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Section B : *Calcul des Probabilités et Statistique.*

# Asymptotic behaviour of the quadratic measure of deviation of multivariate density estimates

by

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RÉSUMÉ. — Nous obtenons un test d'adéquation de la distribution asymptotique de $||\hat{f}_{n,N} - f_n||^2$ et nous prouvons également que la statistique considérée est asymptotiquement gaussienne sous les hypothèses de contiguïté de la forme $f_N = f^0 + \delta_N\phi$, $\phi \in L^2(\mu)$, $\delta_N \downarrow 0$.

ABSTRACT. — We obtain a test of goodness of fit from the asymptotic distribution of $||\hat{f}_{n,N} - f_n||^2$ and we also prove that the statistic under consideration is asymptotically gaussian under contiguous alternatives of the form $f_N = f^0 + \delta_N\phi$, $\phi \in L^2(\mu)$, $\delta_N \downarrow 0$.

Key words and phrases: Multidimensional density estimates, quadratic measure, asymptotic distribution, test of goodness of fit.

## 1. INTRODUCTION

Set $X_1, X_2, \ldots, X_N, \ldots$ be a sequence of independent identically distributed random vector with values in $\mathbb{R}^p$. We shall suppose that their common distribution has a density $f$ with respect to Lebesgue measure and that $f \in L^2(\mu)$, where $\mu$ is a Borel probability measure on $\mathbb{R}^p$ with density $r(x)$ with respect to Lebesgue measure.

If $\{\phi_j\}_{j=1}^{\infty}$ is complete orthonormal system in $L^2(\mu)$, the $n$-th partial sum of the respective Fourier series for $f$ is

$$f_n(x) = \sum_{j=1}^{n} a_j \phi_j(x) \qquad x \in \mathbb{R}^p$$

where

$$a_j = \int_{\mathbb{R}^p} f(x)\phi_j(x)r(x)dx = E\alpha_j(X_1) \qquad \alpha_j(x) = \phi_j(x)r(x)$$

Ĉencov [2] defines the following estimator of $f_n$:

$$\hat{f}_{n,N}(x) = \sum_{j=1}^{n} \hat{a}_j \phi_j(x)$$

where the $\hat{a}_j's$ are estimators of $a_j$ defined by

$$\hat{a}_j = \int_{\mathbb{R}^p} \alpha_j(x)dF_N(x)$$

$F_N$ being, as usual, the empirical distribution function of the sample $X_1, X_2, \ldots, X_N$.

The aim of this paper is to give conditions under which

$$\frac{N}{n^{1/2}} \| \hat{f}_{n,N} - f_n \|^2 .$$

When appropriately centered, has gaussian asymptotic distribution. The method we use is inspired by Naradaya [8], although instead of using the strong approximation of the empirical process by a Brownian bridge, we approximate the estimator by functions of Gaussian variables with values in $L^2(\mu)$. For these ones, the result follows from the central limit theorem on the real line, and the approximation allows to study the behaviour of $\frac{N}{\sqrt{n}} \| \hat{f}_{n,N} - f_n \|^2$.

The result is applied to various complete orthonormal sets.

Finally, we consider tests of goodness of fit upon $\| \hat{f}_{n,N} - f_n \|^2$ and the behaviour of $g(n) = \| f_n - f \|^2$, which permit together to study the asymptotic behaviour of the statistic $\| \hat{f}_{n,N} - f \|^2$.

## 2. ASSOCIATED GAUSSIAN VARIABLES

We have defined

$$\hat{f}_{n,\mathrm{N}}(x) = \sum_{j=1}^{n} \hat{a}_j \phi_j(x)$$

If we put

$$\mathrm{Y}_{n,k}(x) = \sum_{j=1}^{n} (\alpha_j(\mathrm{X}_k) - a_j)\phi_j(x)$$

it is clear that

$$\sqrt{\mathrm{N}}(\hat{f}_{n,\mathrm{N}} - f_n) = \frac{1}{\sqrt{\mathrm{N}}} \sum_{k=1}^{\mathrm{N}} \mathrm{Y}_{n,k}(x)$$

If we consider the $\mathrm{Y}_{n,k}$ as independent identically distributed random variables with values in $\mathrm{L}^2(\mu)$ we have

$$\mathrm{E}(\mathrm{Y}_{n,k}) = 0$$

and

$$\Gamma_n(g, h) = \mathrm{E}\{(g, \mathrm{Y}_{n,k})(h, \mathrm{Y}_{n,k})\}$$

where $\Gamma_n$ is the covariance of $\mathrm{Y}_{n,k}$ and $(.,\ .)$ is the scalar product in $\mathrm{L}^2(\mu)$.

