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Demiconvergence of processes indexed by two indices

by

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RÉSUMÉ. — Nous étudions les processus indicés par $\mathbb{N} \times \mathbb{N}$. La convergence presque sûre des martingales (1) (et donc des martingales sous la condition d'indépendance conditionnelle (F4) de Cairoli et Walsh) bornées dans L Log L est connue. Ici nous montrons que les martingales (1) positives arbitraires sont demi-convergentes inférieurement, c'est-à-dire que la limite stochastique coïncide p. s. avec lim inf. Nous prouvons la demi-convergence supérieure des sous-martingales (1) (et donc des sous-martingales sous l'hypothèse (F4)) bornées dans L Log L non nécessairement positives, c'est-à-dire l'égalité entre lim sup et la limite stochastique. Ces résultats sont étendus à des processus indicés par $\mathbb{R}_+^2$ pour lesquels nous prouvons la « demi-régularité », et de façon moins concluante aux processus à valeurs dans les espaces de Banach réticulés.

ABSTRACT. — We consider processes indexed by $\mathbb{N} \times \mathbb{N}$. It is known that 1-martingales (hence, under the Cairoli-Walsh conditional indepen-

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dence assumption (F4), martingales) converge almost surely if bounded in L Log L. Here we prove lower demiconvergence for arbitrary positive 1-martingales. By this we mean that the stochastic limit, which is known to exist, is almost surely equal to lim inf. Not necessarily positive L Log L-bounded 1-submartingales (hence under (F4), submartingales) upper demiconverge, that is, lim sup is equal to the stochastic limit. These results extend to the continuous parameter, where we prove « demiregularity », and, less conclusively, to the Banach lattice valued case.

Martingales, submartingales and amarts, when properly bounded, converge in probability whether indexed by one or two (or k) indices; this follows from the topological fact that in a complete metric space a net with a countable cofinal subsequence converges to a limit whenever every increasing cofinal subsequence converges. Almost sure convergence lies deeper: it is more difficult to prove, holds under stricter assumptions on the process (e. g., 1-martingale instead of martingale or submartingale), and under stricter boundedness assumptions (L Log L boundedness instead of L_1 boundedness). The existence of the limit in probability X of a process (X_t) when possibly $\limsup X_t \neq \liminf X_t$ gives rise to the notion of *demiconvergence*. We say that X_t *demiconverges* if $\limsup X_t = X$ (upper demiconvergence) or $\liminf X_t = X$ (lower demiconvergence). G. A. Edgar and the second author [*5*] proved that L Log L-bounded descending 1-submartingales upper demiconverge. Here the same result is obtained in the ascending case. The proof is somewhat longer because, unlike the descending case, it does not easily reduce to an amart theorem, but is derived from an « approximate amart » theorem (proposition 1.2). Cairoli's pioneering maximal inequalities [*3*] are slightly extended, to show that processes under study are « approximate amarts ». Upper demiconvergence of L Log L-bounded 1-submartingales at once implies the lower demiconvergence of positive (integrable) 1-martingales. There is an easy extension to the continuous case; unlike the discrete case, our « demiregularity » does not imply the much more difficult regularity properties of L Log L-bounded martingales proved in [*2*] and [*9*]. There is also a Banach lattice version of the submartingale result; this gives an extension to two parameters of Heinich's convergence theorem for positive submartingales [*6*].

Finally, we observe that the prefix « 1 » is used here in the new sense of [*9*] and [*5*], i. e., a 1-submartingale [1-martingale] is always assumed to be a submartingale [martingale]. (The integrability is also a part of

the definition.) Under the Cairoli-Walsh conditional independence assumption (F4) [*4*], also called the *commutation* assumption [*10*], every submartingale [martingale] is a 1-submartingale [1-martingale]. However, there exist non-trivial applications of descending 1-submartingales to laws of large numbers when (F4) fails; see [*5*].

1. DISCRETE PARAMETER: GENERALITIES

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space. Let I be a directed set filtering to the right, and let $(\mathscr{F}_t, t \in I)$ be a filtration of sub-sigma-algebras of $\mathscr{F}$. Let T denote the set of simple (i. e., taking on finitely many values) stopping times for $(\mathscr{F}_t)$. Given an adapted process (X_t), let $s \lim X_t$ denote its limit in probability, $e \limsup X_t$ [$e \lim X_t$] its essential upper limit [essential limit]. The filtration $(\mathscr{F}_t)$ satisfies the *Vitali condition* V if for every adapted family of sets (A_t) and for every $\varepsilon > 0$, there exists $\tau \in T$ such that $P(e \limsup A_t \backslash A_\tau) < \varepsilon$. If B is a Banach space, a map $\pi : B \to \mathbb{R}^+$ is called *subadditive* if it is continuous, $\pi(0) = 0$, and $\pi(x+y) \leq \pi(x) + \pi(y)$ for all x, y in B.

