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Large deviations in spaces of stable type

by

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SOMMAIRE. — On caractérise les espaces de Banach de type p-stable, $1 \leq p < 2$, comme ceux qui vérifient la propriété de « grandes déviations » suivante : il existe $C < \infty$ telle que pour toute v. a. X symétrique, a valeurs dans B, $\| X \|$ dans le domaine d'attraction d'une loi p-stable, et pour toute suite $\gamma_n \uparrow \infty$,

$$\overline{\lim}_{n\to\infty} P\{ \| S_n(X) \| > \gamma_n \}/nP\{ \| X \| > \gamma_n \} \leq C.$$

(Le $\underline{\lim}$ est strictement positif si et seulement si $\overline{\lim}_{n\to\infty} nP\{ \| X \| > \gamma_n \} < \infty$). On considère aussi le cas non-symétrique. L'outil principal est une caractérisation des espaces de type p-stable en termes de suites d'une certaine classe de v. a. a valeurs dans B, indépendantes et équidistribuées. Finalement, on examine la relation entre ces résultats et la conduite presque sûre des sommes partielles.

1. INTRODUCTION

Let ξ_i be i. i. d. real symmetric random variables not in the domain of partial attraction (DPA) of a normal law. Then Heyde (1967 *a*, *b*, 1968, 1969) proved: (1)

(1.1) $$\overline{\lim}_{n\to\infty} P\{ | \textstyle\sum_{i=1}^{n} \xi_i | > \gamma_n \}/nP\{ | \xi_1 | > \gamma_n \} < \infty$$

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whenever $\gamma_n \uparrow \infty$; and (2)

$$(1.2) \qquad \overline{\lim}_{n\to\infty} \mid \Sigma_{i=1}^n \xi_i \mid / \gamma_n = 0 \qquad \text{or} \qquad = +\infty$$

according as

$$(1.3) \qquad \sum_{i=1}^\infty P\{ \mid \xi_i \mid > \gamma_n \} < \infty \qquad \text{or} \qquad = +\infty .$$

This second result (with absolute value replaced by norm) also holds in type 2 Banach spaces (Kuelbs and Zinn (1981), part of Theorem 2), where as in a general Banach space condition (1.3) must be replaced by

$$(1.3') \qquad \sum_{i=1}^\infty P\{ \| \Sigma_{i=1}^n \xi_i \| > \gamma_n \}/n < \infty \qquad \text{or} \qquad = +\infty$$

(Kuelbs (1979)). These results of Kuelbs and Kuelbs and Zinn suggest then the problem of characterizing those Banach spaces where (1) and or (2) hold. The main result of this article is that the result (1) holds for every B-valued r. v. X such that $\| X \|$ is in the domain of attraction of a stable law, $1 \leqslant p < 2$ if and only if B is of type p-stable. It seems plausible that a similar result holds for (2), but I do not know. The proof of this characterization of type p-stable spaces is based on another interesting property of these spaces. Rosinski (1980) proved that a Banach space B is of type p-stable ($1 \leq p < 2$) if and only if there exists a constant $C < \infty$ such that for any set of independent symmetric B-valued r. v. s' X_i,

$$(1.4) \qquad \Lambda_p^p(\Sigma_i X_i) \leq C \Sigma_i \Lambda_p^p(X_i)$$

where

$$(1.5) \qquad \Lambda_p(X) = \sup_{t>0} [t^p P\{ \| X \| > t \}]^{1/p} .$$

The property in question is that it suffices to check (1.4) for i. i. d. X_i belonging to a certain class of random variables $\lambda_p(B)$ to be defined below (the corresponding result for type p-Rademacher spaces is proved in Pisier (1975), Proposition 5.1). This result is strengthened by the following: B is of type p-stable, $1 \leq p < 2$, if and only if every B-valued r. v. X such that $\lim_{t\to\infty} t^p P\{ \| X \| > t \} < \infty$ satisfies

$$(1.6) \qquad \overline{\lim}_{n\to\infty} \Lambda_p\left(\sum_{i=1}^n X_i/n^{1/p}\right) < \infty ,$$

where the X_i are independent copies of X (Theorem 2.3). A similar result (with moments instead of Λ_p) holds also in the case $p = 2$.

The last mentioned properties of type p-stable spaces are given in Section 2, the large deviation results in Section 3, and Section 4 contains some comments on the almost sure behavior of $\| S_n \|$ in type p-stable Banach spaces for r. v. 's X such that $t^p P\{ \| X \| > t \}$ is a slowly varying function of t.

