}} Y(\cdot, \mathcal{J}) d\mathcal{J}_{\mathcal{C}^0} \right)^2 d\mathcal{J}_{\mathcal{C}^1} \right]^{p/2} \right).
 \end{aligned}$$

*Proof.* — Due to the density of  $\mathcal{E}_{\mathcal{C}}$  in  $L_{(\mathcal{C}, \phi)}$ , the asserted inequalities need to be established only for  $Y_0 \in \mathcal{E}_{\mathcal{C}}$ . Evidently, we can assume  $\mathcal{C}^1 = \mathcal{C}$ . Using the notations of lemma 3, put

$$V_n(Y_0) := \sum_{\frac{1}{\bar{e}} \leq j_{\bar{e}} \leq r(n)_{\bar{e}}} (\Delta_{J_{\bar{e}}}^{j_{\bar{e}}, n} I_{0, \dots, 1, \bar{e}}^{(\mathcal{C}, \phi)}(Y_0))^2, \quad n \in \mathbb{N}.$$

(2.1) implies

$$(2.5) \quad A_p E([V_n(Y_0)]^{p/2}) \leq E(|I^{(\mathcal{C}, \phi)}(Y_0)|^p) \leq B_p E([V_n(Y_0)]^{p/2}), \quad n \in \mathbb{N}.$$

Moreover, by (2.5), the sequence  $(V_n(Y_0))^{p/2}$ ,  $n \in \mathbb{N}$ , is uniformly integrable for  $p > 1$ , and, by lemma 3, it converges at least in probability. Therefore, Vitali's theorem completes the proof.  $\square$

*Remarks.* — 1. Doob's maximal inequalities can be used to sharpen the right inequality of proposition 1.

2. For  $1 < p < \infty$ ,  $(\mathcal{C}, \phi) \in \Psi_U$ ,  $Y \in \mathcal{M}(\mathcal{P}^U, \mathcal{B}(\mathbb{R}))$ , proposition 1 yields the weaker inequality  $E(|I^{(\mathcal{C}, \phi)}(Y^{\mathcal{C}})|^p) \leq B_p E\left(\left[\int_{\Pi} |Y|^2(., s) ds\right]^{p/2}\right)$ .

### 3. CONTINUITY OF THE LOCAL TIME OF $W$ IN $(t, x)$ , DIFFERENTIABILITY IN $x$

We now come back to the study of smoothness properties of local time and its distributional derivatives  $K^{(k)}$ . As will be seen, this amounts essentially to the study of the finiteness of the moments of local time. The following « moment lemma » plays a central role.

LEMMA 4. — Let  $-dv - 2N < l \in \mathbb{R}$ ,  $p \in \mathbb{N}$ ,  $0 < u^0 \in \overline{\mathbb{I}}$  be given. Then there exists  $c_1 \in \mathbb{R}$  such that for all  $J = ]s, t]$   $s \geq u^0$ ,  $x \in \mathbb{R}^d$

$$E\left(\int_{J^p} \prod_{1 \leq i \leq p} |W_{u^i} - x|^l \prod_{1 \leq i \leq p} du^i\right) \leq \begin{cases} c_1 \lambda^N(J)^p (1 + |x|^{lp}), & \text{if } l \geq 0, \\ c_1 \lambda^N(J)^{p(1 + l/2N)}, & \text{if } l < 0. \end{cases}$$

*Proof.* — Since  $l > -d$ ,  $\beta_l := E(|W_1|^l)$  is finite. In case  $l \geq 0$ , note that Hölder's inequality implies for  $x \in \mathbb{R}^d$ ,  $u^i \in \mathbb{I}$ ,  $1 \leq i \leq p$

$$\begin{aligned} E\left(\prod_{1 \leq i \leq p} |W_{u^i} - x|^l\right) &\leq \prod_{1 \leq i \leq p} [E(|W_{u^i} - x|^{lp})]^{1/p} \\ &\leq \prod_{1 \leq i \leq p} [2^{lp-1} (E(|W_{u^i}|^{lp}) + |x|^{lp})]^{1/p}. \end{aligned}$$

The desired conclusion follows easily. Let  $l < 0$ . First observe that it is enough to show

(3.1) there exists  $c_2 \in \mathbb{R}$  such that for  $u^0 \leq u^i \in \mathbb{I}$ ,  $1 \leq i \leq p$ , with pairwise different coordinates  $u_j^i$ ,  $1 \leq j \leq N$ ,  $1 \leq i \leq p$ , and  $x \in \mathbb{R}^d$

$$E\left(\prod_{1 \leq i \leq p} |W_{u^i} - x|^l\right) \leq c_2 \prod_{1 \leq i \leq p} \prod_{1 \leq j \leq N} (u_j^i - r_j^i)^{l/2N},$$

where  $r_j^i := \max \{u_j^q : 1 \leq q \leq p, u_j^q < u_j^i\} \vee u_j^0$ ,  $1 \leq j \leq N$ ,  $1 \leq i \leq p$ . Indeed, integrating (3.1) over  $J^p$  gives the desired conclusion: introduce new variables  $v_j^i := u_j^i - r_j^i$ ,  $1 \leq j \leq N$ ,  $1 \leq i \leq p$ , observe  $l > -2N$  and keep in mind that the set of all  $(u^1, \dots, u^p)$ , not all of whose coordinates are pairwise different, is a zero-set w. r. t.  $\lambda^{Np}$ .

To prove (3.1), we proceed by induction on  $p$ . For  $u > u^0$  we first decompose  $W_u$  in the following way. Consider the  $\sigma$ -fields

$$\mathcal{G} := \sigma\left(\Delta_K W : \mathcal{J} \ni K \subset \bigcup_{T \in \Pi_N, |T| \neq 1} ]u^0, \underline{1}]^T\right),$$

$$\mathcal{G}^j := \sigma(\Delta_K W : \mathcal{J} \ni K \subset ]u^0, \underline{1}]^{[j]}), \quad 1 \leq j \leq N$$

and write

$$W_u = V^0(u) + \sum_{1 \leq j \leq N} V^j(u), \quad \text{putting } V^j(u) := \Delta_{[\mu^0, u]^{[j]}} W, \quad 1 \leq j \leq N.$$

Then

$$(3.2) \quad V^0(u) \in \mathcal{M}(\mathcal{G}, \mathcal{B}(\mathbb{R}^d)), \quad V^j(u) \in \mathcal{M}(\mathcal{G}^j, \mathcal{B}(\mathbb{R}^d)), \quad (\mathcal{G}, \mathcal{G}^1, \dots, \mathcal{G}^N)$$

is independent.

Now let  $p = 1$ . For  $1 \leq j \leq N$  set  $a_j := \left[ \prod_{1 \leq q \leq N, q \neq j} u_j^0(u_j^1 - u_j^0) \right]^{1/2}$ . Infer

from (3.2) that  $\sum_{1 \leq j \leq N} V^j(u^1)$  is centered Gaussian with variance  $\sum_{1 \leq j \leq N} a_j^2$ .