We at first prove the following rather technical maximal inequality.

LEMMA 1.1. — Let I be a directed set filtering to the right with a countable cofinal subset, $(\mathscr{F}_t, t \in I)$ a filtration which satisfies the Vitali condition V, B a Banach space, $\pi : B \to \mathbb{R}^+$ a subadditive map. Let $(X_t, t \in I)$ be an adapted B-valued Bochner integrable process, let (Y_n) and Y be B-valued Bochner integrable random variables, and suppose that there exists a cofinal sequence (t_n) of indices such that

$$Y = s \lim Y_n = s \lim E(Y_n \mid \mathscr{F}_{t_n}).$$

Then for every $a > 0$ one has

$$P(e \limsup_t \pi(X_t - Y) \geq a) \leq \frac{a+1}{a} \limsup_n \limsup_{\tau \in T} E[\pi(X_\tau - Y_n)].$$

Proof. — Fix $\varepsilon > 0$ with $0 < \varepsilon < a$. The subadditivity of π implies that for all n

$$\begin{aligned} P(e \limsup_t \pi(X_t - Y) \geq a) &\leq P(e \limsup_t \pi[X_t - E(Y_n \mid \mathscr{F}_{t_n})] \geq a - \varepsilon) \\ &\quad + P(\pi[E(Y_n \mid \mathscr{F}_{t_n}) - Y] \geq \varepsilon). \end{aligned}$$

The last term can be made less than ε for large n. Apply the maximal inequality ([8], Theorem 3.1) to the positive process (Z_t) defined by

$$Z_t = \frac{\pi[X_t - E(Y_n \mid \mathscr{F}_{t_n})]}{1+\pi[X_t - E(Y_n \mid \mathscr{F}_{t_n})]} \quad \text{if} \quad t \geq t_n, \qquad Z_t = 0 \text{ otherwise}.$$

Since the function $\rho(x) = \dfrac{x}{x+1}$ is increasing,

$$P(e \limsup_t \pi[X_t - E(Y_n \mid \mathscr{F}_{t_n})] \geq a-\varepsilon) \leq P\left(e \limsup_t Z_t \geq \frac{a-\varepsilon}{1+a-\varepsilon}\right)$$
$$\leq \frac{1+a-\varepsilon}{a-\varepsilon} \limsup_{\tau \in T} EZ_\tau .$$

The subadditivity of π implies that of $\dfrac{\pi}{1+\pi}$, hence

$$EZ_\tau \leq E\frac{\pi(X_\tau - Y_n)}{1+\pi(X_\tau - Y_n)} + E\frac{\pi[Y_n - E(Y_n \mid \mathscr{F}_{t_n})]}{1+\pi[Y_n - E(Y_n \mid \mathscr{F}_{t_n})]}.$$

The last term can be made less than ε for large n since the sequence $Y_n - E(Y_n \mid \mathscr{F}_{t_n})$ converges to zero in probability. Hence for every fixed $\varepsilon > 0$ there exists N such that $n \geq N$ implies

$$P(e \limsup_t \pi(X_t - Y) \geq a) \leq \varepsilon + \frac{1+a-\varepsilon}{a-\varepsilon} \limsup_{\tau \in T} E[\pi(X_\tau - Y_n)] + \varepsilon \frac{1+a-\varepsilon}{a-\varepsilon}.$$

The announced maximal inequality follows by letting $\varepsilon \to 0$. □

An adapted B-valued Bochner integrable process $(X_t, \mathscr{F}_t, t \in I)$ satisfies the *condition* C_π if there exists a cofinal sequence of indices (t_n) and a B-valued random variable X such that:

(*i*) $$s \lim X_{t_n} = X ;$$

(*ii*) $$\lim_n \limsup_{\tau \in T} E[\pi(X_\tau - X_{t_n})] = 0 .$$

The following is an « approximate amart » theorem.

PROPOSITION 1.2. — Let I be a directed set filtering to the right with a countable cofinal subset, and let $(\mathscr{F}_t, t \in I)$ be a filtration. If $(\mathscr{F}_t)$ satisfies the Vitali condition V, then for every Banach space B, for every subadditive map $\pi : B \to \mathbb{R}_+$, and for every adapted process $(X_t, \mathscr{F}_t, t \in I)$ which satisfies the condition C_π, one has that $\pi(X_t - X)$ converges essentially to zero. Conversely, if V fails, then there exists a subadditive map π and an adapted positive process (X_t) which satisfies C_π, such that $e \limsup \pi(X_t - X) > 0$ on a set of positive measure.

