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Bochner property in Banach spaces

by

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ABSTRACT. — This paper is a study of the relation between the geometry of Banach spaces and the topological solutions to Bochner-type theorems. We obtain two theorems which extend some results of Sazanov, Gross, Mustari and Kuelbs.

(1) A real separable Banach space E with b. a. p. is of stable type p and embeddable in $L^p[0, 1]$ (and hence in $L^0(\Omega, P)$) iff the continuity of a positive definite function Φ (with $\Phi(0) = 1$) in a certain topology defined on the topological dual E′ is a necessary and sufficient condition for Φ to be a characteristic functional of a Borel probability measure on E. (Here $0 < p \leq 2$).

(2) A real separable Banach space E is of cotype 2 iff the equicontinuity of a family $\{ \hat{\mu}_\alpha, \alpha \in I \}$ of characteristic functionals in the topology generated by the Gaussian ch. f's is a sufficient condition for the corresponding family $\{ \mu_\alpha, \alpha \in I \}$ of Borel probability measures on E to be tight.

0. INTRODUCTION

In this paper we introduce the following property of real separable Banach spaces.

BOCHNER PROPERTY I. — A real separable Banach space E is said to have Bochner Property I if there exists a topology τ on its topological dual E′ such that a positive definite function Φ with $\Phi(0) = 1$, is continuous in τ iff it is a characteristic functional (cf. f.) of a probability measure on the borel sets B(E) of E.

From the results of Sazanov [*28*], Gross [*10*] and Mustari [*23*], it follows that a real separable Banach space is isomorphic to a Hilbert space iff it has Bochner Property I with respect to topology τ generated by the ch. F. 's of Gaussian measures. Our purpose here is to study embeddable Banach spaces with bounded approximation property (b. a. p.) which have Bochner Property I with respect to topology τ_p generated by forms associated with ch. f. 's of stable measures of order p. A complete characterization of such spaces can be given. Since a real separable Hilbert space is embeddable and has b. a. p. this work constitutes an extension of the result mentioned above.

In ([*22*], Theorem 1 *c*)) Mustari has shown that embeddability in $L^0(\Omega, P)$ is a sufficient condition for real separable spaces with b. a. p. to have Bochner Property I. We observe that the main tool of his proof is an inequality first proved by Lévy for which the above two hypotheses of embeddability and bounded approximation property seem to be tailor-made. In order to bring this point across, using techniques of [*13*] we give a simpler proof of Mustari's result in Section 2. This proof together with results of [*19*] enable us to obtain explicit topological solutions for Banach spaces with a certain geometric structure.

In section 3, using a result of [*15*] it is shown that Banach spaces having Bochner Property I with respect to certain explicit topologies are necessarily embeddable in $L^0(\Omega, P)$, giving a partial converse of Mustari's result ([1]).

In the final section we consider a related problem. We denote by $\tilde{\tau}_2$ the topology on E′ generated by ch. f. of Gaussian probability measures. Sazanov and Prohorov's [*28*] [*25*] result shows that for real separable Hilbert spaces, the equicontinuity in $\tilde{\tau}_2$ of ch. f. 's of a family of probability measures implies the tightness of the family. It is shown that the validity of this result, in fact, characterizes the cotype 2 spaces, a class larger than the class of Hilbert spaces.

1. PRELIMINARIES AND NOTATION

A « Banach Space » E will mean a real separable complete normed linear space with norm $\| \ \|_E$. We will denote its topological dual by E′.

([1]) See addendum.

A Banach space E is said to be of Rademacher type p if for every sequence $\{ x_i \}_{i=1}^{\infty} \subset E$ with $\sum_{i=1}^{\infty} \| x_i \|^p < \infty$ we have $\Sigma x_i \mathscr{E} i$ converges a. e., where $\{ \mathscr{E} i \}$ is a sequence of i. i. d. Bernoulli random variables

$$\left(P(\mathscr{E} i = -1) = P(\mathscr{E} i = 1) = \frac{1}{2} \right).$$

A Banach space E is said to be of cotype 2 if for every sequence $\{ x_i \} \subset E$ satisfying $\Sigma x_i \mathscr{E} i$ converges a. e. we have $\Sigma \| x_i \|^2$ is finite.

1.1. REMARK. — Spaces of Rademacher type 2 and cotype 2 can be equivalently defined by replacing the Bernoulli random variables by the standard Gaussian r. v. (i. e. variables having ch. f. e^{-t^2}). We will use this definition.

A Banach space E is said to be of stable type p, $1 \leq p \leq 2$ if for every sequence $\{ x_j \}_{j=1}^{\infty} \subset E$ with $\Sigma \| x_j \|^p < \infty$ we have $\Sigma x_j \eta_j$ converges a. e. where η_j are i. i. d. symmetric stable r. v. of index p (i. e. the ch. f. of each η_j is of the form $\exp(- | t |^p)$ for real t). For $p = 2$, this notion is equivalent to Rademacher type 2.