It follows that

$$\Gamma_n(\phi_i, \phi_j) = \int_{\mathbb{R}^p} f(x)\phi_i(x)\phi_j(x)r^2(x)dx - a_i a_j \qquad i, j = 1, 2, \ldots, n.$$

Define the centered gaussian random variable $\mathrm{Z}_{1,n}$ with values in $\mathrm{L}^2(\mu)$ by

$$\mathrm{Z}_{1,n} = \sum_{i=1}^{n} \xi_i \phi_i$$

in such a way that $\mathrm{Z}_{1,n}$ and $\mathrm{Y}_{n,k}$ have the same covariance. Now, if $\xi_0$ is a normalized gaussian real random variable, independent from $\{\xi_i\}_{i=1}^{n}$, consider the random variable (with values in $\mathrm{L}^2(\mu)$):

$$\mathrm{Z}_{2,n} = \mathrm{Z}_{1,n} + \xi_0 \sum_{i=1}^{n} a_i \phi_i.$$

Them $Z_{2,n}$ is gaussian, $E(Z_{2,n}) = 0$ and its covariance $\Gamma_n^{(2)}$ satisfies

$$\Gamma_n^{(2)}(\phi_i, \phi_j) = \int_{\mathbb{R}^p} \phi_i(x)\phi_j(x)r^2(x)f(x)dx$$

LEMMA 2.1. — If $\| fr \|_\infty < \infty$, then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} E \mid \| Z_{2,n} \|^2 - \| Z_{1,n} \|^2 \mid = 0$$

*Proof.* — $E \mid \| Z_{2,n} \|^2 - \| Z_{1,n} \|^2 \le \sum_{j=1}^{n} a_j^2 + 2E \mid \xi_0 \mid E \mid \sum_{j=1}^{n} \xi_j a_j \mid$
and since $E(\mid \xi_0 \mid) < 1$,

$$\left( E \left| \sum_{j=1}^{n} \xi_j a_j \right| \right)^2 \le \int \left( \sum_{i=1}^{n} \phi_i(x) a_i \right)^2 r^2(x) f(x) dx \le \| fr \|_\infty \| f \|^2 .$$

The result now follows from

$$E \mid \| Z_{2,n} \|^2 - \| Z_{1,n} \|^2 \mid \le \| f \| \left( \| f \| + 2 \| fr \|_\infty^{1/2} \right).$$

Before studying the asymptotic behaviour of $Z_{1,n}$ let us define:

*i)* $A_n = (\Gamma_n^{(2)}(\phi_i, \phi_j)) = (C_{ij})$

*ii)* $\Delta_n = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^p} \alpha_i^2(x) f(x) dx = \frac{1}{n} \operatorname{tr}(A_n)$

*iii)* $S_m(n) = \sum_{i_1=1}^{n} \dots \sum_{i_m=1}^{n} C_{i_1 i_2} C_{i_2 i_3} \dots C_{i_m i_1}$, evidently $S_m(n) = \operatorname{Tr}(A_n^m)$.

*iv)* $\sigma_n^2 = \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_{\mathbb{R}^p} \phi_i(x)\phi_j(x) r^2(x) f(x) dx \right)^2$

Note that

$$\sigma_n^2 = \frac{2}{n} S_2(n) = \frac{2}{n} \sum_{i=1}^{n} \lambda_{i,n}^2$$

if $\{ \lambda_{i,n} \}_{i=1}^n$ are the eigenvalues of $A_n$.

LEMMA 2.2. — Suppose that there exists $m \ge 3$ such that $\frac{1}{n} S_m = 0(1)$. Then, if $\lim_{n\to\infty} \sigma_n^2 = \sigma^2 > 0$, we have

$$\lim_{n\to\infty} \frac{n\sigma_n^2}{(\max_{1 \le i \le n} \lambda_{i,n})^2} = \infty$$

*Proof.* — $(\sup_{1 \leq i \leq n} \lambda_{i,n})^m \leq \sum_{i=1}^{m} \lambda_{i,n}^m = S_m(n)$

So that

$$\frac{n\sigma_n^2}{(\sup_{1 \leq i \leq n} \lambda_{i,n})^2} \geq \frac{n^{\left(\frac{m-2}{m}\right)} \sigma_n^2}{\left(\frac{1}{n} S_m(n)\right)^{2/m}}.$$

The result follows letting $n \to \infty$.

