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The class of Banach spaces, which do not have c_0 as a spreading model, is not L^2-hereditary

by

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1. INTRODUCTION

In [*4*] the problem was raised wether the fact, that a Banach space E does not have c_0 as a spreading model, implies that $L^2([0,1]; E)$ has the same property. It was conjectured that the answer is no, as the property of not having c_0 as a spreading model is somewhat dual to the Banach-Saks property (see [*2*]) and for this latter property J. Bourgain has constructed a counterexample ([*3*]).

The present author has constructed independently of J. Bourgain another space E with the Banach-Saks property and $L^2(E)$ failing it ([*6*]) and it turns out that the dual E′ gives a counterexample to the problem raised in the title.

2. THE EXAMPLE

Let $\gamma = \{ n_1, n_2, \ldots, n_k \}$ an increasing finite sequence of natural numbers. Write $n_i = 2^{u_i} + v_i$ where this expression is unique, if we require that $\leqq v_i < 2^{u_i}$. As in [*6*] we associate to every n_i the real number $t(n_i) = v_i/2^{u_i} \in [0, 1[$ and call γ *admissible* if

(1) $k \leqq n_1$.

(2) For every $0 \leqq j < 2^{u_1+1}$ there is only one i such that $t(n_i) \in [j/2^{u_1+1}, (j+1)/2^{u_1+1}[$.

For an admissible $\gamma = (n_1, \ldots, n_k)$ and $x \in \mathbb{R}^{(\mathbb{N})}$, the space of finite sequences, we define

$$\| x \|_\gamma = \sum_{i=1}^{k} | x_{n_i} | .$$

For our purposes it will this time be convenient, not to use interpolation but to follow Baernstein's original definition ([*1*]): For $x \in \mathbb{R}^{(\mathbb{N})}$ define

$$\| x \|_{\mathrm{E}} = \sup \left\{ \left(\sum_{l=1}^{\infty} \| x \|_{\gamma_l}^2 \right)^{1/2} \right\}$$

where the sup is taken over all increasing sequences $\{ \gamma_l \}_{l=1}^{\infty}$ of admissible sets (i. e. the last member of γ_l is smaller than the first member of γ_{l+1}).

Let $(\mathrm{E}, \| \ \|_{\mathrm{E}})$ be the completion of $\mathbb{R}^{(\mathbb{N})}$ with respect to this norm. In an analogous way as in [*6*] we shall show that E has the Banach-Saks property, i. e. that it does not have a spreading model isomorphic to l^1; we shall also show that E′ does not have a spreading model isomorphic to c_0. In fact we shall prove a much stronger result.

PROPOSITION 1. — *a*) Every spreading model based on a normalized weak null sequence $(x_n)_{n=1}^{\infty}$ of E is isomorphic to l^2.

b) Every spreading model based on a normalized weak null sequence $(y_n)_{n=1}^{\infty}$ of E′ is isomorphic to l^2.

Proof. — *a*) Let $(x_n)_{n=1}^{\infty}$ be a normalized weak null sequence in E. As $(x_n)_{n=1}^{\infty}$ converges to zero coordinatewise, we may assume (by a standard perturbation argument) that the x_n's are supported by disjoint blocks, i. e. there is an increasing sequence $(r(n))_{n=1}^{\infty}$ of natural numbers, such that with $r(0) = 0$

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i .$$

Then for every sequence $\alpha_1, \ldots, \alpha_k$ of scalars and $n_1 < \ldots < n_k$, the estimate

$$\left\| \sum_{i=1}^{k} \alpha_i x_{n_i} \right\|_{\mathrm{E}} \geqq \left(\sum_{i=1}^{k} | \alpha_i |^2 \right)^{1/2} \tag{1}$$

holds trivially in view of the definition of $\| \, . \, \|_{\mathrm{E}}$.