It is known that an analogous definition of stable cotype p, $1 \leq p < 2$ does not restrict the class of Banach spaces, since if η_j are i. i. d. symmetric stable r. v. of index p, $1 \leq p < 2$, then the a. e. convergence of $\Sigma x_j \eta_j$ implies $\Sigma \| x_j \|^p$ is finite.

For the definitions of Gaussian, stable and infinitely divisible distributions on a Banach space E, and also for some properties of their ch. f. we refer the reader to [*19*].

A sequence of probability measures $\{ \mu_n \}$ on B(E) is said to converge weakly to the probability measure μ on B(E) if $\int_E f d\mu_n \to \int_E f d\mu$ for every bounded continuous real valued function defined on E. If μ_n converges weakly to μ, we write $\mu_n \Rightarrow \mu$. For other concepts related to the weak convergence of measures we use the terminology as in [*4*].

A Banach space E is said to be embeddable in a linear metric space Y, if there exists a linear topological isomorphism of E into Y.

We denote by $L^0(\Omega, \mu)$ the equivalence classes of real valued measurable functions where functions equal μ a. e., are identified. (Throughout this work we consider $L^0(\Omega, \mu)$ with μ a probability measure and the topology to be that of convergence in probability.) For $1 \leq p < \infty$, $L^p(\Omega, \mu)$ is the Banach space (with norm $\| \ \|_p$) consisting of all $f \in L^0(\Omega, \mu)$ such that

$$\| f \|_p^p = \int | f |^p d\mu < \infty .$$

We denote this space by L^p if $\Omega = [0, 1]$ and μ is the Lebesgue measure and by l^p if $\Omega = Z^+$ (the natural numbers) and μ is the counting measure.

Let Ψ denote the embedding of E into $L^0(\Omega, P)$ (with (Ω, P) a probability space). Then $\hat{P}_\Psi(x) = \int_\Omega e^{i\Psi(x)} dP$ defines a continuous complex-valued positive definite function on E and hence a cylinder measure P_Ψ on E′ ([*3*], Exposé 1.2). It has been a folklore that the embeddability of E in $L^0(\Omega, P)$ is equivalent to the accessibility of norm of E with respect to a positive definite function, as introduced by Kuelbs in [*13*], (i. e. there exists a real continuous positive definite function $\Phi(x)$ on E, with $\Phi(0) = 1$, such that for any $\mathscr{E} > 0$ and $\| x \|_E > \mathscr{E}$ we have $1 - \Phi(x) > h(\mathscr{E})$ where $h(\mathscr{E}) > 0$). For the sake of completeness we include the proof of this equivalence here.

Suppose norm of E is accessible by the continuous, real-valued positive definite function Φ, with $\Phi(0) = 1$. It is known that Φ is a ch. f. of a probability measure μ on cylinder sets of the algebraic dual E^a of E. Define $\Psi : E \to L^0(E^a, \mu)$ as

$$\Psi(x)(y) = \langle y, x \rangle \qquad \text{where} \qquad y \in E^a .$$

Then Ψ is an embedding.

Conversely suppose Ψ is an embedding of E into $L^0(\Omega, P)$. Without loss of generality we can assume that for each $x \in E$, $\Psi(x)$ is a symmetric random variable and is of the form $V\tilde{\Psi}(x)$ where $\tilde{\Psi}$ embeds E into a subset of symmetric random variables of some $L^0(\Omega', P')$, and V is a uniform r. v. on $[-1, 1]$ independent of $\{ \tilde{\Psi}(x), x \in E \}$. Then the norm of E is accessible with respect to the real-valued positive definite function $\hat{P}_\Psi(x) = \int_\Omega e^{i\Psi(x)} dP$.

Suppose not, then for some $\mathscr{E}_0 > 0$ and for each positive integer n, there exists an $\chi_n \in E$ such that $\| \chi_n \|_E > \mathscr{E}_0$ but $1 - \hat{P}_\Psi(\chi_n) \to 0$. Now

$$(1.2) \qquad \hat{P}_\Psi(\chi_n) = \int e^{iV\tilde{\Psi}(\chi_n)} dP = \frac{1}{2} \int_{-1}^{1} \left(\int e^{it\tilde{\Psi}(\chi_n)} dP' \right) dt .$$

We note that $\tilde{\Psi}(\chi_n)$ being symmetric, its ch. f. $f_n(t) = \int e^{it\tilde{\Psi}(\chi_n)}d\mathrm{P}'$ is real valued. Thus from (1.2) we get that

$$\frac{1}{2}\int_{-1}^{1} |1 - f_n(t)|\,dt = 1 - \hat{\mathrm{P}}_\Psi(\chi_n) \to 0 .$$

This implies that there exists a subsequence $\{ f_{n_k} \}$ such that $f_{n_k}(t) \to 1$ a. e. (lebesgue) in $[-1, 1]$. By ([*17*], p. 197) we get that $f_{n_k}(t) \to 1$ for all real t, which in turn implies that $\tilde{\Psi}(\chi_{n_k}) \Rightarrow 0$. Thus $\tilde{\Psi}(\chi_{n_k}) \to 0$ in probability, therefore, $\chi_{n_k} \to 0$ in E since $\tilde{\Psi}$ is an embedding. This contradicts that $\| \chi_n \|_{\mathrm{E}} > \mathscr{E}_0$ for all n, and thus we get the result.