Proof. — Assume that V holds and apply Lemma 1.1 to the process (X_t), the cofinal sequence given in the condition C_π, and the sequence $Y_n = X_{t_n}$. One obtains that $e \lim \pi(X_t - X) = 0$. Conversely, let $B = \mathbb{R}$. To prove the validity of V it suffices to consider an arbitrary fixed adapted process (X_t) such that the net $(X_\tau, \tau \in T)$ converges in probability to some random variable X, and to show that $X = e \lim X_t$ ([7] or [8], Theorem 3.1). Let (t_n) be any cofinal sequence. The assumptions (*i*) and (*ii*) are clearly satisfied by (X_t), (t_n) and $\pi(x) = |x|/(1 + |x|)$. Hence $e \lim \pi(X_t - X) = 0$, so that $e \lim X_t = X$. □

The filtration $(\mathscr{F}_t)$ satisfies the *regularity condition* (R_1) [8] if for every $\varepsilon > 0$, and for every adapted family of sets (A_t), there exists a simple stopping time τ such that $P(A_\tau) \geq P(e \sup A_t) - \varepsilon$. It was proved [8] that a filtration $(\mathscr{F}_1)$ satisfies (R_1) if and only if for every adapted positive process (X_t) and every $\lambda > 0$, one has that

$$P(e \sup X_t \geq \lambda) \leq \frac{1}{\lambda} \sup_{\tau \in T} EX_\tau .$$

We give here a short proof. Suppose that (R_1) holds. Give $\lambda > 0$, fix ε with $0 < \varepsilon < \lambda$, set $A_t = \{X_t > \lambda - \varepsilon\}$, and choose $\tau \in T$ such that $P(A_\tau) \geq P(e \sup A_t) - \varepsilon$. Then

$$P(A_\tau) = P(X_\tau > \lambda - \varepsilon) \leq \frac{1}{\lambda - \varepsilon} EX_\tau \leq \frac{1}{\lambda - \varepsilon} \sup_\tau EX_\tau,$$

hence

$$\begin{aligned} P(e \sup X_t \geq \lambda) &\leq P(A_\tau) + \varepsilon \\ &\leq \frac{1}{\lambda - \varepsilon} \sup_{\tau \in T} EX_\tau + \varepsilon . \end{aligned}$$

The maximal inequality follows on letting $\varepsilon \to 0$. To get the converse implication, apply the maximal inequality to the process $(X_t) = (1_{A_t})$, and $\lambda = 1$.

It is easy to see that a filtration $(\mathscr{G}_t)$ totally ordered by set-inclusion satisfies the regularity condition (R_1). Indeed, $e \sup A_t$ is a union of countably many sets A_t. Given $\varepsilon > 0$, choose finitely many indices, say $t_1, t_2, \ldots, t_n$, such that $P\left(e \sup A_t \setminus \bigcup_{i \leq n} A_{t_i}\right) \leq \varepsilon$. Choose an index t_{n+1} with $\mathscr{G}_{t_i} \subset \mathscr{G}_{t_{n+1}}$ for $1 \leq i \leq n$, and set $A_{t_0} = \emptyset$. Recordering the t_i's, we may and do assume that $\mathscr{G}_{t_1} \subset \mathscr{G}_{t_2} \subset \ldots \subset \mathscr{G}_{t_n}$. For every $i = 1, \ldots, n$, set $\tau = t_i$ on $A_{t_i} \setminus \bigcup_{0 \leq j < i} A_{t_j}$, and set $\tau = t_{n+1}$ on $\left(\bigcup_{j \leq n} A_{t_j}\right)^c$; then τ is the desired stopping time.

In the sequel I will denote a subset of $\mathbb{R}^2$ with the usual order:

$$s = (s_1, s_2) \leq (t_1, t_2) = t \qquad \text{if} \qquad s_1 \leq t_1 \quad \text{and} \quad s_2 \leq t_2 .$$

Let $(\mathscr{F}_t,\ t \in \mathrm{I})$ be an increasing filtration, and for every $t \in \mathrm{I}$ set

$$\mathscr{F}_t^1 = \bigvee_{s \in \mathrm{I}, s_1 \leq t_1} \mathscr{F}_s .$$

$\mathscr{F}_t^2$ is defined analogously. The filtration $(\mathscr{F}_t^1)$ is totally ordered. A stopping time is *simple* if it takes on only finitely many values. Let T^1 denote the set of simple 1-*stopping times*, i. e., simple stopping times for $(\mathscr{F}_t^1)$. Let D_1 and D_2 be countable subsets of $\mathbb{R}$. We give an abstract version of the basic maximal inequality established by Cairoli [*3*] for positive submartingales. This is a technical result which will be needed later.