THEOREM 2.3. — Under the same hypothesis of Lemma 2.2 we have

$$W - \lim_{n\to\infty} \frac{1}{n^{1/2}}(\| Z_{2,n} \|^2 - E \| Z_{2,n} \|^2) = N(0, \sigma^2).$$

*Proof.* — Since $Z_{2,n}$ is gaussian we can find a basis $\{ e_1, \ldots, e_n \}$ such that

$$Z_{2,n} = \lambda_{1,n}^{1/2} \gamma_1 e_1 + \ldots + \lambda_{n,n}^{1/2} \gamma_n e_n .$$

where $\gamma_1, \ldots, \gamma_n$ are normalized independent gaussian random variables. So

$$\| Z_{2,n} \|^2 = \lambda_{1,n}\gamma_1^2 + \ldots + \lambda_{n,n}\gamma_n^2$$

and

$$n^{-1/2}(\| Z_{2,n} \|^2 - E \| Z_{2,n} \|^2) = n^{-1/2} \sum_{i=1}^{n} \lambda_{i,n}(\gamma_i^2 - 1).$$

The result follows from

$$\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \lambda_{i,n}^2 = \sigma^2 .$$

Together with Lemma 2.2 and Lindeberg's Theorem on the line.

REMARK. — From Lemma 2.1 and $n^{-1/2}E \| Z_{2,n} \|^2 = \sqrt{n}\Delta_n$ we obtain

$$W - \lim_{n\to\infty} (n^{-1/2} \| Z_{1,n} \|^2 - n^{1/2}\Delta_n) = N(0, \sigma^2).$$

## 3. MAIN THEOREM

The following Theorem is due to Kuelbs and Kurtz [*7*] see also Giné and León [*4*]. The statement is adapted to our present needs.

THEOREM 3.1. — Let $\{ Y_i \}_{i=1}^n$ be independent identically distributed random variables with values in $L^2(\mu)$, $E(Y_1) = 0$, $E(\| Y_1 \|^3) < \infty$ and $Z_1$ a centered gaussian variable with the same covariance as $Y_1$. Then, for each $t$ and $\delta > 0$,

$$\left| P\left\{ \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} Y_i \right\| \leq t \right\} - P\{ \| Z_1 \| < t \} \right| = 0\left( \delta^{-3} \frac{E \| Y_1 \|^3}{\sqrt{N}} \right) + P\{ | \| Z_1 \| - t | \leq \delta \}$$

holds true.

We now prove our main result:

THEOREM 3.2. — Suppose that for some $\alpha > 0$

$$E(\| Y_{n,1} \|^3) = o(N^{1/2}) \qquad \text{if} \qquad n = 0(N^\alpha)$$

If, additionally, $\| fr \|_\infty < \infty$, $\sigma_n^2 \to \sigma^2 > 0$ and $\frac{1}{n} S_m(n) = 0(1)$ for some $m \geq 3$, them

$$W - \lim_{n \to \infty} \left[ \frac{N}{n^{1/2}} \| \hat{f}_{n,N} - f_n \|^2 - n^{1/2} \Delta_n \right] = N(0, \sigma^2)$$

*Proof.* — Define $G_{n,N}(t)$ and $G_n(t)$ in the following way:

$$G_{n,N}(t) = P\left\{ \frac{N}{n^{1/2}} \| \hat{f}_{n,N} - f_n \|^2 - n^{1/2} \Delta_n \leq t \right\}$$
$$= P\left\{ \left\| \frac{1}{\sqrt{N}} \sum_{k=1}^{n} \frac{Y_{n,k}}{n^{1/4}} \right\| \leq (t + n^{1/2} \Delta_n)^{1/2} \right\}$$
$$G_n(t) = P\left\{ \frac{1}{n^{1/2}} \| Z_{1,n} \|^2 - n^{1/2} \Delta_n \leq t \right\}$$
$$= P\left\{ \left\| \frac{1}{n^{1/4}} Z_{1,n} \right\| \leq (t + n^{1/2} \Delta_n)^{1/2} \right\}.$$