Some notation. B will denote a separable Banach space, X a B-valued r. v., X_i independent copies of X and $S_n(X) := \Sigma_{i=1}^n X_i$. B is of type p-stable, $0 < p \leq 2$, if there exists a constant $C < \infty$ such that for any finite set of points $x_i \in B$ and i. i. d. standard symmetric p-stable real random variables θ_i (i. e. for every $t \in \mathbb{R}$, $Ee^{it\theta_1} = e^{-c|t|^p}$, with $c = 1$ for $p < 2$ and $c = \frac{1}{2}$ for $p = 2$), $E \| \Sigma_i \theta_i x_i \|^p \leq C\Sigma_i \| x_i \|^p$. Every Banach space is of type p-stable for $p < 1$. If θ_i are replaced by ε_i in the previous definition ($\{ \varepsilon_i \}$ a sequence of i. i. d. random variables with $P\{ \varepsilon_i = 1 \} = P\{ \varepsilon_i = -1 \} = \frac{1}{2}$) then B is of type p-Rademacher. We will use the following result of Maurey and Pisier (1976) : B is of type p-stable, $1 \leq p < 2$, if and only if it is of type $(p + \varepsilon)$-Rademacher for some $\varepsilon > 0$. Type 2-stable is equivalent to type 2-Rademacher. Finally, if μ is a finite measure on B, Pois μ will denote the probability measure $e^{-\mu(B)}\sum_{n=0}^{\infty} \mu^n/n!$, $\mu^n = \mu * \ldots n) * \mu$, where $*$ is the product of convolution.

2. TYPE p-STABLE BANACH SPACES

We consider here a class of random variables similar to those used by Pisier in the proof of Proposition 5.1 (1975).

2.1. Definition. — $\lambda_p(B)$, $0 < p < 2$, will denote the set of B-valued r. v. 's of the form $X = \Sigma_{\text{finite}} \xi_i \theta_i x_i$ where $\{ x_i \} \subset B$, the ξ_i are identically distributed Bernoulli r. v. 's with disjoint supports (support of $\xi_i = \{ \omega : \xi_i(\omega) \neq 0 (\text{and} = 1) \}$), the θ_i are i. i. d. standard p-stable, and the family of random variables $\{ \xi_i \}$ is independent of the family $\{ \theta_i \}$. $\bar{\lambda}_p(B)$, $0 < p < 2$, will denote the set of B-valued r. v. 's of the form $X = \Sigma_{i=1}^{\infty} \xi_i \theta_i x_i$, where the ξ_i are Bernoulli r. v. 's with disjoint supports and $\{ \theta_i \}$ and $\{ x_i \}$ are as above (with $\{ \xi_i \}$ independent of $\{ \theta_i \}$). The definitions for $p = 2$ are analogous, but in this case $\{ \theta_i \}$ is a Rademacher sequence.

Remark. — Note that if $X \in \lambda_p(B)$ then X is in the domain of normal attraction of a p-stable law. If $X \in \bar{\lambda}_p(B)$ then we still have that $\| X \|$ and $f(X)$, $f \in B'$, are in domains of normal attraction of p-stable laws in $\mathbb{R}$, but X itself may not unless B is of type p-stable. In fact, the candidate for the limiting p-stable law, $Y := \Sigma_{i=1}^{\infty} p_i^{1/p} \theta_i x_i$ (with θ_i replaced by independent N(0, 1) in the case $p = 2$) may not exist; but it exists if B is of type p-stable and $\Sigma_i p_i \| x_i \|^p < \infty$.

2.2. THEOREM. — B is of type p-stable, $1 \leq p < 2$, if and only if there exists a constant $C < \infty$ such that for every $n \in \mathbb{N}$ and $X \in \lambda_p(B)$,

$$(2.1) \qquad \Lambda_p^p(\Sigma_{i=1}^n X_i/n^{1/p}) \leq C\Lambda_p^p(X).$$

Proof. — By Rosinski's previously mentioned result, it is enough to prove the if part. Let $\{x_i\} \subset B$ be such that $\Sigma \| x_i \|^p < \infty$. By the three series and Ito-Nisio's theorems it is enough to show that $\{\mathscr{L}(\Sigma_{i=1}^n \theta_i x_i)\}_{n=1}^\infty$ is a uniformly tight sequence of laws. Set $Y_i = \theta_i x_i$ and note that there exists a positive constant c_p such that for all n,

$$(2.2) \qquad \Sigma_{i=1}^n \Lambda_p^p(Y_i) = c_p \Sigma_{i=1}^n \| x_i \|^p < \infty.$$

Let now $Y_j^{n,m}$, $j = 1, \ldots, m - n$, be i. i. d. B-valued r. v. 's with law

$$\mathscr{L}(Y_j^{n,m}) = (m - n)^{-1} \Sigma_{i=n+1}^m \mathscr{L}(Y_i).$$

Then, $Y_j^{n,m} \in \lambda_p(B)$ and

$$\Lambda_p^p(Y_j^{n,m}) \leq (m - n)^{-1} \Sigma_{i=n+1}^m \Lambda_p^p(Y_i).$$

Therefore, (2.1) and (2.2) give

$$(2.3) \qquad \Lambda_p^p(\Sigma_{j=1}^{m-n} Y_j^{n,m}) \leq C c_p \Sigma_{i=n+1}^m \| x_i \|^p.$$

It then follows from (2.3) that

$$(2.4) \qquad pr - \lim_{n\to\infty} \Sigma_{j=1}^{m-n} Y_j^{n,m} = 0 \qquad \text{uniformly in} \quad m > n$$

(where $pr - \lim$ denotes limit in probability).