For the converse let $1 > \varepsilon > 0$ and choose inductively an increasing sequence $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ and infinite subsets M_k of $\mathbb{N}$: Let $M_0 = \mathbb{N}$ and $n_1 = 1$ and suppose M_{k-1} and n_k are defined. Let p_k be such that $2^{p_k - 1} \leqq r(n_k) < 2^{p_k}$ and consider the partition of $[0, 1[$ into the intervals $[j/2^{p_k}, (j+1)/2^{p_k}[$, $j = 0, \ldots, 2^{p_k - 1}$. For $n \geqq n_k$ define

$$\mu_j^{(n)} = \max \{ \mid \lambda_i^{(n)} \mid : t(i) \in [j/2^{p_k}, (j+1)/2^{p_k}[\}.$$

Note that, for every n,

$$\sum_{j=0}^{2^{p_k}-1} \mu_j^{(n)} \leqq 1$$

in view of the definition of the norm of E and the fact that $\| x_n \|_E = 1$. Find an infinite subset $\overline{M}_k$ of $M_{k-1} \cap [n_k + 1, n_k + 2, \ldots, \infty[$ such that for every $j = 0, \ldots, 2^{p_k} - 1$ the sequence $(\mu_j^{(n)})_{n \in \overline{M}_k}$ converges, to μ_j say. Clearly

$$\sum_{j=0}^{2^{p_k}-1} \mu_j \leqq 1 .$$

Finally let M_k be the subset of $\overline{M}_k$ consisting of those n for which for every $j = 0, \ldots, 2^{p_k} - 1$

$$\mu_j^{(n)} \leqq \mu_j + 2^{-p_k} . \varepsilon/3$$

and let n_{k+1} be the first element of M_k. This completes the induction.

Note that for an admissible $\gamma = (m_1, \ldots, m_q)$ and $k \in \mathbb{N}$ such that $\inf(\gamma) = m_1 \leqq r(n_k)$ and for every choice of scalars $\alpha_{k+1}, \ldots, \alpha_{k+l}$

$$\left\| \sum_{i=1}^{l} \alpha_{k+i} x_{n_{k+i}} \right\|_{\gamma} \leqq (1 + \varepsilon/3) \sup_{1 \leq i \leq l} \{ \mid \alpha_{k+i} \mid \} \leqq \sqrt{1+\varepsilon} \left(\sum_{i=1}^{l} \mid \alpha_{k+i} \mid^2 \right)^{1/2} \tag{2}$$

Indeed, as $m_1 \leqq r(n_k) < 2^{p_k}$, we see that γ may contain for every $j = 0, \ldots, 2^{p_k - 1}$ at most one index m_r $(1 \leqq r \leqq q)$ with $t(m_r) \in [j/2^{p_k}, (j+1)/2^{p_k}[$; by construction the m_r'th entry of each $x_{n_{k+i}}$ $(1 \leqq i \leqq l)$ is bounded in absolut value by $\mu_j + 2^{-p_k} . \varepsilon/3$. As the x_{n_i} are disjointly supported we get

$$\left| \sum_{i=1}^{l} \alpha_{k+1} x_{n_{k+i}}(m_r) \right| \leqq \sup_{1 \leq i \leq l} \{ \mid \alpha_{k+i} \mid \} (\mu_j + 2^{-p_k} . \varepsilon/3)$$

Summing over j and recalling the definition of $\| . \|_{\gamma}$ we get the first inequality of (2), while the second is trivial.