Consequently,

$$E(|W_{u^1} - x|^l) \leq E\left(\left|\sum_{1 \leq j \leq N} V^j(u^1)\right|^l\right) \quad \begin{array}{l} ((3.2), \quad E(|\xi - y|^l) \leq E(|\xi|^l), \\ y \in \mathbb{R}^d, \\ \text{for a Gaussian unit vector } \xi) \end{array}$$

$$= \beta_l \left[ \sum_{1 \leq j \leq N} a_j^2 \right]^{l/2} \leq \beta_l \prod_{1 \leq j \leq N} a_j^{lN} N^{l/2} \quad \begin{array}{l} (\ll \text{arithm. mean} \gg \\ \leq \ll \text{geom. mean} \gg). \end{array}$$

This is (3.1) for  $p = 1$ .

Now assume (3.1) is valid for  $p$ . Set  $T := \{j: 1 \leq j \leq N, u_j^{p+1} = \max_{1 \leq i \leq p+1} u_j^i\}$  and let  $q_j, r_j$  be chosen such that  $u_j^{q_j} = \max \{u_j^q: 1 \leq q \leq p, u_j^q < u_j^{p+1}\} \vee u_j^0$ ,  $u_j^{r_j} = \min \{u_j^q: 1 \leq q \leq p, u_j^q > u_j^{p+1}\}$ , if  $j \notin T$ . For  $1 \leq j \leq N$ ,  $V^j(u^{p+1})$  can be derived from  $V^j(u^{q_j})$  and  $V^j(u^{r_j})$  by « interpolation » resp. « extrapolation » with some Gaussian unit vector  $\xi_j$  such that (cf. (3.2))

$$(3.3) \quad (\mathcal{G}, \mathcal{G}^1, \dots, \mathcal{G}^N, \xi_1, \dots, \xi_N) \text{ is independent,}$$

$$\text{and, putting } b_j := \left[ \prod_{1 \leq q \leq N, q \neq j} u_q^0 (u_j^{r_j} - u_j^{p+1}) (u_j^{p+1} - u_j^{q_j}) (u_j^{r_j} - u_j^{q_j})^{-1} \right]^{1/2} \text{ for } j \notin T,$$

$$\text{resp. } b_j := \left[ \prod_{1 \leq q \leq N, q \neq j} u_q^0 (u_j^{p+1} - u_j^{q_j}) \right]^{1/2} \text{ for } j \in T,$$

$$(3.4) \quad (V^j(u^i): 1 \leq i \leq p+1, 1 \leq j \leq N) \text{ is equal in law to}$$

$$(V^j(u^i), b_j \xi_j + d_j^1 V^j(u^{r_j}) + d_j^2 V^j(u^{q_j}), \quad 1 \leq i \leq p, \quad 1 \leq j \leq N)$$

with suitable  $d_j^k \in \mathbb{R}$ .

Now we are ready for the induction step. We proceed in a similar way

as for  $p = 1$ , the role of  $\sum_{1 \leq j \leq N} V^j(u^1)$  being taken by  $\sum_{1 \leq j \leq N} b_j \xi_j$ :

$$\begin{aligned} & E \left( \prod_{1 \leq i \leq p+1} |W_{u^i} - x|^l \right) \\ &= E \left( \prod_{1 \leq i \leq p} |W_{u^i} - x|^l \left| \sum_{1 \leq j \leq N} b_j \xi_j + d_j^1 V^j(u^{r_j}) + d_j^2 V^j(u^{q_j}) + V^0(u^{p+1}) - x \right|^l \right) \\ & \quad \quad \quad ((3.4)) \\ &\leq E \left( \prod_{1 \leq i \leq p} |W_{u^i} - x|^l \right) E \left( \left| \sum_{1 \leq j \leq N} b_j \xi_j \right|^l \right) \quad ((3.3), \text{ cf. } \ll p=1 \gg) \\ &\leq E \left( \prod_{1 \leq i \leq p} |W_{u^i} - x|^l \right) \beta_l \prod_{1 \leq j \leq N} b_j^l N^{l/2} \quad (\text{cf. } \ll p=1 \gg) \end{aligned}$$

To complete the proof, it remains to apply the induction hypothesis and to look at the definition of  $b_j$ ,  $1 \leq j \leq N$ .  $\square$

*Remark.* — Essential use is made of the hypothesis «  $u^0 > 0$  » in the proof of lemma 4. This is the reason why our smoothness results (theorems 2

and 3) contain no statement for intervals which « touch » the boundary  $\partial \mathbb{R}_+^N \cap \mathbb{I}$ .

As a direct consequence of lemma 4 we can prove now (by a rather crude estimation) that the moments of  $K^{(k)}$  are bounded.

**PROPOSITION 2.** — Let  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  be such that  $2|k| + d < 2N$ . Further, let  $p \in \mathbb{N}$ ,  $0 < u^0 \in \mathbb{I}$ , and a product  $\rho$  on  $\mathcal{B}(\mathbb{I})$  of finite measures  $\rho_i$ ,  $1 \leq i \leq N$ , be given. Then there exist  $c_i \in \mathbb{R}$ ,  $i = 1, 2$ ,  $c_2 > 0$ , such that for  $\rho^2 \times \lambda^d$  — a. e.  $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$ ,  $s \geq u^0$ ,

$$E(|K^{(k)}(., s, t, x)|^p) \leq c_1 \exp(-c_2|x|^2),$$

where  $K^{(k)}$  is given by theorem 1.

*Proof.* — We proceed in two steps. First we use Tanaka's formula and Burkholder's inequalities in order to establish

(3.5) there exists  $c_3 \in \mathbb{R}$  such that for  $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$ ,  $s \geq u^0$ ,

$$E(|M^{(k)}(., s, t, x)|^{2p}) \leq c_3(1 + |x|^{2p(2N-d-|k|+1)}).$$

For  $(\mathcal{C}, \phi) \in \Lambda$  with order  $m$ ,  $J = ]s, t] \in \mathcal{J}$ ,  $s \geq u^0$ ,  $u \geq u^0$ , remark 2 after proposition 1 yields

$$\begin{aligned} E(|I^{(\mathcal{C}, \phi, u_{\underline{e}})}([1_{\Omega \times J_{\underline{e}}} D^{(k)} D^{(\mathcal{C}, \phi)} F^{N,d}(x, W_{(., u_{\underline{e}})})]_{\mathcal{C}})^{2p}) \\ \leq B_{2p} E\left(\left[\int_{J_{\underline{e}}} 1_{J_{\underline{e}}} (D^{(k)} D^{(\mathcal{C}, \phi)} F^{N,d}(x, W_{(., u_{\underline{e}})}))^2 d\lambda_{|\underline{e}|}\right]^p\right). \end{aligned}$$