Given $\varepsilon > 0$ let n_1 be such that

$$P\{\| \Sigma_{j=1}^{m-n} Y_j^{n,m} \| > \varepsilon/2\} < \varepsilon/4 \qquad \text{for all} \quad n > n_1 \quad \text{and} \quad m > n$$

(n_1 exists by (2.4)). Let $Y_0^{n,m} = 0$, let j in $Y_j^{n,m}$ run up to ∞ and let N_{m-n} be a Poisson real r. v. with parameter $m - n$, independent of $\{Y_j^{n,m}\}_{j=1}^\infty$. Then, Pois $(\Sigma_{j=1}^{m-n} \mathscr{L}(Y_j^{n,m})) = \mathscr{L}(\Sigma_{j=0}^{N_{m-n}} Y_j^{n,m})$, and the argument in exercise 2, p. 122 of Araujo and Giné (1980) shows that for $n > n_1$ and $m - n > 4/\varepsilon$,

$$P\{\| \Sigma_{j=1}^{m-n} Y_j^{n,m} - \Sigma_{j=0}^{N_{m-n}} Y_j^{n,m} \| > \varepsilon/2\}$$
$$\leq 2P\{\| \Sigma_{j=1}^{m-n} Y_j^{n,m} \| > \varepsilon/2\} + P\{N_{m-n} > 2(m - n)\} < 3\varepsilon/4.$$

Hence

$$(2.5) \qquad P\{\| \Sigma_{j=0}^{N_{m-n}} Y_j^{n,m} \| > \varepsilon\} < \varepsilon \qquad \text{for} \quad n > n_1 \quad \text{and} \quad m - n \geq 4/\varepsilon.$$

Let h be a natural number such that $e^{-1} \Sigma_{r=h+1}^\infty 1/(r-1)! < \varepsilon^2/8$ and let n_2 be such that for $n > n_2$ and $m > n$,

$$(2.6) \qquad P\{\| Y_j^{n,m} \| > \varepsilon^2/4h\} < \varepsilon^2/8$$

(n_2 exists by (2.4)). Then, for $n > n_2$ and $m - n < 4/\varepsilon$ we have:

(2.7) $$\text{Pois}\,(\Sigma_{j=1}^{m-n}\mathscr{L}(Y_j^{n,m}))\{\,\|x\| > \varepsilon\,\}$$
$$\leq (m-n)[\text{Pois}\,\mathscr{L}(Y_j^{n,m})]\{\,\|x\| > \varepsilon/(m-n)\,\}$$
$$\leq 4\varepsilon^{-1}e^{-1}\Sigma_{r=1}^{\infty}P\{\,\|Y_j^{n,m}\| > \varepsilon^2/4r\,\}/(r-1)\,!$$
$$\leq 4\varepsilon^{-1}e^{-1}(\Sigma_{r=1}^{h}1/(r-1)\,!)\varepsilon^2/8 + 4\varepsilon^{-1}\varepsilon^2/8 < \varepsilon.$$

Then, (2.5) and (2.7) give

(2.8) $$\text{Pois}\,\{\,\Sigma_{i=n+1}^{m}\mathscr{L}(Y_i)\,\} = \text{Pois}\,\{\,\Sigma_{j=1}^{m-n}\mathscr{L}(Y_j^{n,m})\,\} \to_w \delta_0$$
$$\text{as} \quad n \to \infty \quad \text{uniformly in } m > n\,.$$

Let finally Z_i be independent r. v. 's with law Pois $\mathscr{L}(Y_i)$, $i = 1, \ldots$. Then (2.8) implies that the sequence of r. v. 's $\{\,\Sigma_{i=1}^{n}Z_i\,\}_{n=1}^{\infty}$ is Cauchy in probability, hence convergent. This, by LeCam's theorem (e. g. Theorem 3.4.8 in Araujo and Giné (1980)), gives the uniform tightness of $\{\,\mathscr{L}(\Sigma_{i=1}^{n}Y_i)\,\}_{n=1}^{\infty}$. □

The following theorem gives additional information.

2.3. Theorem. — If every B-valued r. v. X in $\bar{\lambda}_p(B)$ satisfies

(2.9) $$\overline{\lim}_{n\to\infty}\Lambda_p(S_n(X)/n^{1/p}) < \infty\,,$$

then B is of type p-stable.