By theorem 3.1.

$$| G_{n,N}(t) - G_n(t) | = 0\left( \delta^{-3} \frac{E \| Y_{n,1} \|^3}{n^{3/4} N^{1/2}} \right) + P\left\{ \left| \frac{1}{n^{1/4}} \| Z_{1,n} \| - (t + n^{1/2} \Delta_n)^{1/2} \right| \leq \delta \right\}.$$

But

$$P\left\{ \left| \frac{1}{n^{1/4}} \| Z_{1,n} \| - (t + n^{1/2} \Delta_n)^{1/2} \right| \leq \delta \right\} = P\left\{ \delta^2 - 2\delta \sqrt{t + n^{1/2} \Delta_n} + t \leq \frac{1}{n^{1/2}} \| Z_{1,n} \|^2 - n^{1/2} \Delta_n \leq \delta^2 + 2\delta \sqrt{t + n^{1/2} \Delta_n} + t \right\}.$$

If we choose $\delta = \delta(n)$ such that $\delta^2 n^{1/2} \to 0$, the remark following theorem 2.3 and the fact that $\Delta_n$ is bounded, since

$$\Delta_n \leq \left(\frac{1}{n}\sum_{i=1}^{n} \lambda_{i,n}^2\right)^{1/2} \to \frac{1}{\sqrt{2}}\sigma,$$

imply:

$$\mathrm{P}\left(\left|\frac{1}{n^{1/4}}\,||\,\mathrm{Z}_{1,n}\,||-(t+n^{1/2}\Delta_n)^{1/2}\right| \leq \delta\right) \to \phi_\sigma(t)-\phi_\sigma(t)=0.$$

Where $\phi_\sigma$ denotes the normal $(0, \sigma^2)$ distribution.

Finally, if we choose $\delta(n) = n^{-1/4}\theta_n$ with $\theta_n \to 0$, $\delta^2 n^{1/2} = \theta_n \to 0$ holds, and the theorem will follow if the first term in (3.1) tends to zero. This is achieved if $\theta_n$ tends to zero slowly enough.

COROLLARY 3.3. — If in addition to the hypothesis of theorem 3.2, one has $g(n) = ||\, f_n - f\,||^2 = o(n^{1/2}\mathrm{N}^{-1})$, them

$$\mathrm{W}-\lim_{n\to\infty}\left[\frac{\mathrm{N}}{n^{1/2}}\,||\,\hat{f}_{n,\mathrm{N}}-f\,||^2 - n^{1/2}\Delta_n\right] = \mathrm{N}(0, \sigma^2).$$

We shall use the following theorem to verify that

$$\mathrm{E}(\,||\,\mathrm{Y}_{n,1}\,||^3) = o(\mathrm{N}^{1/2}).$$

THEOREM 3.4. — Define $v_i = \sup_x |\,\alpha_i(x)\,|$. Suppose that

$$\sum_{i=1}^{n} v_i^2 = o(n^{-2}\mathrm{N}) \qquad \text{and} \qquad ||\, fr\,||_\infty < \infty$$

them

$$\mathrm{E}(\,||\,\mathrm{Y}_{n,1}\,||^3) = o(\mathrm{N}^{1/2})$$

*Proof.* —

$$\mathrm{E}(\,||\,\mathrm{Y}_{n,1}\,||^3) = \mathrm{E}\left(\sum_{j=1}^{n}(\alpha_j(\mathrm{X}_1)-a_j)^2\right)^{3/2} \leq 2^{5/2}\mathrm{E}\left(\sum_{j=1}^{n}\alpha_j^2(\mathrm{X}_1)\right)^{3/2}$$

now

$$\sum_{j=1}^{n}\alpha_j^2(\mathrm{X}_1) \leq \sum_{j=1}^{n} v_j^{2/3}\alpha_j^{4/3}(\mathrm{X}_1)$$

on applying Hölder's inequality it follows that

$$E\Big(\sum_{j=1}^{n} \alpha_j^2(X_1)\Big)^{3/2} \leq \Big(\sum_{j=1}^{n} v_j^2\Big)^{1/2} E\Big(\sum_{j=1}^{n} \alpha_j^2(X_1)\Big)$$

but

$$E\Big(\sum_{j=1}^{n} \alpha_j^2(X_1)\Big) = 0(n)$$

then

$$E(\| Y_{n,1} \|^3) \leq K\Big(n^2 \sum_{j=1}^{n} v_j^2\Big)^{1/2} = o(N^{1/2})$$

according to the hypothesis.