Therefore, by (1.4) and Tanaka's formula (3.5) follows once we have shown that

(3.6) for  $0 < \delta < 1/2$ ,  $(\mathcal{C}, \phi) \in \Lambda$  with order  $m$ ,  $l := 2(2N - d - |k| - m \pm \delta)$

there exists  $c_4 \in \mathbb{R}$  such that for  $J = ]s, t] \in \mathcal{J}$ ,  $s \geq u^0$ ,  $u \geq u^0$

$$E\left(\left[\int_{J_{\underline{e}}} 1_{J_{\underline{e}}} |W_{(., u_{\underline{e}})} - x|^{2l} d\lambda_{|\underline{e}|}\right]^p\right) \leq \begin{cases} c_4(1 + |x|^{pl}) \lambda_{|\underline{e}|}^{|\underline{e}|} (J_{\underline{e}})^p, & \text{if } l \geq 0, \\ c_4 \lambda_{|\underline{e}|}^{|\underline{e}|} (J_{\underline{e}})^{p(1 + l/2|\underline{e}|)}, & \text{if } l < 0. \end{cases}$$

But  $l > -d \vee -2|\underline{e}|$ . Consequently, (3.6) follows from scaling ( $u \geq u^0$ ) and lemma 4. Now remember that  $K^{(0)}(., s, t, .)$  is a local time of  $W$  over  $]s, t]$  for  $\rho^2$  — a. e.  $(s, t) \in \hat{\mathbb{I}}^2$ . Using Fubini's theorem, we infer from this

$$K^{(0)}(., s, t, x) = K^{(0)}(., s, t, x) 1_{\left\{ \sup_{s < u \leq t} |W_u| \geq 1/2|x| \right\}}$$

for  $\rho^2 \times \lambda^d$  — a. e.  $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$ .

Apply theorem 1, i) and iii) and the inequality of Cauchy-Schwartz. Thus

$$\begin{aligned} E(|K^{(k)}(., s, t, x)|^p) \\ \leq [E(|M^{(k)}(., s, t, x)|^{2p})]^{1/2} [P(\sup_{s < u \leq t} |W_u| \geq 1/2 |x|)]^{1/2} \end{aligned}$$

for  $\rho^2 \times \lambda^d - a. e. (s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$

But from Paranjape, Park [15] we have

$$(3.7) \quad P(\sup_{0 \leq u \leq 1} |W_u| \geq 1/2 |x|) \leq c_5 \exp\left(-1/2d \left|\frac{x}{2}\right|^2\right).$$

Combine (3.5) with (3.7) to complete the proof.  $\square$

To be able to verify Kolmogorov's criterion for  $M^{(k)}$  we need to investigate the Hölder continuity of

$$x \rightarrow D^{(k)} D^{(\bar{\epsilon}, \phi)} F^{N,d}(x, y), \quad y \in \mathbb{R}^d, \quad (\bar{\epsilon}, \phi) \in \Lambda, \quad k \in \mathbb{N}_0^d.$$

LEMMA 5. — Let  $q \in \mathbb{N}_0^d$ ,  $0 < \delta, \eta < 1$ . For  $y, z \in \mathbb{R}^d$  put  $A_{y,z} := \{x \in \mathbb{R}^d : |x - z| \geq 2|y - z|\}$ , for  $\gamma > 0$  put  $g_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $r \rightarrow r^{2N-d-\gamma+\delta} + r^{2N-d-\gamma-\delta}$ . Then there exists  $c_1 \in \mathbb{R}$  such that for  $y, z \in \mathbb{R}^d$

$$\begin{aligned} |D^{(q)} F^{N,d}(., y) - D^{(q)} F^{N,d}(., z)| \leq c_1 [ & (g_{|q|}(|. - z|) + g_{|q|}(|. - y|)) 1_{\overline{A_{y,z}}} \\ & + |y - z|^\eta g_{|q|+\eta}(|. - z|) 1_{A_{y,z}} ]. \end{aligned}$$

*Proof.* — Fix  $y, z \in \mathbb{R}^d$ . On  $\overline{A_{y,z}}$ , use (1.3). Let  $x \in A_{y,z}$ . Then for each  $w$  on the line segment connecting  $y-x$  and  $z-x$  we have

$$(3.8) \quad 1/2 |x - z| \leq |w| \leq 3/2 |x - z|.$$

Therefore,

$$\begin{aligned} & |D^{(q)}(F^{N,d}(x, y) - F^{N,d}(x, z))| \\ & \leq c_2 [g_{|q|}(|x - y|) + g_{|q|}(|x - z|)]^{1-\eta} \\ & \quad \left| \sum_{1 \leq j \leq d} (y_j - z_j) \int_0^1 D^{(q+e_j)} F^{N,d}(x, y + s(z - y)) ds \right|^\eta \quad ((1.3)) \\ & \leq c_3 |y - z|^\eta g_{|q|}(|x - z|)^{1-\eta} \left[ \int_0^1 g_{|q|+1}(|x - (y + s(z - y))|) ds \right]^\eta \quad ((1.3), (3.8)) \\ & \leq c_4 |y - z|^\eta g_{|q|}(|x - z|)^{1-\eta} g_{|q|+1}(|x - z|)^\eta \quad ((3.8)), \end{aligned}$$

with  $c_2, \dots, c_4$  independent of  $x, y, z \in \mathbb{R}^d$ .

This gives the desired inequality on  $A_{y,z}$ .  $\square$

We are now prepared to verify Kolmogorov's criterion for  $M^{(k)}$ . This will be done separately for the time (proposition 3) and space (proposition 4) variables.

PROPOSITION 3. — Let  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  be such that  $2|k| + d < 2N$ . Further, let  $p \in \mathbb{N}$ ,  $0 < u^0 \in \mathbb{I}$ ,  $0 < \eta < 1/2$  be given. Then there exists  $c_1 \in \mathbb{R}$  such that for all  $x \in \mathbb{R}^d$ ,  $(s, t), (s', t') \in \hat{\mathbb{I}}^2$ ,  $s, s' \geq u^0$

$$E(|M^{(k)}(., s, t, x) - M^{(k)}(., s', t', x)|^{2p}) \leq c_1 |(s, t) - (s', t')|^{p\eta/N}.$$

*Proof.* — Fix  $0 < \delta$  such that  $\delta + \eta < 1/2$  and put

$$g^{(\bar{\mathcal{C}}, \phi)}(s, t, x) := \Delta_{J_{\bar{\mathcal{C}}}} I^{(\bar{\mathcal{C}}, \phi, \cdot)}([1_{\Omega \times J_{\bar{\mathcal{C}}}} D^{(k)} D^{(\bar{\mathcal{C}}, \phi)} F^{N, d}(x, W_{(\cdot, \cdot)})]^{(\bar{\mathcal{C}})}), \quad J = ]s, t] \in \mathcal{J}, \\ x \in \mathbb{R}^d, \quad (\bar{\mathcal{C}}, \phi) \in \Lambda.$$