LEMMA 1.3. — Let $(\mathscr{F}_t, t \in \mathrm{D}_1 \times \mathrm{D}_2)$ be a fixed filtration. Let $\mathscr{C}$ be a class of adapted positive processes. Assume that for every $(\mathrm{X}_t) \in \mathscr{C}$ there exists an increasing sequence $a_n \in \mathrm{D}_1$ such that $a_n \to \infty$ and :

(*i*) For every $\tau = (\tau_1, \tau_2) \in \mathrm{T}^1$, there exists an n such that

$$\tau_1 \leq a_n \qquad \text{and} \qquad \mathrm{EX}_\tau \leq \mathrm{EX}_{a_n, \tau_2} ;$$

(*ii*) There exists a function φ: $\mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_n \sup_{b \in \mathrm{D}_2} \mathrm{E}[\varphi(\mathrm{X}_{a_n, b})] \leq \delta \quad \text{implies} \quad \sup_n \mathrm{E}[\sup_{b \in \mathrm{D}_2} \mathrm{X}_{a_n, b}] \leq \varepsilon .$$

Then for every $(\mathrm{X}_t) \in \mathscr{C}$ with $\sup_n \sup_{b \in \mathrm{D}_2} \mathrm{E}[\varphi(\mathrm{X}_{a_n, b})] \leq \delta$, one has that for every $\lambda > 0$, $\lambda \mathrm{P}(\sup_t \mathrm{X}_t \geq \lambda) \leq \varepsilon$.

Proof. — Since $(\mathscr{F}_t^1, t \in \mathrm{D}_1 \times \mathrm{D}_2)$ is totally ordered, it satisfies the regularity condition (R_1), and one has for each $\lambda > 0$

$$\lambda \mathrm{P}(\sup_t \mathrm{X}_t \geq \lambda) \leq \sup_{\tau \in \mathrm{T}^1} \mathrm{EX}_\tau .$$

Let $\tau = (\tau_1, \tau_2)$ be a simple 1-stopping time, and let $a_n \geq \tau_1$ be such that

$$\mathrm{EX}_\tau \leq \mathrm{EX}_{a_n, \tau_2} \leq \mathrm{E}[\sup_{b \in \mathrm{D}_2} \mathrm{X}_{a_n, b}] .$$

By the choice of δ, $\mathrm{EX}_\tau \leq \varepsilon$. $\square$

We now consider a particular class of two-parameter submartingales.

An adapted integrable process $(X_t, t \in \mathbb{N}^2)$ is a 1-*submartingale* if it is a submartingale, i. e., $E(X_t \mid \mathscr{F}_s) \geq X_s$ if $s \leq t$, and in addition

$$(i) \qquad E(X_t \mid \mathscr{F}_s^1) \geq X_{s_1,t_2}, \qquad \forall s \leq t.$$

Recall that the filtration $(\mathscr{F}_t)$ satisfies the *conditional independence assumption* (F4) [*4*], if for each t, $\mathscr{F}_t^1$ is independent of $\mathscr{F}_t^2$ given $\mathscr{F}_t$. It is easy to see that under (F4) every submartingale is a 1-submartingale. Let $(X_t, t \in \mathbb{N}^2)$ be a uniformly integrable 1-submartingale. Denote by $X_{\infty,\infty} = X$ the L_1-limit of the net (X_t). For every fixed $a \in D_1$ $[a \in D_2]$, let $X_{a,\infty}$ $[X_{\infty,a}]$ denote the almost sure (and L_1) limit of the one-parameter submartingale $(X_{a,b}, \mathscr{F}_{a,b}, b \in D_2)$ $[(X_{b,a}, \mathscr{F}_{b,a}, b \in D_1)]$. It is easy to see that the extended process $(X_t, \mathscr{F}_t, t \in (D_1 \cup \{+\infty\}) \times (D_2 \cup \{+\infty\}))$ is a 1-submartingale. The following lemma is very close to a classical result of Doob (cf. [*11*], p. 70); for a proof of this lemma, see e. g. [*9*], p. 23.

LEMMA 1.4. — Let D be a countable subset of $\mathbb{R}$. For every $\delta > 0$ and for every positive submartingale $(X_t, t \in D)$ which is bounded in L Log L, one has that

$$E[\sup_t X_t] \leq \frac{e}{e-1} [\delta + |\operatorname{Log} \delta| \sup_t EX_t + \sup_t E(X_t \operatorname{Log}^+ X_t)].$$

We now prove a maximal inequality for positive 1-submartingales. It slightly generalizes Cairoli's theorem [*3*]; see also Smythe [*13*].