We now shall pass to the general case. Fix a sequence $\alpha_1, \dots, \alpha_k$ of scalars. We shall show

$$\left\| \sum_{i=1}^{k} \alpha_i x_{n_i} \right\|_E \leqq \sqrt{6 + 3\varepsilon}. \left(\sum_{i=1}^{k} | \alpha_i |^2 \right)^{1/2} \tag{3}$$

which (in view of (1) and the arbitrarness of $\varepsilon > 0$) will readily prove (*a*). So fix an increasing sequence $\gamma_1 < \gamma_2 < \dots < \gamma_l$ of admissible sets. For brevity we write

$$x = \sum_{i=1}^{k} \alpha_i x_{n_i} .$$

For $i = 1, \dots, k$ let $J(i)$ be the set of $j \in \{ 1, \dots, l \}$ such that the last element of γ_j lies in $]r(n_{i-1}), r(n_i)]$. If $J(i)$ is not empty denote $j(i)$ the first element of $J(i)$ and let $s(j(i))$ be the element $s \in \{ 1, \dots, k \}$, such that the first element of $\gamma_{j(i)}$ lies $]r(n_{s-1}, r(n_s)]$. Note that for the $j \in J(i), j > j(i)$ the first and the last element of γ_j lie in $]r(n_{i-1}), r(n_i)]$, while for $j(i)$ in general only the last element lies in $]r(n_{i-1}), r(n_i)]$. So we may estimate

$$\left(\sum_{j=1}^{l} \| x \|_{\gamma_j}^2 \right)^{1/2} = \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^{k} \sum_{j \in J(i)} \| x \|_{\gamma_j}^2 \right)^{1/2}$$

$$= \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^{k} \left(\| x \|_{\gamma_{j(i)}}^2 + \sum_{\substack{j \in J(i) \\ j > j(i)}} \| x \|_{\gamma_j}^2 \right) \right)^{1/2}$$

$$= \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^{k} \left(\left\| \sum_{s=s(j(i))}^{i} \alpha_s x_{n_s} \right\|_{\gamma_{j(i)}}^2 + \sum_{\substack{j \in J(i) \\ j > j(i)}} \| \alpha_i x_{n_i} \|_{\gamma_{j(i)}}^2 \right) \right)^{1/2}$$

$$\leqq \left(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^{k} \left(\left(\| \alpha_{s(j(i))} x_{n_{s(j(i))}} \|_{\gamma_{j(i)}} + \left\| \sum_{s=s(j(i))+1}^{i-1} \alpha_s x_{n_s} \right\|_{\gamma_{j(i)}} + \| \alpha_i x_{n_i} \|_{\gamma_{j(i)}} \right)^2 \right.\right.$$

$$\left.\left. + \sum_{\substack{j \in J(i) \\ j > j(i)}} \| \alpha_i x_{n_i} \|_{\gamma_{j(i)}}^2 \right) \right)^{1/2} .$$

Using (2) and the fact that $(a^{1/2} + b^{1/2} + c^{1/2})^2 \leqslant 3(|a|+|b|+|c|)$ we get

$$\leqq \Bigg(\sum_{\substack{i=1 \\ J(i) \neq \emptyset}}^{k} \Bigg(\Bigg(\Big(\| \alpha_{s(j(i))} x_{n_{s(j(i))}} \|^2_{\gamma_{j(i)}} \Big)^{\frac{1}{2}} + \sqrt{1+\varepsilon} \Big(\sum_{s=s(j(i))+1}^{i-1} |\alpha_s|^2 \Big)^{\frac{1}{2}} + \big(\| \alpha_i x_{n_i} \|^2_{\gamma_{j(i)}} \big)^{\frac{1}{2}} \Bigg)^2 + \sum_{\substack{j \in J(i) \\ j > j(i)}} \| \alpha_i x_{n_i} \|^2_{\gamma_{j(i)}} \Bigg) \Bigg)^{\frac{1}{2}}$$

$$\leqq \Bigg(\sum_{\substack{i=1 \\ J(i) \neq O}}^{k} 3 \Big(|\alpha_{s(j(i))}|^2 + (1+\varepsilon) \sum_{s=s(j(i))+1}^{i-1} |\alpha_s|^2 + \sum_{j \in J(i)} \| \alpha_i x_{n_i} \|^2_{\gamma_{j(i)}} \Big) \Bigg)^{\frac{1}{2}}$$

$$\leqq \Big(3(1+1+\varepsilon) \sum_{i=1}^{k} |\alpha_i|^2 \Big)^{\frac{1}{2}}$$

$$= \sqrt{6+3\varepsilon}.\Big(\sum_{i=1}^{k} |\alpha_i|^2 \Big)^{\frac{1}{2}}.$$

Hence we have proved (*3*) and thus part (*a*) of proposition 1.