Proof. — If B is not of type p-stable, Theorem 2.2 implies that for any sequence $C_k \uparrow \infty$ there exist random variables $X_k \in \lambda_p(B)$ such that $\sup_{t>0} t^p P\{\,\|X_k\| > t\,\} = 1$, and positive integers $n_k \uparrow \infty$ such that

(2.10) $$\Lambda_p^p(S_{n_k}(X_k)/n_k^{1/p}) \geq C_k\,.$$

Notice that we must have $n_k^p \geq C_k$. This property allows us to choose X_k, C_k and n_k, and define m_k and p_k as follows: let $K \in (0, \infty)$ be such that for all natural number n and $p \in [0, 1]$,

$$\text{binomial}\,(n, p)\{\,x > np\,\} \geq \frac{1}{2} - K(np(1-p))^{-\frac{1}{2}},$$

which exists by the Berry-Esseen theorem, and then let X_k, C_k, n_k, m_k and p_k be such that

i) $$K(n_k(1 - C_k^{-\frac{1}{2}}))^{-\frac{1}{2}} < \frac{1}{4},$$

ii) $$\Sigma_{k=1}^{\infty}C_k^{-\frac{1}{2}} = 1 \qquad \text{and} \qquad p_k = C_k^{-\frac{1}{2}} \qquad (C_k \geq 0)\,,$$

iii) m_k an integer such that $n_k - 1 < m_k p_k \leq n_k$, *iv*) the r. v. 's X_k satisfy $\Lambda_p(X_k) = 1$ and (2.10) for C_k and n_k, and are independent ($k = 1, \ldots$).

Let $\{ \eta_k \}$ be a sequence of real r. v. 's with disjoint supports, independent of the sequence $\{ X_k \}$, and such that $p_k = P\{ \eta_k = 1 \} = 1 - P\{ \eta_k = 0 \}$. Define a r. v. Y as

$$Y = \Sigma_{k=1}^{\infty} \eta_k X_k$$

and let $\{ Y_i = \Sigma_{k=1}^{\infty} \eta_k^i X_k^i \}_{i=1}^{\infty}$ be independent copies of Y. Then, the properties of n_k, m_k, p_k, C_k give that for all $t > 0$ and natural number k,

$$t^p P\{ \| S_{m_k}(Y) \| > m_k^{1/p} t \} = t^p P\{ \| \Sigma_{r=1}^{\infty} (\Sigma_{i=1}^{m_k} \eta_r^i X_r^i) \| > m_k^{1/p} t \}$$
$$\geq \frac{1}{2} t^p P\{ \| \Sigma_{i=1}^{m_k} \eta_k^i X_k^i \| > m_k^{1/p} t \}$$
$$\geq \frac{1}{2} t^p P\{ \| \Sigma_{i=1}^{m_k} \eta_k^i X_k^i \| > m_k^{1/p} t, \quad \# \{ \eta_k^i = 1, \quad i = 1, \ldots, m_k \} > m_k p_k \}$$
$$\geq 16^{-1} t^p P\{ \| S_{n_k}(X_k) \| > n_k^{1/p} t / p_k^{1/p} \}$$
$$= 16^{-1} p_k (t/p_k^{1/p})^p P\{ \| S_{n_k}(X_k) \| > n_k^{1/p} t / p_k^{1/p} \}.$$

(The second inequality above is obvious, the third just uses one of the Lévy's inequalities, and the first one follows by the following argument on symmetric variables: if the conditional law of U given V is symmetric, then we have

$$P\{ \| U + V \| > c \} = \int P\{ \| U + v \| > c \mid V = v \} d\mathscr{L}(V)(v)$$
$$= \int P\{ \| U - v \| > c \mid V = v \} d\mathscr{L}(V)(v) \geq \frac{1}{2} \int P\{ \| U \| > c \mid V = v \} d\mathscr{L}(V)(v)$$
$$= \frac{1}{2} P\{ \| U \| > c \};$$

now take $U = \Sigma_{i=1}^{m_k} \eta_k^i X_k^i$ and $V = \Sigma_{r \neq k} \Sigma_{i=1}^{m_k} \eta_r^i X_r^i$). Therefore, by this inequality and the properties (*i*)-(*iv*), it follows that

$$\Lambda_p^p(S_{m_k}(Y)/m_k^{1/p}) \geq 16^{-1} p_k \Lambda_p^p(S_{n_k}(X_k)/n_k^{1/p}) \geq p_k C_k / 16 \to \infty.$$

Hence Y satisfies

(2.11) $$\overline{\lim}_{n \to \infty} \Lambda_p(S_n(Y)/n^{1/p}) = \infty.$$

Finally, it is obvious that, by construction, Y can be written as

$$Y = \Sigma_{k=1}^{\infty} \zeta_k \theta_k x_k,$$

with $\{ \zeta_k \}$ independent of $\{ \theta_k \}$, with disjoint supports, and such that