Let $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing right-continuous function such that $\varphi(t) > 0$ if $t > 0$. Set $\Phi(t) = \int_0^t \varphi(s)ds$. Then Φ is a continuous convex one-to-one Orlicz function. We say that Φ is co-moderate ([*10*], p. 16) if $\bar{p} = \inf_{t>0} [t\varphi(t)/\Phi(t)] > 1$. Let q be such that $1/\bar{p} + 1/q = 1$.

THEOREM 1.5. — Let $(X_t, t \in D_1 \times D_2)$ be a positive 1-submartingale which converges to X in L_1.

(α) Suppose that (X_t) is bounded in L Log L. Then for every $\lambda > 0$ and for every $\delta > 0$, one has

$$P(\sup_t X_t \geq \lambda) \leq \frac{1}{\lambda} \frac{e}{e-1} [\delta + |\operatorname{Log} \delta| EX + E(X \operatorname{Log}^+ X)].$$

(β) Let Φ be a co-moderate Orlicz function, and suppose that $E[\Phi(X)] < \infty$. Then for every $\lambda > 0$,

$$P(\sup_t X_t \geq \lambda) \leq \frac{E[\Phi(X)]}{\Phi\left(\frac{\lambda}{q}\right)}.$$

Proof. — (α) The condition (i) of Lemma 1.3 is clearly satisfied by the class $\mathscr{C}$ of positive 1-submartingales and any cofinal subsequence (a_n) of D_1. Indeed, for every $\tau \in T^1$ and $a \geq \tau_1$, any positive 1-submartingale (X_t) satisfies the inequalities

$$EX_\tau = \sum_t E[1_{\{\tau=t\}}X_t] \leq \sum_t E[1_{\{\tau=t\}}X_{a,t_2}] = EX_{a,\tau_2}.$$

Hence it suffices to check that the property (ii) of Lemma 1.3 holds. Fix $a \in D_1$, and apply Lemma 1.4 to the one-parameter positive submartingale $(X_{a,b}, \mathscr{F}_{a,b}, b \in D_2)$, to obtain that for every $\alpha > 0$

$$E[\sup_{b\in D_2} X_{a,b}] \leq \frac{e}{e-1}[\alpha + |\operatorname{Log}\alpha| \sup_t EX_t + \sup_t E(X_t \operatorname{Log}^+ X_t)].$$

This proves the required property for $\varphi(x) = x + x \operatorname{Log}^+ x$.

(β) Since Φ is increasing, one-to-one, and continuous, given any $\lambda > 0$,

$$P(\sup X_t \geq \lambda) = P\left[\Phi\left(\frac{\sup X_t}{q}\right) \geq \Phi\left(\frac{\lambda}{q}\right)\right]$$
$$= P\left[\sup \Phi\left(\frac{X_t}{q}\right) \geq \Phi\left(\frac{\lambda}{q}\right)\right].$$

Since Φ is convex, Jensen's inequality implies that the process $Y_t = \Phi\left(\frac{X_t}{q}\right)$ is a 1-submartingale. Fix $a \in D_1$, and let $X_{a,\infty}$ denote the L_1-limit as $b \to \infty$ of the uniformly integrable submartingale $(X_{a,b}, b \in D_2)$. Then

$$X_{a,\infty} = \lim_{b\in D_2} X_{a,b} \leq \lim_{b\in D_2} E(X \mid \mathscr{F}_{a,b})$$
$$= E(X \mid \mathscr{F}_a^1).$$

The following maximal inequality follows from a result attributed by P. A. Meyer to C. Dellacherie (cf. [*10*], p. 16); a less precise result is proved in [*11*], p. 217);

$$E[\sup_{b\in D_2} Y_{a,b}] \leq E[\Phi(X_{a,\infty})] \leq E[\Phi(X)].$$

Hence the class of positive 1-submartingales $Y_t = \Phi\left(\frac{X_t}{q}\right)$ satisfies the condition (ii) of Lemma 1.3. In part (α) of the proof we showed that any

positive 1-submartingale satisfies the condition (i) of Lemma 1.3. Hence

$$P\left[\sup_t Y_t \geq \Phi\left(\frac{\lambda}{q}\right)\right] \leq \frac{E[\Phi(X)]}{\Phi\left(\frac{\lambda}{q}\right)},$$

which concludes the proof of the theorem. □

2. DEMICONVERGENCE

We now obtain demiconvergence theorems for L Log L-bounded 1-submartingales, and for positive 1-martingales. Recall that D_1 and D_2 are countable subsets of $\mathbb{R}$.