Proof of (*b*). — It is easily seen using (*a*) that the unit vector basis $(e_i)_{i=1}^{\infty}$ of E is shrinking and boundedly complete (see [*1*] or [*6*]), hence the dual unit vectors $(e_i')_{i=1}^{\infty}$ form a basis of E′. So let $(y_n)_{n=1}^{\infty}$ be a normalized sequence tending weakly (and therefore coordinatewise) to zero. Similarly as in (*a*) we may suppose that there is an increasing sequence $(r(n))_{n=1}^{\infty}$ such that

$$y_n = \sum_{i=r(n-1)+1}^{r(n)} \rho_i^{(n)} e_i' .$$

Now choose a sequence $(x_n)_{n=1}^{\infty}$ in E, $\| x_n \| = \langle x_n, y_n \rangle = 1$, which clearly implies that x_n is of the form

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i .$$

As in the prove of (*a*) find a subsequence $(n_k)_{k=1}^{\infty}$ such that $(x_{n_k})_{k=1}^{\infty}$ spans a space $\sqrt{6+\varepsilon}$ isomorphic to l^2.

Now fix a sequence $\beta_1, \ldots, \beta_k$ of scalars and find a sequence $\alpha_1, \ldots, \alpha_k$ such that

$$\sum_{i=1}^{k} |\alpha_i|^2 = 1 \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i\beta_i = \left(\sum_{i=1}^{k} |\beta_i|^2\right)^{\frac{1}{2}}.$$

Denote

$$x = \sum_{i=1}^{k} \alpha_i x_{n_i}$$

$$y = \sum_{i=1}^{k} \beta_i y_{n_i}.$$

By (a) we know that

$$\| x \|_{\mathrm{E}} \leqq \sqrt{6 + \varepsilon}.$$

Hence

$$\begin{aligned} \| y \|_{\mathrm{E}'} &= \sup \{ |\langle \xi, y \rangle| : \xi \in \mathrm{E}, \| \xi \| \leqq 1 \} \\ &\geqq (6 + \varepsilon)^{-1/2} |\langle x, y \rangle| \\ &= (6 + \varepsilon)^{-1/2} . \sum_{i=1}^{k} \alpha_i\beta_i \langle x_{n_k}, y_{n_k} \rangle \\ &= (6 + \varepsilon)^{-1/2} \left(\sum_{i=1}^{k} |\beta_i|^2\right)^{\frac{1}{2}}. \end{aligned}$$

On the other hand the reverse inequality

$$\| y \|_{\mathrm{E}'} \leqq \left(\sum_{i=1}^{k} |\beta_i|^2\right)^{\frac{1}{2}}$$

is again easily checked directly from the definition of $\| . \|_{\mathrm{E}}$. This proves (b) and therefore proposition 1. □

Remark. — Consider the sequence of unit-vectors $(e_{2^n-1})_{n=1}^{\infty}$ in E (resp. $(e'_{2^n-1})_{n=1}^{\infty}$ in E').

It may be checked that every spreading model based on $(e_{2^n-1})_{n=1}^{\infty}$ (resp. $(e'_{2^n-1})_{n=1}^{\infty}$) is isometric to a countable l^2-sum of 2-dimensional l^1's, hence in this case the Banach-Mazur distance equals precisely $\sqrt{2}$.

To show that $\mathrm{L}^2(\mathrm{E}')$ does have c_0 as spreading model we need a trivial probabilistic lemma, whose proof is left to the reader.