$P\{\zeta_k = 1\} = 1 - P\{\zeta_k = 0\}$, $\Sigma_k P\{\zeta_k = 1\} = 1$, $\{\theta_k\}$ i. i. d. standard p-stable, and $\{x_k\} \subset B$. □

As mentioned above, Pisier (1975) proved that type 2 is equivalent to the existence of $C < \infty$ such that for all $n > 0$,

$$(2.12) \qquad E \| \Sigma_{i=1}^n X_i \| / n^{\frac{1}{2}} \leq C(E \| X_1 \|^2)^{\frac{1}{2}}$$

for i. i. d. r. v. 's $X_i \in \lambda_2(B)$. A proof similar to that of Theorem 2.3, which will be omitted, gives:

2.4. THEOREM. — If every $X \in \bar{\lambda}_2(B)$ satisfies

$$\overline{\lim}_{n\to\infty} E \| S_n(X) \| / n^{\frac{1}{2}} < \infty ,$$

then B is of type 2.

3. LARGE DEVIATIONS

The following theorem generalizes to type p-stable Banach spaces some results of Heyde (1967 *a*, *b*).

3.1. THEOREM. — *a*) Let B be a type p-stable Banach space, $0 < p < 2$, and X a symmetric B-valued r. v. such that for some $\varepsilon > 0$ and every $p' \in (p, p + \varepsilon)$,

$$(3.1) \qquad \overline{\lim}_{t\to\infty} E \| X \|^{p'} I_{(\|X\| \leq t)} / t^{p'} P\{ \| X \| > t \} = K_{p'} < \infty .$$

Then, there exists a constant $C < \infty$ depending only on $K_{p'}$ and the type p'-Rademacher constant of B (for some $p' \in (p, p + \varepsilon)$) such that for every sequence $\gamma_n \uparrow \infty$,

$$(3.2) \qquad \overline{\lim}_{n\to\infty} P\{ \| S_n(X) \| > \gamma_n \} / nP\{ \| X \| > \gamma_n \} \leq C .$$

If B is of type 2, then (3.1) with $p' = 2$ implies (3.2).

b) If B is not of type p-stable, $1 \leq p < 2$, then there exists a B-valued symmetric r. v. Y such that $\| Y \|$ and $f(Y)$, $f \in B'$, are in the domains of normal attraction of p-stable laws, and a sequence $\tau_n \uparrow \infty$ such that

$$(3.3) \qquad \overline{\lim}_{n\to\infty} P\{ \| S_n(Y) \| > \tau_n n^{1/p} \} / nP\{ \| Y \| > \tau_n n^{1/p} \} = \infty .$$

c) If B is not of type 2, there exists a B-valued symmetric r. v. Y not in the domain of partial attraction of any Gaussian law which satisfies (3.3) with $p = 2$.

Proof. — *a*). Let, for $\gamma_n \uparrow \infty$, $T_n = \Sigma_{k=1}^n X_k I_{(\|X_k\| \leq \gamma_n)}$ and $a_n = EXI_{(\|X\| \leq \gamma_n)}$ ($a_n = 0$ if X is symmetric, but we will impose symmetry only when needed).

Let $p' \in (p, p + \varepsilon)$ be such that B is of type p'-Rademacher and let K'_p be its type p' constant (such a p' exists by a previously mentioned result of Maurey and Pisier). Then we have:

$$\begin{aligned} P\{\|S_n(X) - na_n\| > \gamma_n\} &\leq nP\{\|X\| > \gamma_n\} + P\{\|T_n - na_n\| > \gamma_n\} \\ &\leq nP\{\|X\| > \gamma_n\} + K'_{p'}n\gamma_n^{-p'}E\|XI_{(\|X\|\leq\gamma_n)} - a_n\|^{p'} \\ &\leq nP\{\|X\| > \gamma_n\} + 2^{p'\wedge 1}K'_{p'}n\gamma_n^{-p'}E\|XI_{(\|X\|\leq\gamma_n)}\|^{p'}, \end{aligned}$$

and by (3.1),

$$(3.4)\quad \overline{\lim}_{t\to\infty} P\{\|S_n(X)-na_n\| > \gamma_n\}/nP\{\|X\| > \gamma_n\} \leq 1+2^{p'\wedge 1}K'_{p'}K_{p'}.$$

In the symmetric case, $na_n = 0$ and (3.4) is just (3.2). The case $p = 2$ is proved similarly (using that type 2-Rademacher is equivalent to type 2-stable).

b) Let Y and m_k be as in the proof of Theorem 2.3. Then it follows from that proof that there exists a sequence $\tau_{m_k} \uparrow \infty$ (we may assume monotonicity by passing to a subsequence if necessary) such that

$$\lim_{k\to\infty} \tau_{m_k}^p P\{\|S_{m_k}(Y)\| > m_k^{1/p}\tau_{m_k}\} = \infty .$$