## 4. EXAMPLES AND APPLICATIONS

*a)* Let us consider the space $L^2((-\pi, \pi)^2)$ with respect to Lebesgue measure, and the complete orthonormal set $\left\{ \frac{1}{2\pi} e^{i(mx+ny)} \right\}_{(m,n)\in Z^2}$. With the obvious modifications in the previous sections to be able to include complex valued functions $f$, we may study the estimators

$$\hat{f}_{n,N}(x, y) = \sum_{j,k=-n}^{n} \hat{C}_{k,j} e^{i(kx+jy)}$$

(Here, $C_{k,j}$ are the Fourier coefficients of $f$, and $\hat{C}_{k,j}$ their estimators).
We easily verify

$$\Delta_n = \frac{2}{4\pi^2}$$

and

$$\sigma_n^2 = \frac{1}{2\pi^2(2n+1)^2} \sum_{-2n \leq j,k \leq 2n} (2n+1-|k|)(2n+1-|j|)|C_{k,j}|^2$$

and by Beppo-Levi's Theorem:

$$\sigma_n^2 \to \frac{1}{2\pi^2} \| f \|^2$$

The bound $\sup_x |D_n(x)| = 0(\log n)$, for the Dirichlet-Kernel ([10], p. 151) gives:

$$\frac{S_m(n)}{n} \leq (\text{const})\left(\frac{\log n}{n}\right)^{2m} = 0(1) \qquad (m \geq 3)$$

Moreover $v_{k,j} = 1$ and if we put $v = (2n+1)^2$

$$\sum_{-n \leq k,j \leq n} v_{k,j}^2 = o(v^{-2}N) \text{ is verified if } n = N^\alpha \text{ with } \alpha < 1/6.$$

Finally if $f$ is periodic continuously differentiable in $C((-\pi, \pi)^2$ until the second order and it has three derivatives in $L^2$ with respect to each variable i. e. $\| f_{xxx} \|_2 < \infty$ and $\| f_{yyy} \|_2 < \infty$, then is easy to verify that $g(n) = 0(n^{-6}) = (v^{1/2}N^{-1})$ if $n = N^\alpha$ with $\alpha > \frac{1}{7}$.

Then, under the above conditions for $f$, we get

$$W - \lim_{N \to \infty} \left[\frac{N}{2n+1} \| \hat{f}_{n,N} - f \|^2 - \frac{2n+1}{4\pi^2}\right] = N(0, \sigma^2)$$

if $n = N^\alpha$, $\frac{1}{7} < \alpha < 1/6$, and $\sigma^2 = \frac{1}{2\pi^2} \| f \|^2$.

*Note.* — This result was obtained by Naradaya [8] in the univariate case, although our method improves the choice of the exponent $\alpha$. With minor changes the same proof applies to densities on $\mathbb{R}^p$.

*b*) As a second example we consider an asymptotic test of goodness of fit for uniform distribution on a sphere.

The basis for $L^2(S^2)$ with the measure invariant by rotations is denoted by $\{ Y_n^m(\theta, \phi) \}_{m=-n}^n$ $n = 0, 1, \ldots$ $\left(0 \leq \theta < 2\pi, -\frac{\pi}{2} \leq \phi < \pi/2\right)$ and constructed from the spherical harmonics ([3], p. 511).

We put, with the obvious notations

$$\hat{f}_{n,N}(\theta, \phi) = \sum_{k=0}^{n} \sum_{m=-k}^{k} \hat{C}_{m,k} Y_k^m(\theta, \phi)$$

$$\hat{C}_{m,k} = \frac{1}{N} \sum_{i=1}^{N} Y_k^m(\theta_i, \phi_i)$$

where $(\theta_1, \phi_1), \ldots, (\theta_N, \phi_N)$ is the observed sample.