THEOREM 2.1. — (i) Let $(X_t, \mathscr{F}_t, t \in D_1 \times D_2)$ be an L Log L-bounded 1-submartingale, and let X denote its limit in probability. Then $\limsup X_t = X$ a. s. (ii) Let $(X_t, \mathscr{F}_t, t \in D_1 \times D_2)$ be a positive (integrable) ascending or descending 1-martingale, and let X denote its limit in probability. Then $\liminf X_t = X$ a. s.

Proof. — (i) Let (t_n) be an increasing cofinal subsequence of $D_1 \times D_2$. By Fatou's lemma X belongs to the Zygmund (Orlicz) space L Log L. Set $\alpha(x) = x \operatorname{Log}^+ x$, and suppose that (X_t) is positive. Then by Jensen's inequality the net $\alpha(X_t)$ is dominated by the net $E[\alpha(X) \mid \mathscr{F}_t]$, and hence it is uniformly integrable (cf. [9], p. 22). The filtration $(\mathscr{F}_t^1)$ is totally ordered, and therefore satisfies the Vitali condition V. Set $\pi(x) = x^+$; then $\pi(0) = 0$, π is subadditive and continuous. For every n the positive process $((X_t - X_{t_n})^+, \mathscr{F}_t, t \geq t_n)$ is a 1-submartingale. Hence for every $\tau \in T^1$ with $\tau \geq t_n$, one has that

$$E[(X_\tau - X_{t_n})^+] \leq \frac{e}{e-1}\{\delta + |\operatorname{Log}\delta| E(X - X_{t_n})^+ + E[(X - X_{t_n})^+ \operatorname{Log}^+ [(X - X_{t_n})^+]]\}$$

for every $\delta > 0$. Choose a sequence $\delta_n \to 0$ such that

$$\lim_n |\operatorname{Log}\delta_n| E(X - X_{t_n})^+ = 0,$$

to obtain

$$\lim_n \limsup_{\tau \in T^1} E[(X_\tau - X_{t_n})^+] = 0.$$

Then Proposition 1.2 applied to the filtration $(\mathscr{F}_t^1, t \in D_1 \times D_2)$, the process (X_t) and $\pi(x) = x^+$ yields the almost sure convergence of $(X_t - X)^+$ to zero. This shows that $\limsup X_t \leq X$, and the converse inequality

obviously holds since the sequence X_{t_n} has a subsequence which converges to X almost surely. The theorem is thus proved for positive 1-submartingales, and hence for 1-submartingales which are bounded from below by a constant. Let (X_t) be a (not necessarily positive) 1-submartingale, which is bounded in L Log L and converges to X in probability. Fix $M \in \mathbb{R}$; the process $(X_t \vee M, \mathscr{F}_t, t \in D_1 \times D_2)$ is also a 1-submartingale, and $\limsup X_t \vee M = X \vee M$ by the first part of the proof. Since

$$\limsup_t X_t \vee (-n) = X \vee (-n) \qquad \text{for every} \quad n \geq 0,$$

and since $\limsup X_t > -\infty$ a. s. by Fatou's lemma, we have that $\limsup X_t = X$ a. s. (*ii*) Let (X_t) be a positive (integrable) 1-martingale. X_t converges to a random variable X in probability (since X_t is automatically L_1-bounded). Fix $a > 0$; the process $(X_t \wedge a; \mathscr{F}_t, t \in D_1 \times D_2)$ is an L_∞-bounded 1-supermartingale. Hence the part (*i*), reformulated for supermartingales by changing the signs, implies that $\liminf_t X_t \wedge a = X \wedge a$ a. s. Fatou's lemma implies that $\liminf X_t < \infty$ a. s., so that $\liminf X_t = X$ a. s. The descending case follows similarly from the descending 1-submartingale result proved in [*5*]. □

Upper demiconvergence is the best result one can obtain for L Log L-bounded 1-submartingales. Indeed, assume that the conditional independence condition (F4) holds. Then it is known that there exists a positive L_1-bounded martingale (X_t) which does not converge almost surely (cf. Cairoli [*3*]). It follows, as is known and was pointed out to us by R. Cairoli, that the positive L_∞-bounded submartingale (e^{-X_t}) does not converge a. s. (Another counter-example is $(1/(1 + X_t))$).