Since $\lim_{n\to\infty} \tau_{m_k}^p m_k P\{\|Y\| > m_k^{1/p}\tau_{m_k}\} < \infty$, it follows that

$$(3.5)\quad \lim_{k\to\infty} P\{\|S_{m_k}(Y)\| > m_k^{1/p}\tau_{m_k}\}/m_kP\{\|Y\| > m_k^{1/p}\tau_{m_k}\} \to \infty .$$

c) Let Y be as in the proof of Theorem 2.4. By Hoffmann-Jorgensen's inequality (see e. g. Araujo and Giné (1980), p. 107), there exist $c_1, c_2 \in (0, \infty)$ such that

$$E\|S_n(Y)/n^{\frac{1}{2}}\| \leq c_1 E \max_{i\leq n}\|Y_i\|/n^{\frac{1}{2}} + c_2 t_n$$

where

$$t_n = \inf\,[t : P\{\|S_n(Y)/n^{\frac{1}{2}}\| > t\} < 1/48].$$

Since $E\|S_{m_k}(Y)/m_k^{\frac{1}{2}}\| \to \infty$ and $E\max_{i\leq n}\|Y_i\|/n^{\frac{1}{2}} \leq 1$, it follows that $t_{m_k} \to \infty$. We may take $t_{m_k} \uparrow \infty$ by passing to a subsequence. Then, since $E\|Y\|^2 < \infty$,

$$\begin{aligned} P\left\{\|S_{m_k}(Y)\| > \frac{1}{2}t_{m_k}m_k^{\frac{1}{2}}\right\}/m_kP\left\{\|Y\| > \frac{1}{2}t_{m_k}m_k^{\frac{1}{2}}\right\} \\ \geq 1/48m_kP\left\{\|Y\| > \frac{1}{2}t_{m_k}m_k^{\frac{1}{2}}\right\} \to \infty . \quad \square \end{aligned}$$

A typical computation shows that if $t^pP\{\|X\| > t\} \to c \in (0, \infty)$, $0 < p < 2$ (in fact, if this function is slowly varying), then X satisfies (3.1) for $p' > p$. Hence:

3.2. COROLLARY. — A Banach space B is of type p-stable, $0 < p < 2$, if and only if every B-valued symmetric r. v. X such that $\| X \|$ is in the domain of (normal) attraction of a p-stable law satisfies the large deviation result (3.2).

Remark (and questions). — It is easy to show using elementary methods (exercises 3.7.11 and 12 in Araujo and Giné, loc. cit.) that the random variable Y in the proof of 3.1 (*b*) satisfies

$$\lim_{t\to\infty} t^p \mathrm{P}\{ \| \mathrm{S}_n(\mathrm{Y}) \| > n^{1/p} t \} = \lim_{t\to\infty} t^p \mathrm{P}\{ \| \mathrm{Y} \| > t \} \in (0, \infty).$$

So, if $\tau_n \uparrow \infty$ fast enough, the limit in (3.3) is 1. It would be interesting to know if $\tau_n \uparrow \infty$ can be chosen so that (3.5) holds and also $\mathrm{S}_n(\mathrm{Y})/n^{1/p}\tau_n \to 0$ in probability or, at least, is bounded in probability. Regarding (*c*), the example obtained satisfies $\mathrm{E}\| \mathrm{Y} \|^2 = 1$; it would be interesting to obtain an example such that $\| \mathrm{Y} \|$ were not in the DPA of any Gaussian law (in the present example, Y is not in the DPA of any Gaussian law, but obviously $\| \mathrm{Y} \|$ is).

We may ask whether

$$\underline{\lim}_{n\to\infty} \mathrm{P}\{ \| \mathrm{S}_n(\mathrm{X}) \| > \gamma_n \}/n\mathrm{P}\{ \| \mathrm{X} \| > \gamma_n \} > 0 \tag{3.6}$$

in the situation of Theorem 3.1. Obviously, for this lim inf not to be zero, it is necessary that

$$\overline{\lim}_{n\to\infty} n\mathrm{P}\{ \| \mathrm{X} \| > \gamma_n \} < \infty, \tag{3.7}$$

and these are the interesting γ_n's (for instance as in (3.3)). It is easy to prove that (3.7) is also sufficient for (3.6).

3.3. PROPOSITION. — Let B be any Banach space, X a symmetric B-valued r. v. and $\gamma_n \uparrow \infty$. Then (3.6) holds for X and $\{ \gamma_n \}$ if and only if (3.7) holds.