We will show for each  $(\bar{\mathcal{C}}, \phi) \in \Lambda$  with order  $m$

(3.9) there exists  $c_2 \in \mathbb{R}$  such that for  $x \in \mathbb{R}^d$ ,  $(s, t), (s', t') \in \hat{\mathbb{I}}^2$ ,  $s, s' \geq u^0$

$$E(|g^{(\bar{\mathcal{C}}, \phi)}(s, t, x) - g^{(\bar{\mathcal{C}}, \phi)}(s', t', x)|^{2p}) \leq c_2 |(s, t) - (s', t')|^{p\eta/|\bar{\mathcal{C}}|}$$

Once this is done, the assertion follows from Tanaka's formula. Compare  $g^{(\bar{\mathcal{C}}, \phi)}(s, t, x)$  and  $g^{(\bar{\mathcal{C}}, \phi)}(s', t', x)$  coordinatewise in  $s, s', t, t'$  to conclude that it is enough to find a constant  $c_3$  such that  $E(|g^{(\bar{\mathcal{C}}, \phi)}(s, t, x)|^{2p})$  can be estimated by  $c_3 |s_i - t_i|^{p\eta/|\bar{\mathcal{C}}|}$  for  $x \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ , and all  $1 \leq i \leq N$ . Hereby it is essential to distinguish between  $i \in \bar{\mathcal{C}}$  and  $i \notin \bar{\mathcal{C}}$ . Therefore, like in the proof of proposition 2, an application of Burkholder's inequalities reduces (3.9) to

(3.10) there exists  $c_4 \in \mathbb{R}$  such that for  $x \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $s \geq u^0$ ,  $u \geq u^0$

$$E\left(\left[\int_{J_{\bar{\mathcal{C}}}} 1_{J_{\bar{\mathcal{C}}}} (D^{(k)} D^{(\bar{\mathcal{C}}, \phi)} F^{N, d}(x, W_{(\cdot, u_{\bar{\mathcal{C}}}})))^2 d\lambda^{|\bar{\mathcal{C}}|}\right]^p\right) \leq c_4 |s_i - t_i|^{p\eta/|\bar{\mathcal{C}}|}, \quad \text{if } i \in \bar{\mathcal{C}}, \\ E\left(\left[\int_{J_{\bar{\mathcal{C}}}} 1_{J_{\bar{\mathcal{C}}}} [D^{(k)} D^{(\bar{\mathcal{C}}, \phi)} (F^{N, d}(x, W_{(\cdot, u_{\bar{\mathcal{C}}}}(t_i, t_i)) - F^{N, d}(x, W_{(\cdot, u_{\bar{\mathcal{C}}}}(t_i, s_i)))]^2 d\lambda^{|\bar{\mathcal{C}}|}\right]^p\right) \leq c_4 |s_i - t_i|^{p\eta/|\bar{\mathcal{C}}|}, \\ \text{if } i \notin \bar{\mathcal{C}}, \quad \text{where } J = ]s, t].$$

First consider the case  $i \in \bar{\mathcal{C}}$ . Estimate the integrand with the help of (1.4) and conclude by (3.6), observing that  $l = 2(2N - d - |k| - m \pm \delta) > 2\eta - 2|\bar{\mathcal{C}}|$ . The case  $i \notin \bar{\mathcal{C}}$  is more difficult, since lemma 5 has to be used for estimating the integrand. As  $u \geq u^0 > 0$ , by scaling we may suppose  $\bar{\mathcal{C}} = \{i\}$ . Then, according to lemma 5, (3.10) is a consequence of

(3.11) there exists  $c_5 \in \mathbb{R}$  such that for  $x \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $s \geq u^0$

$$i) \\ E\left(\left[\int_{J_{\bar{\mathcal{C}}}} 1_{J_{\bar{\mathcal{C}}}} |W_{(\cdot, s_i)} - W_{(\cdot, t_i)}|^{2\eta} |W_{(\cdot, s_i)} - x|^{l - 2\eta} d\lambda^{|\bar{\mathcal{C}}|}\right]^p\right) \leq c_5 |s_i - t_i|^{p\eta/|\bar{\mathcal{C}}|},$$

ii)

$$E\left(\left[\int_{\mathbb{J}_{\underline{e}}} 1_{\mathbb{J}_{\underline{e}}} |W_{(\cdot, s_i)} - x|^l 1_{\{|W_{(\cdot, s_i)} - x| < 2|W_{(\cdot, t_i)} - W_{(\cdot, s_i)}|\}} d\lambda^{|\underline{e}|}\right]^p\right) \leq c_5 |s_i - t_i|^{pn/|\underline{e}|},$$

iii)

$$E\left(\left[\int_{\mathbb{J}_{\underline{e}}} 1_{\mathbb{J}_{\underline{e}}} |W_{(\cdot, t_i)} - x|^l 1_{\{|W_{(\cdot, s_i)} - x| < 2|W_{(\cdot, t_i)} - W_{(\cdot, s_i)}|\}} d\lambda^{|\underline{e}|}\right]^p\right) \leq c_5 |s_i - t_i|^{pn/|\underline{e}|},$$

where  $\mathbb{J} = ]s, t]$ .

To argue (3.11), i), use independence of increments to single out a factor  $|t_i - s_i|^{2pn}$  and observe that the remainder can be treated by lemma 4, since by choice of  $\delta$ ,  $l - 2\eta > -d \vee -2|\underline{e}|$ . To argue (3.11), ii) and iii), we make use of the boundedness of the moments of local time (proposition 2). To infer ii), we will show

(3.12) there exists  $c_6 \in \mathbb{R}$  such that for  $x \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $q \in \mathbb{R}_+$

$$E\left(\left[\int_{\mathbb{J}_{\underline{e}}} 1_{\mathbb{J}_{\underline{e}}} |W_{(\cdot, s_i)} - x|^l 1_{\{|W_{(\cdot, s_i)} - x| < q\}} d\lambda^{|\underline{e}|}\right]^p\right) \leq c_6 q^{p(l+d)}, \text{ with } \mathbb{J} = ]s, t].$$