It can be also shown that there is no demiconvergence for not necessarily positive uniformly integrable martingales. Indeed, let $A \in \mathscr{F}_{M,N}$ for some $(M, N) \in \mathbb{N}^2$ be such that $0 < P(A) < 1$. Consider the probability spaces $(A, \mathscr{F}_{|A}, Q)[(A^c, \mathscr{F}_{|A^c}, Q')]$ where $Q(.) = P(. \mid A)[Q'(.) = P(. \mid A^c)]$, with the increasing filtrations $(\mathscr{F}_{t|A})[(\mathscr{F}_{t|A^c})]$. By Cairoli's theorem [*3*], there exists a positive uniformly integrable martingale $X_t = E(X \mid \mathscr{F}_t)$ on $A[(X'_t = E(X' \mid \mathscr{F}_t)$ on $A^c]$ not bounded in L Log L and such that $X_t[X'_t]$ does not converge almost surely. Hence by Theorem 2.1, $\limsup X_t \neq X$ and $\limsup X'_t \neq X'$. The process $Y_t = E[1_A X - 1_{A^c} X' \mid \mathscr{F}_t]$ is a uniformly integrable martingale. For $t \geq (M, N)$, $Y_t = X_t 1_A - X'_t 1_{A^c}$, hence neither $\limsup Y_t = s \lim Y_t$, nor $\liminf Y_t = s \lim Y_t$.

We now give an application of Theorem 2.1 to derivation theory. Let $(\mathscr{F}_{ij}, i \geq 0, j \geq 0)$ be an increasing family of σ-algebras with the conditional independence property (F4). Let Q be a positive finite measure on $(\Omega, \mathscr{F})$, not necessarily absolutely continuous with respect to P. For

every (i, j) ,let $Q = f_{ij} . P + R_{ij}$ be the Lebesgue decomposition of the restriction of Q to $\mathscr{F}_{ij}$ with respect to P, R_{ij} and P being singular. Similarly, let $Q = f . P + R$, with R and P singular.

THEOREM 2.2. — (*i*) Suppose that Q is absolutely continuous with respect to P. Then (f_{ij}) is a 1-martingale, and $\liminf f_{ij} = f$ (P a. s.).

(*ii*) Suppose that Q is not absolutely continuous with respect to P, but that (f_{ij}) is bounded in L Log L. Then (f_{ij}) is a 1-supermartingale, and $\liminf f_{ij} = f$ (P a. s.).

Proof. — Under the assumption (*i*) [(*ii*)] (f_{ij}) is known to be a martingale [supermartingale] which converges to f in P-probability. Since the condition (F4) holds, (f_{ij}) is a 1-martingale [1-supermartingale], and Theorem 2.1 applies. □

3. BANACH LATTICES

We now prove a Banach lattice version of Theorem 2.1.

THEOREM 3.1. — Let B be a Banach lattice with the Radon-Nikodým property. Let $(X_t, \mathscr{F}_t, t \in D_1 \times D_2)$ be a positive B-valued 1-submartingale which is bounded in L Log L. Then there exists one B-valued random variable X such that $X = s \lim X_t$, and

$$\lim \| (X_t - X)^+ \| + \liminf \| (X_t - X)^- \| = 0 \quad \text{a. s.}$$

Proof. — The Radon-Nikodým property implies that the Banach lattice B is order-continuous. For every $A \in \cup \mathscr{F}_t$, the countable net $E(1_A X_t)$ is increasing and norm nounded, hence converges strongly to a limit, say $\lambda(A)$. The set-function λ is strongly bounded and of bounded variation, therefore it can be extended to a measure λ on $\mathscr{F} = \sigma(\cup \mathscr{F}_t)$, and λ is absolutely continuous with respect to P. The Radon-Nikodým property implies the existence of a B-valued random variable X such that $\lambda(A) = E(1_A X)$ for $A \in \mathscr{F}$. Given any increasing cofinal sequence of indices (t_n), Heinich's theorem [*6*] implies that the one-parameter positive submartingale $(X_{t_n}, \mathscr{F}_{t_n}, n \geq 0)$ converges a. s., hence in probability, necessarily to X. This in turn implies that $X = s \lim X_t$, and $\liminf \| (X_t - X)^- \| = 0$ a. s.. To prove that $\lim \| (X_t - X)^+ \| = 0$, we fix an increasing cofinal sequence of indices (t_n) and prove that (X_t) satisfies the condition C_π of Proposition 1.2 for this sequence (t_n), the function π: $B \to \mathbb{R}_+$ defined by $\pi(x) = \| x^+ \|$, and the totally ordered filtration $(\mathscr{F}_t^1)$. Fix $n > 0$, let $\tau \geq t_n$ be a simple 1-stopping time, and set

$$Y_t = \| (X_t - X_{t_n})^+ \| \quad \text{if} \quad t \geq t_n,$$
$$Y_t = 0 \quad \text{otherwise}.$$

Then (Y_t) is a real positive 1-submartingale which is bounded in L Log L. The net $\| X_t \| \operatorname{Log}^+ \| X_t \|$ is uniformly integrable by Jensen's inequality. The argument of Theorem 2.1 shows that $\lim_n \limsup_{\tau \in T^1} E[\pi(X_t - X_{t_n})] = 0$, and the use of Proposition 1.2 concludes the proof. □

Since one-parameter martingales are known not to converge for the order, on cannot obtain in Theorem 3.1 $X = \limsup X_t$.