The statistic $T_{n,N} = \| \hat{f}_{n,N} - 1 \|^2_{L^2(S^2)}$ can be used to test uniformity. One easily verifies that $A_n = Id$, so that $\Delta_n = 1$, $\sigma_n^2 = 2$, $\frac{S_m(n)}{n} = 1$. Moreover

$$\sum_{j=j}^{n} v_j^2 = 0(n^3) = o(v^{-2}N) \text{ (with } v = n^2 + 1) \text{ for } n = N^\alpha \text{ and } \alpha < 1/7.$$

Hence, Theorem 3.2 gives

$$W - \lim_{N \to \infty} \left[ \frac{N}{\sqrt{n^2 + 1}} T_{n,N} - (n^2 + 1)^{1/2} \right] = N(0, 2)$$

*c*) As a final example, let $f \in L^2(-1, 1)$, $r(x) = 1$ and $\phi_j = (2j + 1)^{1/2} p_j, p_j$ the sequence of Legendre polynomials

$$v_j \leq \sqrt{2j + 1}$$

So that

$$\sum_{j=1}^{n} v_j^2 = 0(n^{-2}) = o(n^{-2}N) \quad \text{for } n = N^\alpha \quad \text{and } \alpha < 1/4.$$

The remaining conditions can be verified using the following Theorem ([5], p. 116).

THEOREM 4.1. — With the above notations and $f \in C[-1, 1]$, consider Toeplitz matrices.

$$A_n(f) = \left( \int_{-1}^{1} \phi_i(x)\phi_j(x)f(x)dx, \quad i, j = 1, \ldots, n \right)$$

If $\lambda_i^{(n)}$ $(i = 1, \ldots, n)$ are the eigenvalues of $A_n(f)$, them for each $m \geq 1$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\lambda_i^{(n)})^m = \frac{1}{\pi} \int_{-1}^{1} f^m(x) \frac{1}{\sqrt{1 - x^2}} dx$$

holds true.

In our case, we get

$$\lim_{n \to \infty} \sigma_n^2 = \frac{2}{\pi} \int_{-1}^{1} f^2(x) \frac{1}{\sqrt{1 - x^2}} dx$$

and

$$S_m(n) = \sum_{i=1}^{m} (\lambda_i^{(n)})^m = 0(n).$$

If assume that $f^4 \in C[-1, 1]$, then $g(n) = 0(n^{-5})$ (see [6], p. 209) and $g(n) = o(n^{1/2}N^{-1})$ if $n = N^\alpha$, $\alpha > 2/11$.

Summing up, if $f^4 \in C[-1, 1]$, $n = N^\alpha$, $\frac{2}{11} < \alpha < 1/4$, then

$$W - \lim_{N\to\infty} \left[ \frac{N}{n^{1/2}} \| \hat{f}_{n,N} - f \|^2 - n^{1/2}\Delta_n \right] = N(0, \sigma^2)$$

with

$$\sigma^2 = \frac{2}{\pi} \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1 - x^2}} dx .$$

## 5. ASYMPTOTIC BEHAVIOUR UNDER CONTIGUOUS ALTERNATIVES

Suppose that one wants to test the null hypothesis

$$H_0 : f = f^0$$

against the sequence of alternatives

$$H_N : f^N(x) = f^0(x) + \delta_N \Phi(x)$$

where $\Phi$ is a fixed function in $L^2(\mu)$ and $\delta_N \to 0$.

The following Theorem states that $T_{n,N} = \frac{N}{n} \| f_{n,M} - f_n \|^2$ is asymptotically gaussian under $H_N$. The proof follows the lines of [1], th. 4.2 and [8], th. 4.2, and the result can be applied to the previous examples.

We must define before

$$\tilde{\sigma}_n^2 = \frac{2}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int_{\mathbb{R}^p} \phi_i(x)\phi_j(x) r^2(x) f^N(x) dx \right)^2$$

THEOREM 5.1. — Under $H_N$, if $\Delta_n = \Delta + o\left(\frac{1}{\sqrt{n}}\right)$, $\tilde{\sigma}_n^2 \to \sigma_0^2 > 0$, suppose also that the hypothesis of the Theorem 3.2 are satisfied for $n = N^\alpha$, $0 < \alpha < \alpha_0$ and $\delta_N = (N^{-\frac{(2-\alpha)}{4}})$ then

$$W - \lim_{N\to\infty} \sqrt{n} \left( \frac{T_{n,N} - \Delta}{\sigma_0^2} \right) = N\left( \frac{1}{\sigma_0} \| \Phi \|^2, 1 \right)$$

*Proof.* — Denote

$$E_{H_N}(\hat{f}_{n,N}) = f_n^N .$$

Where $E_{H_N}$ denotes the expectation when the true underlying distribution has density $f^N$.