Note first that (3.12) implies (3.11), ii). Indeed,  $W_{(\cdot, t_i)} - W_{(\cdot, s_i)}$  is independent of  $W_{(\cdot, s_i)}$ . Consequently, by (3.12), the left side of (3.11), ii) is less or equal to

$$E\left(\sup_{u_{\underline{e}} \in \mathbb{J}_{\underline{e}}} |W_{(u_{\underline{e}}, t_i)} - W_{(u_{\underline{e}}, s_i)}|^{p(l+d)}\right),$$

which, by Doob's inequality, is  $c_7 |t_i - s_i|^{p(l+d)/2}$ , with a suitable  $c_7 \in \mathbb{R}$ . But

$$(3.13) \quad l + d \geq 2N - d - 2|k| \pm 2\delta \geq 1 \pm 2\delta > 2\eta$$

evidently implies (3.11), ii). To prove (3.12), for familiar reasons, we may and do assume  $\overline{\underline{e}} = \emptyset$ . Let  $L(\cdot, s, t, \cdot)$  be a local time of  $W$  over  $\mathbb{J} = ]s, t]$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $s \geq u^0$ . For  $x \in \mathbb{R}^d$ , (0.1) gives

$$\int_{\mathbb{I}} 1_{\mathbb{J}}(u) |W_u - x|^l 1_{\{|W_u - x| < q\}} du \leq \left(\prod_{1 \leq i \leq N} u_i^0\right)^{1-N} \int_{K_q(0)} |z|^l L(\cdot, s, t, x+z) dz.$$

Therefore, proposition 2 (with  $\rho = \bigtimes_{1 \leq i \leq N} (\varepsilon_{\{s_i\}} + \varepsilon_{\{t_i\}})$ ,  $\varepsilon_v$  being the point mass in  $v \in \mathbb{I}$ ) yields a constant  $c_8 \in \mathbb{R}$ , such that for  $x \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $q \in \mathbb{R}_+$

$$\mathbb{E} \left( \left[ \int_1^{\infty} 1_J |W_u - x|^l 1_{|W_u - x| < q} du \right]^p \right) \leq c_8 \left( \int_{K_q(0)} |z|^l dz \right)^p.$$

As  $l > -d$ , the integral on the right side exists and (3.12) follows.

Finally, for (3.11), iii) independence of increments can be used in nearly the same way as it has just been done. Consider the process

$$X(\omega, u) := u_i W(\omega, 1/u_i, u_{\{i\}}), \quad \omega \in \Omega, \quad u \in \mathbb{R}_+^N,$$

which is again an  $(N, d)$ -Wiener process.

Observe that

$$\begin{aligned} \{ |W_{(\cdot, s_i)} - x| < 2 |W_{(\cdot, t_i)} - W_{(\cdot, s_i)}| \} \\ \subset \{ |W_{(\cdot, t_i)} - x| < 3 |W_{(\cdot, t_i)} - W_{(\cdot, s_i)}| \}, \quad s_i, t_i \in \mathbb{I}, \end{aligned}$$

use this to write (3.11), iii) in terms of  $X$  and carry out an analogous calculation to the one which proved ii). This gives the desired conclusion.  $\square$

**PROPOSITION 4.** — Let  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  be such that  $2|k| + d < 2N$ . Further, let  $p \in \mathbb{N}$ ,  $0 < u^0 \in \mathbb{I}$ ,  $0 < \eta < 1/2$  be given. Then there exists  $c_1 \in \mathbb{R}$  such that for  $x, y \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $s \geq u^0$

$$\mathbb{E}(|M^{(k)}(\cdot, s, t, x) - M^{(k)}(\cdot, s, t, y)|^{2p}) \leq c_1 |x - y|^{2p\eta}.$$

*Proof.* — Fix  $0 < \delta$  such that  $\delta + \eta < 1/2$ . Following the proof of proposition 3, we can, due to Tanaka's formula, fix  $(\mathcal{C}, \phi) \in \Lambda$  with order  $m$  and apply Burkholder's inequalities (in the form of remark 2 after proposition 1) to see that it suffices to show

(3.14) there exists  $c_2 \in \mathbb{R}$  such that for  $x, y \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$ ,  $s \geq u^0$ ,  $u \geq u^0$

$$\begin{aligned} \mathbb{E} \left( \left[ \int_{\mathbb{I}_{\bar{e}}} 1_{J_{\bar{e}}} [D^{(k)} \mathbf{D}^{(\mathcal{C}, \phi)} (F^{N, d}(x, W_{(\cdot, u_{\bar{e}})}) - F^{N, d}(y, W_{(\cdot, u_{\bar{e}})}))]^2 d\lambda_{|\bar{e}|} \right]^p \right) \\ \leq c_2 |x - y|^{2p\eta}, \quad \text{where } J = ]s, t]. \end{aligned}$$

Put again  $l := 2(2N - d - |k| - m \pm \delta)$ . Use lemma 5 to estimate the integrand, and scaling, in order to trace back (3.14) to

(3.15) there exists  $c_3 \in \mathbb{R}$  such that for  $x, y \in \mathbb{R}^d$ ,  $(s, t) \in \hat{\mathbb{I}}^2$

$$\begin{aligned}
i) \quad & \mathbb{E} \left( \left[ \int_{\mathbb{I}} 1_J |W-x|^{l-2\eta} |x-y|^{2\eta} d\lambda^N \right]^p \right) \leq c_3 |x-y|^{2p\eta}, \\
ii) \quad & \mathbb{E} \left( \left[ \int_{\mathbb{I}} 1_J |W-x|^l 1_{\{|W-x| < 2|x-y|\}} d\lambda^N \right]^p \right) \leq c_3 |x-y|^{2p\eta}, \\
iii) \quad & \mathbb{E} \left( \left[ \int_{\mathbb{I}} 1_J |W-y|^l 1_{\{|W-x| < 2|x-y|\}} d\lambda^N \right]^p \right) \leq c_3 |x-y|^{2p\eta},
\end{aligned}$$

where  $J = ]s, t]$ .

By choice of  $\delta, l - 2\eta > -d \vee -2N$ . Therefore, *i*) follows from lemma 4, whereas *ii*) and *iii*) are consequences of (3.12) and (3.13)  $\square$

We are now ready to state the first smoothness theorem for the local time of  $W$ .

**THEOREM 2.** — Let  $d < 2N$ . Then for each  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2N$  there exists  $L^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  which satisfies

- i)  $L^{(k)}(., s, t, x) = M^{(k)}(., s, t, x)$  for  $\lambda^{2N+d}$ -a. e.  $(s, t, x) \in \hat{\mathbb{I}}^2 \times \mathbb{R}^d$ ,
- ii)  $(s, t, x) \rightarrow L^{(k)}(\omega, s, t, x)$  is continuous on  $\hat{\mathbb{I}}_0^2 \times \mathbb{R}^d$  for  $P$ -a. e.  $\omega \in \Omega$ ,
- iii)  $D^{(k)}L^{(0)}(., s, t, x) = L^{(k)}(., s, t, x)$  for all  $(s, t, x) \in \hat{\mathbb{I}}_0^2 \times \mathbb{R}^d$ .