Proof. — Only the sufficiency requires proof. Set $\alpha_n = \mathrm{P}\{ \| \mathrm{X} \| > \gamma_n \}$. Then, by Lévy's inequality,

$$\mathrm{P}\{ \| \mathrm{S}_n(\mathrm{X}) \| > \gamma_n \} \geq \frac{1}{2} \mathrm{P}\{ \max_{k\leq n} \| \mathrm{X}_k \| > \gamma_n \} = \frac{1}{2} [1 - (1 - \alpha_n)^n] = \frac{1}{2} n\alpha_n (1 - \beta_n)^{n-1},$$

where $0 < \beta_n < \alpha_n$. So,

$$\underline{\lim}_{n\to\infty} \mathrm{P}\{ \| \mathrm{S}_n(\mathrm{X}) \| > \gamma_n \}/n\mathrm{P}\{ \| \mathrm{X} \| > \gamma_n \} \geq \frac{1}{2} \underline{\lim}_{n\to\infty} (1 - \beta_n)^n \geq \frac{1}{2} \underline{\lim}_{n\to\infty} (1 - \alpha_n)^n$$

which is strictly positive if (3.7) holds. □

As seen in the derivation of (3.4), the symmetry assumption has only the effect of killing the centering; it can thus be replaced by other assumptions having the same effect, such as $S_n(X)/\gamma_n \to 0$ in probability, or we may directly assume $na_n \to 0$. Here there is a set of minimal conditions for (3.2) and (3.6) to hold in the absence of symmetry (below we discuss some particular cases).

3.4. PROPOSITION. — *a*) In any Banach space, if X and $\{\gamma_n\}$ satisfy

$$\overline{\lim}_{n\to\infty} \max_{k\le n} P\{\|S_n(X)-S_k(X)\| > \gamma_n\} < 1, \tag{3.8}$$

and

$$\overline{\lim}_{n\to\infty} nP\{\|X\| > 4\gamma_n\} < \infty, \tag{3.9}$$

then (3.6) holds. *b*) If B is of type *p*-stable, $p < 2$, and X and $\gamma_n \uparrow \infty$ satisfy: (3.1), and

$$\lim_{n\to\infty} n\,\|EXI_{(\|X\|\le\gamma_n)}\| = 0, \tag{3.10}$$

and

$$(3.11)_{n\to\infty}\ P\{\|X\| > \alpha\gamma_n\}/P\{\|X\| > \gamma_n\} > 0 \quad \text{for some} \quad \alpha > 1,$$

then (3.2) holds for some constant C depending on the same quantities as in Theorem 3.1 (*a*) and on (3.11). If B is of type 2, then this statement is also true with p' replaced by 2 in (3.1).

Proof. — *a*) As in Proposition 3.3, but using the Lévy-Ottaviani inequality, namely that

$$P\{\|S_n(X)\| > \gamma_n\}$$
$$\ge \frac{1}{2}(1-\max_{k\le n} P\{\|S_n(X)-S_k(X)\| > \gamma_n\})P\{\max_{k\le n}\|X_k\| > 4\gamma_n\}$$

(see e. g. Araujo and Giné, loc. cit., p. 110-111). *b*) As in Theorem 3.1(*a*) up to inequality (3.4); (3.2) then follows easily from (3.4), (3.10) and (3.11). □

Remarks. — 1) Note that (3.11) is satisfied if $\|X\|$ is in the domain of attraction of a stable law. (2) If B is of type p'-Rademacher with constant K, then (3.8) and (3.9) (hence (3.6)) hold if

$$\overline{\lim}_{n\to\infty} [nP\{\|X\| > \gamma_n\} + Kn\gamma_n^{-p'}E\|X\|^{p'}I_{(\|X\|\le\gamma_n)}] < 1.$$

This is the case if $\|X\|$ is in the domain of attraction of a *p*-stable law.

$p < p'$, with norming constants a_n, and $\gamma_n/a_n \to \infty$. The same is true for $p = 2$, with $p' = 2$. (3) (3.8) and (3.9) are also satisfied if $S_n(X)/\gamma_n \to 0$ in probability.

4. ALMOST SURE BEHAVIOR

Probably the simplest lemma on almost sure behavior of S_n in $\mathbb{R}$ is the result of Heyde stated as (2) in the introduction. The proof (see e. g. Stout (1974), p. 328-329) extends to type 2 Banach spaces (Kuelbs and Zinn (1981)), part of Theorem 2). We just make here the simple remark that if in fact $\|X\|$ is nearly in the domain of attraction of a p-stable law, and if B is of type p-stable, then the result is also true. This is proved without assumptions on symmetry. We end up this section with a remark on the relation of this result with those in Section 3.