Let $\{\phi_i\}$ a complete orthonormal basis for $L^2(\mu)$ and $\gamma_i = (\Phi, \phi_i)$ the Fourier coefficients of the function $\Phi$. Define

$$\tilde{T}_{n,N} = \frac{N}{n} \| \hat{f}_{n,N} - f_n^N \|^2 .$$

We have

$$\| \hat{f}_{n,N} - f_n^0 \|^2 = \| \hat{f}_{n,N} - f_n^N \|^2 + 2 \langle \hat{f}_{n,N} - f_n^N , f_n^N - f_n^0 \rangle + \| f_n^N - f_n^0 \|^2$$

So that:

$$\sqrt{n}\left(\frac{T_{n,N} - \Delta_0}{\sigma_0}\right) = \sqrt{n}\left(\frac{\tilde{T}_{n,N} - \Delta_0}{\sigma_0}\right) + \frac{2}{\sqrt{n}}\frac{1}{\sigma_0}\langle \hat{f}_{n,N} - f_n^N , f_n^N - f_n^0 \rangle + \frac{N}{\sigma_0\sqrt{n}}\| f_n^N - f^0 \|^2$$

but

$$\frac{N}{\sigma_0\sqrt{n}}\| f_n^N - f^0 \|^2 = \frac{\delta_N^2 N}{\sigma_0\sqrt{n}}\| \Phi_n \|^2 \underset{n\to\infty}{\longrightarrow} \frac{\| \Phi \|^2}{\sigma_0}$$

moreover

$$E\left[\frac{2}{\sqrt{n}}\frac{1}{\sigma_0}\langle \hat{f}_{n,N} - f_n^N , f_n^N - f_n^0 \rangle\right] = 0 .$$

and

$$E\left[\frac{2}{\sqrt{n}}\frac{1}{\sigma_0}\langle \hat{f}_{n,N} - f_n^N , f_n^N - f_n^0 \rangle\right]^2 = \frac{4N\delta_N^2}{n\sigma_0^2} E(\sqrt{N}(\hat{f}_{n,N} - f_n^N), \Phi_n)^2$$

$$= \frac{4N\delta_N^2}{\sqrt{n\sigma_0^2}}\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\int \phi_i\phi_j r^2 f^N dx - a_i^N a_j^N\right)\gamma_i\gamma_j\right)$$

$$\leq k\frac{1}{\sqrt{n}}\| rf_N \|_\infty \| \Phi_n \|^2 \leq k\frac{1}{\sqrt{n}}\| rf \|_\infty \| \Phi \|^2$$

then we can conclude

$$\frac{2}{\sqrt{n}}\frac{1}{\sigma_0}\langle \hat{f}_{n,N} - f_n^N , f_n^N - f_n^0 \rangle \underset{N\to\infty}{\overset{P}{\longrightarrow}} 0$$

and finally applying Theorem 3.2

$$W - \lim_{N\to\infty}\left(\frac{\tilde{T}_{n,N} - \Delta}{\sigma_0}\right) = N(0, 1) .$$

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## REFERENCES

[1] P. J. BICKEL and M. ROSENBLATT, On some measures of the deviations of density functions estimates. *The Ann. of Statistics*, t. **1**, n° 6, 1973, p. 1071-1095.

[2] N. N. CENCOV, Evaluations of an unknown distribution density from observations. *Soviet Math.*, t. **3**, 1962, p. 1559-1562.

[3] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, t. **1**, New York, 1953.

[4] E. GINÉ, J. LEÓN, On the central limit theorem in Hilbert Spaces. *Stochastica*, t. **IV**, n° 1, 1980, p. 43-71.

[5] U. GRENANDER, Szegö. Toeplitz forms and their application. University of California Press. Berkeley. 1958.

[6] E. ISACCSON, H. KELLER, « Analysis of Numerical Methods » John Wiley and Sons. Inc. New York, 1966.

[7] KUELBS, KURTZ, Bery Essen estimates in Hilbert Space and an application to the LIL. *Ann. Probability*, t. **2**, 1974, p. 387-407.

[8] E. A. NARADAYA, On a quadratic measure of deviation of the estimate of a distribution density. *Theo. Prob. and its App.*, n° 4, 1976, p. 844-850.

[9] A. ZYGMUND, Trygonometric series, 2nd Ed. Cambridge University Press. New York, 1959.

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