$L^{(0)}(., s, t, .)$  is a local time of  $W$  over  $]s, t]$  for all  $(s, t) \in \hat{\mathbb{I}}_0^2$ .

*Proof.* — For  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2N$  let  $K^{(k)}$  be given according to theorem 1 with  $\rho = \lambda^N$ . Fix  $p \in \mathbb{N}$ ,  $0 < u^0 \in \mathbb{I}$ ,  $0 < \eta < 1/2$ . Eventually alter  $K^{(k)}$  on a  $P \times \lambda^{2N+d}$ -zero set to infer from propositions 3 and 4: there exists  $c_1 \in \mathbb{R}$  such that for  $x, y \in \mathbb{R}^d$ ,  $(s, t), (s', t') \in \hat{\mathbb{I}}^2$ ,  $s, s' \geq u^0$

$$\mathbb{E}(|K^{(k)}(., s, t, x) - K^{(k)}(., s', t', y)|^{2p}) \leq c_1 |(s, t, x) - (s', t', y)|^{p\eta/N}.$$

As we can take  $p > (2N+d)N/\eta$  in the preceding inequality, we obtain Kolmogorov's continuity criterion for  $K^{(k)}$  (see for example Bernard [4]). Thus there exists  $L_{u^0}^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2 \cap ]u^0, 1]^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  with (in  $(s, t, x)$ ) continuous trajectories such that

$$(3.16) \quad L_{u^0}^{(k)}(., s, t, x) = K^{(k)}(., s, t, x) \quad \text{for } (s, t, x) \in \hat{\mathbb{I}}^2 \cap ]u^0, 1]^2 \times \mathbb{R}^d.$$

Consequently we can ( $P$ -a. s.) uniquely define processes

$L^{(k)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  which coincide with  $L_{u^0}^{(k)}$  on  $\hat{\mathbb{I}}^2 \cap ]u^0, 1] \times \mathbb{R}^d$  and have continuous trajectories in  $(s, t, x) \in \hat{\mathbb{I}}_0^2 \times \mathbb{R}^d$ . *i*) follows from (3.16) and theorem 1, *i*); *iii*) is a consequence of *ii*) and theorem 1, *iii*).  $\square$

*Remark.* — The method used to prove lemma 4 does not allow to extend the results of theorem 2 to all intervals of  $\mathcal{J}$ . Finer estimates, however, should be possible (cf. proof of lemma 1, see also Ehm [8]).

#### 4. DIFFERENTIABILITY OF THE LOCAL TIME OF $W$ IN $(t, x)$

Let  $\emptyset \neq V \in \Pi_N$ ,  $d \in \mathbb{N}$  be such that  $d < 2|V|$ . Then for all  $t_{\bar{V}} \in \mathbb{I}_{\bar{V}}$ , local times for the  $(|V|, d)$ -processes  $W_{(\cdot, t_{\bar{V}})}$  exist. Integrating them over  $t_{\bar{V}}$  produces a local time of the  $(N, d)$ -Wiener process. We will now use this observation to study differentiability of local time in  $t$ . It also makes clear what the  $t$ -derivatives look like. We proceed like in 1. – 3.: starting with an appropriate version of Ito's formula we derive Tanaka's formula for  $(x, t)$ -derivatives and establish Kolmogorov's criterion for continuity.

PROPOSITION 5. — Let  $\emptyset \neq V \in \Pi_N$ ,  $f \in C^{2|V|}(\mathbb{R}^d)$  be such that

$$D^{(\bar{c}, \phi)} f(W) \in L^2(\Omega \times \mathbb{I}, \mathcal{P}, P \times \lambda^N), \quad (\bar{c}, \phi) \in \Psi_V.$$

Further, let a product  $\rho$  on  $\mathcal{B}(\mathbb{I})$  of finite measures  $\rho_i$ ,  $1 \leq i \leq N$ , satisfy

$$(4.1) \quad \int_{\mathbb{I}_{\bar{c}}} [\|D^{(\bar{c}, \phi)} f(W_{(\cdot, t_{\bar{c}})})\|_{(\bar{c}, \phi)}^{t_{\bar{c}}} \|^2 d\rho_{\bar{c}} t_{\bar{c}}] < \infty, \quad (\bar{c}, \phi) \in \Psi, \quad \bar{c} \subset V.$$

Then for each  $(\bar{c}, \phi) \in \Lambda$ ,  $\bar{c} \subset V$ , there exists

$X^{(\bar{c}, \phi)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}_V^2) \times \mathcal{B}(\mathbb{I}_{\bar{V}}), \mathcal{B}(\mathbb{R}))$  such that

$$i) \quad X^{(\bar{c}, \phi)}(\cdot, s_V, \cdot, \cdot) \in \mathcal{M}(\mathcal{P}, \mathcal{B}(\mathbb{R})), \quad s_V \in \mathbb{I}_V,$$

$$ii) \quad X^{(\bar{c}, \phi)}(\cdot, s_V, t_V, u_{\bar{V}})$$

$$= \Delta_{[s_V, t_V]_{\bar{c}} \setminus \bar{V} \times ]0, u_{\bar{V}}]} I^{(\bar{c}, \phi, \cdot)}([1_{\Omega \times ]s_V, t_V]} D^{(\bar{c}, \phi)} f(W_{(\cdot, \cdot)})]^{(\bar{c})})$$

for  $\rho_V^2 \times \rho_{\bar{V}} - a. e. (s_V, t_V, u_{\bar{V}}) \in \hat{\mathbb{I}}_V^2 \times \mathbb{I}_{\bar{V}}$ ,

$$iii) \quad \Delta_{[s_V, t_V]} f(W_{(\cdot, u_{\bar{V}})}) = \sum_{(\bar{c}, \phi) \in \Lambda, \bar{c} \subset V} \frac{1}{2^{|\bar{c}|}} \alpha_{(\bar{c}, \phi)} X^{(\bar{c}, \phi)}(\cdot, s_V, t_V, u_{\bar{V}}) \\ + \frac{1}{2^{|V|}} \int_{[s_V, t_V]} \mathbb{D}^{|V|} f(W_{(\cdot, u_{\bar{V}})}) \prod_{i \in \bar{V}} u_i^{|V|} \prod_{i \in V} u_i^{|V|-1} du_V$$

for  $\rho_V^2 \times \rho_{\bar{V}} - a. e. (s_V, t_V, u_{\bar{V}}) \in \hat{\mathbb{I}}_V^2 \times \mathbb{I}_{\bar{V}}$ , with  $\alpha_{(\bar{c}, \phi)}$  according to (1.1).