LEMMA. — Let $k \in \mathbb{N}$ and $\varepsilon > 0$; there is $\mathrm{N}(k, \varepsilon)$ such that for $\mathrm{M} > \mathrm{N}(k, \varepsilon)$

and for independent randomvariables $X_1, \ldots, X_k$ taking their values in $\{1, \ldots, M\}$ in a uniformly distributed way, we have

$$P\left\{\begin{array}{l}\omega : \text{there is } \ 1 \leqq i < j \leqq k \ \text{ with} \\ \quad X_i(\omega) = X_j(\omega)\end{array}\right\} < \varepsilon$$

PROPOSITION 2. — $L^2_{(0,1)}(E')$has c_0 isometrically as spreading model.

Proof. — Similarly as in [6] we let $\{\vec{f}_u\}_{u=1}^{\infty}$ be an independent sequence in $L^2(E')$ such that $\vec{f}_u$ takes the value e_{2^u+v} with probability 2^{-u} (for $v = 0, \ldots, 2^u - 1$). This times the e_{2^u+v} are unit-vectors in E'.

Clearly $\| \vec{f}_u \|_{L^2(E')} = 1$ and for every sequence $u_1 < u_2 < \ldots < u_k$ and $\varepsilon_i = \pm 1$

$$\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')} \geqq 1$$

Hence the following claim will prove the proposition.

CLAIM. — For every $k \in \mathbb{N}$

$$\lim_{u \to \infty} \sup \left\{ \left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\| : \begin{array}{l} u \leqq u_1 < \ldots < u_k \\ \varepsilon_i = \pm 1 \end{array} \right\} = 1$$

To prove the claim fix k and $\varepsilon > 0$ and let u be such that $2^u > \max(k, \mathbb{N}(k, \varepsilon))$, where the $N(k, \varepsilon)$ is defined in the preceeding lemma. Now fix $u \leqq u_1 < u_2 < \ldots < u_k$ and a sequence of signs $\varepsilon_1, \ldots, \varepsilon_k$.

To apply the above lemma let $X_1, \ldots, X_k$ be the random variables with values in $\{1, \ldots, 2^{u_1+1}\}$ defined by

$$X_i(\omega) = m \qquad \text{if} \qquad \vec{f}_{u_i}(\omega) = e_{2^{u_i}+v}$$

and

$$t(2^{u_i} + v) = v/2^{u_i} \in [(m-1)/2^{u_1+1} \ ; \qquad m/2^{u_1+1}[$$

It follows form the above lemma and the definition of admissible sets γ that there is a subset $A \subseteq [0, 1[$ of measure greater than $1 - \varepsilon$ such that for $\omega \in A$ the set $\gamma_\omega = \{n_1, \ldots, n_k\}$ corresponding to the indices of the unit vectors $\{\vec{f}_{u_1}(\omega), \ldots, \vec{f}_{u_k}(\omega)\}$ is admissible. Hence for $\omega \in A$ we have

$$\begin{aligned}\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega) \right\|_{E'} &= \sup \left\{ \left\langle \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \| x \|_E \leqq 1 \right\} \\ &\leqq \sup \left\{ \left\langle \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i}(\omega), x \right\rangle : \| x \|_{\gamma_\omega} \leqq 1 \right\} \\ &= 1.\end{aligned}$$

Integrating we obtain

$$\left\| \sum_{i=1}^{k} \varepsilon_i \vec{f}_{u_i} \right\|_{L^2(E')}^2 \leqq \int_A \left\| \sum_{i=1}^{k} \vec{f}_{u_i}(\omega) \right\|_{E'}^2 d\omega + \int_{[0,1]\setminus A} \left(\sum_{i=1}^{k} \| \vec{f}_{u_i} \|_{E'} \right)^2 d\omega$$
$$\leqq 1 + k^2\varepsilon .$$

This proves the claim and therefore proposition 2. □

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