4.1. THEOREM. — Let B be a type p-stable Banach space, $0 < p < 2$, and let X be a B-valued r. v. such that condition (3.1) holds. For any sequence $\gamma = \{\gamma_n\}$, $\gamma_n \uparrow \infty$, let $c_n(X, \gamma) = \Sigma_{k=1}^n EXI_{(\|X\| \le \gamma_k)}$. Then,

$$(4.1) \qquad \overline{\lim}_{n\to\infty} \|S_n(X) - c_n(X, \gamma)\|/\gamma_n = 0 \qquad \text{or} \qquad = +\infty$$

according as

$$(4.2) \qquad \Sigma_{n=1}^{\infty} P\{\|X\| > \gamma_n\} < \infty \qquad \text{or} \qquad = +\infty .$$

Proof. — A simple computation using condition (3.1), as in Stout (*loc. cit.*) or in Kuelbs and Zinn (*loc. cit.*), shows that

$$\Sigma_n P\{\|X\| > \varepsilon\gamma_n\} < \infty \quad \text{for some} \quad \varepsilon > 0 \quad \text{if and only if}$$
$$\Sigma_n P\{\|X\| > \varepsilon\gamma_n\} < \infty \quad \text{for every } \varepsilon > 0 .$$

Hence

$$(4.3) \quad \Sigma_n P\{\|X\| > \gamma_n\} = \infty \Rightarrow \Sigma_n P\{\|X\| > (M+1)\gamma_n\} = \infty \quad \text{for all} \quad M > 0$$
$$\Rightarrow \begin{cases} \Sigma_n P\{\|X - EXI_{(\|X\| \le \gamma_n)}\| > M\gamma_n\} = \infty & \text{for all} \quad M > 0 \\ \overline{\lim}_{n\to\infty} \|X_n\|/\gamma_n = \infty & \text{a. s.} \end{cases}$$
$$\Rightarrow \overline{\lim}_{n\to\infty} \|S_n(X) - c_n(X, \gamma)\|/\gamma_n = \infty \quad \text{a. s.}$$
$$\Rightarrow \overline{\lim}_{n\to\infty} \|S_n(X)\|/\gamma_n = \infty \quad \text{a. s.}$$

To prove the convergence part we use again that B is of type p'-Rademacher for some $p' \in (p, p + \varepsilon)$. Set, as in the cited references, $Y_k = X_k I_{(\|X_k\| \le \gamma_k)}$.

If $\Sigma_n P\{ \| X \| > \gamma_n \} < \infty$, then $\Sigma_n P\{ \| X_k - Y_k \| \neq 0 \} < \infty$ and therefore

(4.4) $$\overline{\lim}_{n\to\infty} \| S_n(X) - T_n(X) \| / \gamma_n = 0 \quad \text{a. s.}$$

where $T_n(X) = \Sigma_{k=1}^n Y_k$. Now,

$$E \| Y_k - EY_k \|^{p'} \leq 2^{p' \wedge 1} E \| Y_k \|^{p'} \leq K 2^{p' \wedge 1} \gamma_k^{p'} P\{ \| X_k \| > \gamma_k \}$$

for some $K < \infty$ by (3.1). So, $\Sigma_k E \| Y_k - EY_k \|^{p'} / \gamma_k^{p'} < \infty$, and therefore, since B is of type p'-Rademacher, the series $\Sigma_k (Y_k - EY_k)/\gamma_k$ converges in $L_{p'}$ and a. s. Hence, Kronecker's lemma gives that

$$\lim_{n\to\infty} \| \Sigma_{i=1}^n (Y_k - EY_k) \| / \gamma_k = 0,$$

and this together with (4.4) gives $\overline{\lim}_{n\to\infty} \| S_n(X) - c_n(X, \gamma) \| / \gamma_n = 0$ a. s. □

Remark (on the centering). — By the usual centering considerations in the general CLT, $c_n(X, \gamma)$ can be replaced by the more standard centering $nEXI_{(\|X\| < \gamma_n)}$ in the convergence case, and by zero if $S_n(X)/\gamma_n \to 0$ in probability. $c_n(X, \gamma)$ can always be replaced by zero in the divergence case (as the above proof shows).

Remark (and question). — If B is any Banach space, $S_n(X)/\gamma_n \to 0$ in probability and if $\{ \gamma_n \}$ is regularly varying, then Theorem 3 in Kuelbs (1979) proves that

(4.5) $$\overline{\lim}_{n\to\infty} \| S_n(X) \| / \gamma_n = 0 \quad \text{or} \quad = +\infty$$

according as

(4.6) $$\Sigma_n P\{ \| S_n(X) \| > \gamma_n \} / n < \infty \quad \text{or} \quad = +\infty .$$

Since the divergence part in Theorem 4.1 holds in any B, it follows that this result together with Theorem 3.1 *a*) implies Theorem 4.1. Theorem 3.1 *b*) suggests the question of whether the conclusion in Theorem 4.1 characterizes type p-stable Banach spaces. If the answer were affirmative, the situation would be a familiar one: conditions on S_n can be replaced by conditions on the summands in limit theorems in a Banach space B if and only if B satisfies certain geometric conditions. A possible approach to this question might be to try to find examples as the one in the proof of Theorem 3.1 *b*), but with the constants τ_n in (3.3) satisfying additional properties.

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