*Proof.* — By (4.1), the existence of  $X^{(\bar{c}, \phi)} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\hat{\mathbb{I}}_V^2) \times \mathcal{B}(\mathbb{I}_{\bar{V}}), \mathcal{B}(\mathbb{R}))$

satisfying *i*) and *ii*) follows from lemma 5 of [10] in the same way as the corollary of it. Fix  $(\mathcal{C}, \phi) \in \Lambda$ ,  $\underline{\mathcal{C}} \subset V$ , and  $u_{\bar{V}} \in \mathbb{V}_{\bar{V}}$  such that

$$\int_{\mathbb{V}_{\bar{V}}} [\|D^{(\mathcal{C}, \phi)} f(W_{(\cdot, t_{\bar{V}} \setminus \bar{V}, u_{\bar{V}})})^{\bar{\mathcal{C}}} \|_{(\bar{\mathcal{C}}, \phi)}^{t_{\bar{V}} \setminus \bar{V}}]^2 d\rho_{\bar{\mathcal{C}} \setminus \bar{V}}(t_{\bar{V}} \setminus \bar{V}) < \infty.$$

which is true for  $\rho_{\bar{V}} = a. e. u_{\bar{V}} \in \mathbb{V}_{\bar{V}}$ . Apply theorem 4 of [10] to the  $(|V|, d)$ -Wiener process  $\prod_{i \in \bar{V}} u_i^{-1/2} W_{(\cdot, u_{\bar{V}})}$  to obtain *iii*) (cf. (4.6) in the proof of lemma 6 of [10]).  $\square$

Now let  $\emptyset \neq V \in \Pi_N$ ,  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2|V|$ , a product  $\rho$  on  $\mathcal{B}(\mathbb{I})$  of finite measures  $\rho_i$ ,  $1 \leq i \leq N$ , and  $(\mathcal{C}, \phi) \in \Lambda$ ,  $\underline{\mathcal{C}} \subset V$ , be given. Then, the proof of lemma 1 gives

$$(4.2) \quad (t_{\bar{\mathcal{C}}}, x) \rightarrow \| [D^{(k)} D^{(\mathcal{C}, \phi)} F^{|V|, d}(x, W_{(\cdot, t_{\bar{\mathcal{C}}}}))]^{\bar{\mathcal{C}}} \|_{(\bar{\mathcal{C}}, \phi)}^{t_{\bar{\mathcal{C}}}}$$

is locally bounded on  $\mathbb{V}_{\bar{\mathcal{C}}} \times \mathbb{R}^d$ .

Proposition 5 in place of (1.1) and (1.2) with  $F^{|V|, d}$  instead of  $F^{N, d}$  motivate the following definition, which makes sense in consequence of (4.2).

DÉFINITION 2. — Let  $\emptyset \neq V \in \Pi_N$ ,  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  be such that

$2|k| + d < 2|V|$ . For  $x \in \mathbb{R}^d$ ,  $(s_V, t_V) \in \hat{\mathbb{I}}_V^2$ ,  $u_{\bar{V}} \in \mathbb{V}_{\bar{V}}$  let, setting  $J_V = ]s_V, t_V]$

$$\begin{aligned} M^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V, t_V, x) &:= 2^{|V|} [\Delta_{J_V} D^{(k)} F^{|V|, d}(x, W_{(\cdot, u_{\bar{V}})}) \\ &- \sum_{(\bar{\mathcal{C}}, \phi) \in \Lambda, \underline{\mathcal{C}} \subset V} \frac{1}{2^{|\bar{\mathcal{C}}|}} \alpha_{(\bar{\mathcal{C}}, \phi)} \Delta_{J_{\bar{\mathcal{C}} \setminus \bar{V}} \times ]0, u_{\bar{V}}]} I^{(\bar{\mathcal{C}}, \phi, \cdot)}([1_{\Omega \times (J_V)_{\bar{\mathcal{C}}}} D^{(k)} D^{(\mathcal{C}, \phi)} F^{|V|, d}(x, W_{(\cdot, \cdot)})]^{\bar{\mathcal{C}}})]. \end{aligned}$$

Now observe that the proofs of propositions 3 and 4 go through without essential modifications for  $M^{(k, \bar{V})}$  instead of  $M^{(k)}$ . Therefore, we obtain for  $V$ ,  $d$ ,  $k$  as above,  $p \in \mathbb{N}$ ,  $0 < u^0 \in \mathbb{I}$ ,  $0 < \eta < 1/2$

(4.3) there exists  $c_1 \in \mathbb{R}$  such that for  $x, y \in \mathbb{R}^d$ ,  $u_{\bar{V}}, u'_{\bar{V}} \in \mathbb{V}_{\bar{V}}$ ,

$$\begin{aligned} (s_V, t_V), (s'_V, t'_V) &\in \mathbb{I}_V^2, \quad u_{\bar{V}}, u'_{\bar{V}} \geq u_{\bar{V}}^0, \quad s_V, s'_V \geq u_{\bar{V}}^0 \\ E(|M^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V, t_V, x) - M^{(k, \bar{V})}(\cdot, u'_{\bar{V}}, s'_V, t'_V, y)|^{2p}) \\ &\leq c_1 |(u_{\bar{V}}, s_V, t_V, x) - (u'_{\bar{V}}, s'_V, t'_V, y)|^{pn/|V|}. \end{aligned}$$

With the help of (4.3), the second smoothness theorem for local times can now be proved.

**THEOREM 3.** — For each  $\emptyset \neq V \in \Pi_N$ ,  $\bar{V} \neq \emptyset$ ,  $d \in \mathbb{N}$ ,  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2|V|$  there exists  $L^{(k, \bar{V})} \in \mathcal{M}(\mathcal{F} \times \mathcal{B}(\mathbb{I}_{\bar{V}}) \times \mathcal{B}(\hat{\mathbb{I}}_{\bar{V}}^2) \times \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$  which satisfies

- i)  $L^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V, t_V, x) = M^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V, t_V, x)$   
for  $\lambda^{|\bar{V}|+2N+d}$ -a. e.  $(u_{\bar{V}}, s_V, t_V, x) \in \mathbb{I}_{\bar{V}} \times \hat{\mathbb{I}}_{\bar{V}}^2 \times \mathbb{R}^d$ ,
- ii)  $(u_{\bar{V}}, s_V, t_V, x) \rightarrow L^{(k, \bar{V})}(\omega, u_{\bar{V}}, s_V, t_V, x)$   
is continuous on  $(\mathbb{I}_{\bar{V}})_0 \times (\hat{\mathbb{I}}_{\bar{V}}^2)_0 \times \mathbb{R}^d$ , for  $P$ -a. e.  $\omega \in \Omega$ ,
- iii)  $L^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V, t_V, x) = D^{(k)}L^{(0, \bar{V})}(\cdot, u_{\bar{V}}, s_V, t_V, x)$   
on  $(\mathbb{I}_{\bar{V}})_0 \times (\hat{\mathbb{I}}_{\bar{V}}^2)_0 \times \mathbb{R}^d$ .