4. CONTINUOUS PARAMETER

We now discuss the continuous parameter case. We say that a process $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ is *right upper* [*lower*] *demicontinuous* if there exists a null set N such that for all $\omega \notin N$, the trajectories $t \to X_t(\omega)$ are right upper [lower] demicontinuous, i. e., $\limsup$ [$\liminf$] $\{ X_t(\omega) : t \to s, t \geq s \} = X_s(\omega)$. The following theorem is a continuous analog of Theorem 2.1.

THEOREM 4.1. — Suppose that the filtrations $(\mathscr{F}_t)$ and $(\mathscr{F}_t^1)$ are right-continuous. (*i*) Let $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ be an L Log L-bounded 1-submartingale, then (X_t) has a right upper demicontinuous modification. (*ii*) Let $(X_t, \mathscr{F}_t, t \in \mathbb{R}_+^2)$ be a positive 1-martingale; then (X_t) has a right lower demicontinuous modification.

Proof. — (*i*) First considering a separable modification of (X_t), we may and do assume that (X_t) is separable with separant set $D = D_1 \times D_2$ containing Q_+^2. If $s = (s_1, s_2)$, $t = (t_1, t_2)$ we write $s \gg t$ if $s_1 > t_1$ and $s_2 > t_2$. For every index t, set

$$Y_t(\omega) = \limsup \{ X_s(\omega) : s \to t, s \in D, s \gg t \}.$$

We prove that (Y_t) is a modification of (X_t), and that (Y_t) is right upper demicontinuous. Fix t and let $t(n) = (t(n)_1, t(n)_2) \in D$ be such that $t(n) \to t$, $t(n) \gg t$, and $t(n) \searrow$. The one-parameter descending submartingale $X_{t(n)}$ converges to X_t in L_1. Hence $Y_t \geq X_t$ a. s. Conversely, set $\varphi(x) = x \operatorname{Log}^+ x$. Lemmas 1.3 and 1.4 show that for every $n \geq 0$, $\delta > 0$, and $\lambda > 0$, one has

$$\begin{aligned} P(\sup_{t \leq s \leq t(n), s \in D} (X_s - X_{t(n)})^+ > \lambda) & \\ \leq \frac{1}{\lambda} E[\sup_{s_1 = t(n)_1, t_2 \leq s_2 \leq t(n)_2} (X_s - X_{t(n)})^+] & \\ \leq \frac{1}{\lambda} \cdot \frac{e}{e-1} [\delta + | \operatorname{Log} \delta | E(X_{t(n)} - X_t)^+ + E(\varphi(X_{t(n)} - X_t)^+)]. & \end{aligned}$$

Hence letting $n \to \infty$, one obtains that $X_t \geq Y_t$ a. s., and therefore (Y_t)

is a modification of (X_t). The definition of (Y_t) clearly implies that for every $s \in \mathbb{R}_+^2$ and every $n > 0$ one has that

$$\sup \{ Y_t(\omega) : t \in \mathbb{R}_+^2, s \leq t \ll s + (1/n, 1/n) \} \leq \sup \{ X_t(\omega) : t \in D, s \ll t \ll s + (1/n, 1/n) \}.$$

Hence $\limsup \{ Y_t(\omega) : t \to s, t \geq s \} \leq Y_s(\omega)$. There exists a null set N outside of which the two processes $(X_t, t \in D)$ and $(Y_t, t \in D)$ agree. Given any $s \in \mathbb{R}_+^2$ and any ω, let (t_n) be a sequence in D such that $t(n) \to s$, $t(n) \gg s$, and $Y_s(\omega) = \lim X_{t(n)}(\omega)$. If $\omega \notin N$, one has that $Y_s(\omega) = \lim Y_{t(n)}(\omega)$, and hence $\limsup \{ Y_t(\omega) : t \to s, t \geq s \} \geq Y_s(\omega)$. This completes the proof.

(*ii*) Let (X_t) be a positive 1-martingale (integrable), and define (U_t) by $U_t = e^{-X_t}$. Then (U_t) is a positive L_∞-bounded 1-submartingale, which has a right upper demicontinuous modification, say (V_t). Setting $\operatorname{Log} 0 = -\infty$, define (Y_t) by $Y_t = -\operatorname{Log} V_t$; then (Y_t) is a modification of (X_t), and (Y_t) is clearly right lower demicontinuous. □

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