Let  $L^{(k)}$  be given according to theorem 2. Then

$$iv) \quad L^{(k)}(\cdot, s, t, x) = |\bar{V}|^{|V|} \int_{|s, t|_{\bar{V}}} \int_{\mathbb{I}_V} L^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V \vee u_V, t_V, x) \\ \prod_{1 \leq i \leq N} u_i^{|\bar{V}|-1} du, \quad (s, t) \in \hat{\mathbb{I}}_0^2, \quad x \in \mathbb{R}^d.$$

In particular,  $P$ -a. s. for all  $x \in \mathbb{R}^d$ ,  $0 < s \in \mathbb{I}$

$t \rightarrow L^{(k)}(\cdot, s, t, x)$  is continuously partially differentiable in  $(t_i, i \in \bar{V})$  and

$$\frac{\partial^{|\bar{V}|}}{\partial(t_i, i \in \bar{V})} L^{(k)}(\cdot, s, t, x) \\ = |\bar{V}|^{|V|} \int_{\mathbb{I}_V} L^{(k, \bar{V})}(\cdot, u_{\bar{V}}, s_V \vee u_V, t_V, x) \prod_{i \in \bar{V}} u_i^{|\bar{V}|-1} du_V \prod_{i \in \bar{V}} t_i^{|\bar{V}|-1}.$$

*Proof.* — To argue i)-iii), we proceed like in the proofs of theorems 1 and 2: we make use of an obvious generalization of lemma 2 which rests upon (4.1) instead of lemma 1; (4.3) takes the place of propositions 3 and 4. To prove iv), employing proposition 5 instead of theorem 4 of [10], we derive the following analogon of (1.6)

$$(4.4) \quad \int_{\mathbb{R}^d} L^{(0, \bar{V})}(\cdot, u_{\bar{V}}, s_V \vee u_V, t_V, x) h(x) dx \\ = \int_{|s_V \vee u_V, t_V|} h(W_{(v_V, u_{\bar{V}})}) \prod_{i \in \bar{V}} u_i^{|V|} \prod_{i \in \bar{V}} v_i^{|V|-1} dv_V, \\ h \in C_c^\infty(\mathbb{R}^d), \quad 0 < u \in \mathbb{I}, \quad (s_V, t_V) \in (\hat{\mathbb{I}}_{\bar{V}}^2)_0.$$

Now integrate both sides of (4.4) to get

$$\begin{aligned}
 & |\bar{V}|^{|\bar{V}|} \int_{\mathbb{R}^d} \int_{[s,t]_{\bar{V}}} \int_{\mathbb{I}_{\bar{V}}} L^{(0,\bar{V})}(\cdot, u_{\bar{V}}, s_{\bar{V}} \vee u_{\bar{V}}, t_{\bar{V}}, x) \prod_{1 \leq i \leq N} u_i^{|\bar{V}|-1} du h(x) dx \\
 &= \int_{[s,t]} h(W_u) \prod_{1 \leq i \leq N} u_i^{N-1} du, \quad h \in C_c^\infty(\mathbb{R}^d), \quad (s, t) \in \hat{\mathbb{I}}_0^2.
 \end{aligned}$$

Considering (0.1), this implies the validity of *iv*) for  $k = \underline{0}$ . Apply *iii*) and theorem 2, *iii*) to infer *iv*) for all  $k \in \mathbb{N}_0^d$  such that  $2|k| + d < 2|V|$ . What remains to be done is an easy consequence of *iv*).  $\square$

## REFERENCES

- [1] R. J. ADLER, A Hölder condition for the local time of the Brownian sheet, *Indiana Univ. Math. J.*, t. **29**, 1980, p. 793-798.
- [2] J. AZEMA, M. YOR, Temps locaux, *Astérisque*, t. **52-53**, Société Math. de France, Paris, 1978.
- [3] S. M. BERMAN, Gaussian processes with stationary increments: Local times and sample function properties, *Ann. Math. Stat.*, t. **41**, 1970, p. 1260-1272.
- [4] P. BERNARD, Quelques propriétés des trajectoires des fonctions aléatoires stables sur  $\mathbb{R}^k$ , *Ann. Inst. Henri Poincaré*, t. **6** (2), 1970, p. 131-151.
- [5] K. BICHTLER, Stochastic integration and  $L^p$ -theory of semimartingales. *Ann. Prob.*, t. **9**, 1981, p. 49-89.
- [6] R. CAIROLI, J. WALSH, Stochastic integrals in the plane, *Acta Mathematica*, t. **134**, 1975, p. 111-183.
- [7] R. CAIROLI, J. WALSH, Régions d'arrêt, localisations et prolongements de martingales, *Z. W.*, t. **44**, 1978, p. 279-306.
- [8] W. EHM, Sample function properties of multi-parameter stable processes, *Z. W.*, t. **56**, 1981, p. 195-228.
- [9] D. GEMAN, J. HOROWITZ, Occupation densities, *Ann. Prob.*, t. **8**, 1980, p. 1-67.
- [10] P. IMKELLER, *Ito's formula for continuous (N, d)-processes* (to appear in *Z. W.*).
- [11] P. IMKELLER, *Local times for a class of multi-parameter processes* (to appear in *Stochastics*).
- [12] E. MERZBACH, Processus stochastiques à indices partiellement ordonnés, *Rapp. interne*, t. **55**, École Polytechnique, 1979.
- [13] C. METRAUX, Quelques inégalités pour martingales à paramètre bidimensionnel. Sémin. de Prob. XII, Univ. de Strasbourg, *Lecture Notes in Mathematics*, t. **649**, p. 170-179, Springer, Berlin, 1978.
- [14] P. A. MEYER, Un cours sur les intégrales stochastiques. Sémin. de Prob. X, Univ. de Strasbourg, *Lecture Notes in Mathematics*, t. **511**, Springer, Berlin, 1976.
- [15] S. R. PARANJAPÉ, C. PARK, Laws of the iterated logarithm of multiparameter Wiener processes, *J. Multiv. Anal.*, t. **3**, 1973, p. 132-136.
- [16] L. SCHWARTZ, *Théorie des distributions*, Hermann, Paris, 1966.
- [17] L. T. TRAN, On a problem posed by Orey and Pruitt related to the range of the N-parameter Wiener process in  $\mathbb{R}^d$ , *Z. W.*, t. **37**, 1976, p. 27-33.

- [18] S. USTUNEL, Stochastic integration on nuclear spaces and its applications, *Ann. Inst. Henri Poincaré*, t. **18**, 1982, p. 165-200.
- [19] J. WALSH, The local time of the Brownian sheet, *Astérisque*, t. **52-53**, 1978, p. 47-61.
- [20] E. WONG, M. ZAKAI, Differentiation formulas for stochastic integrals in the plane, *Stoch. Proc. Appl.*, t. **6**, 1978, p. 339-349.

(Manuscrit reçu le 18 juillet 1983).