2. BOCHNER PROPERTY I

Throughout this section we will assume that E is embeddable in $\mathrm{L}^0(\Omega, \mathrm{P})$ and has bounded approximation property. E is separable and has bounded approximation property means that there exists a sequence of finite dimensional operators $\{ \pi_n, n \in \mathrm{Z}^+ \}$ such that $\| \pi_n x - x \|_{\mathrm{E}} \to 0$ for each $x \in \mathrm{E}$. We will denote by π'_n the transpose of π_n.

Let Ψ be the embedding of E into $\mathrm{L}^0(\Omega, \mathrm{P})$, such that norm of E is accessible with respect to $\hat{\mathrm{P}}_\Psi(x) = \int e^{i\Psi(x)}d\mathrm{P}$. Then with the above notation we get,

2.1. **Lemma** (Lévy Inequality). — Any probability measure μ on E satisfies the following inequality, given $\mathscr{E} > 0$ there exists an $h(\mathscr{E}) > 0$ such that

$$\mu\{ x \mid \| (\pi_m - \pi_k)x \| > \mathscr{E} \} \leq \frac{1}{h(\mathscr{E})}\int_{\mathrm{E}'} [1 - \hat{\mu}((\pi_m - \pi_k)'y)]\mathrm{P}_\Psi(dy) .$$

Proof. — Using norm accessibility by P_Ψ and Chebychev's inequality

$$\mu\{ x \mid \| (\pi_m - \pi_k)x \| > \mathscr{E} \}$$
$$\leq \mu\{ x \mid [1 - \hat{\mathrm{P}}_\Psi((\pi_m - \pi_k)x)] > h(\mathscr{E}) \} \leq \frac{1}{h(\mathscr{E})}\int_{\mathrm{E}} [1 - \hat{\mathrm{P}}_\Psi((\pi_m - \pi_k)x)]\mu(dx)$$
$$= \frac{1}{h(\mathscr{E})}\int_{\mathrm{E}}\left[\int_{\mathrm{E}'} (1 - e^{i\langle y,x\rangle})\mathrm{P}_\Psi{}^{(\pi_m - \pi_k)'}(dy)\right]\mu(dx) .$$

where $\mathrm{P}_\Psi{}^{(\pi_m - \pi_k)'}$ denotes the probability measure on E′ with the finite dimensional support $(\pi_m - \pi_k)'\mathrm{E}'$.

Since the measures involved above are probability measures and the function is jointly measurable by Fubini's theorem we get

$$\mu \{ x \mid \| (\pi_m - \pi_k)x \| > \mathscr{E} \} \leq \frac{1}{h(\mathscr{E})} \int_{E'} [1 - \hat{\mu}((\pi_m - \pi_k)'y)] P_\Psi(dy) .$$

Note that $\hat{P}_\Psi$ being real valued all integrals above are real. This completes the proof.

Given a set F of functions on E′, we denote by τ_F the smallest topology with respect to which exactly all functions in F are continuous. In particular for the next lemma we take $F = F_0 = \{ \hat{\mu} \mid \mu$ belongs to a subclass of symmetric probability measures on E, which is closed under convolution $\}$. Then for the corresponding topology τ_{F_0}, a subbasis of neighborhoods at zero is given by the system $\{ V_{\mathscr{O},\mathscr{E}} ; \mathscr{O} \in F_0, \mathscr{E} > 0 \}$, where

$$V_{\mathscr{O},\mathscr{E}} = \{ y \in E' \mid (1 - \mathscr{O}(y)) < \mathscr{E} \} .$$

2.2. LEMMA. — If E has bounded approximation property and is embeddable in $L^0(\Omega, P)$ with embedding Ψ, then the continuity in τ_{F_0} of a positive definite function Φ, with $\Phi(0) = 1$, implies

$$\lim_k \sup_n \lim_m \int_{E'} \text{Re } [1 - \Phi(\pi_n'(\pi_m - \pi_k)'y)] P_\Psi(dy) = 0 .$$

Proof. — Since Φ is continuous in τ_{F_0}, given $\mathscr{E} > 0$, there exists a $\delta > 0$ and a symmetric probability measure ν with $\hat{\nu} \in F_0$, such that

$$\text{Re } (1 - \Phi(y)) < \mathscr{E} \qquad \text{whenever} \qquad 1 - \hat{\nu}(y) < \delta .$$

Using the fact that $\text{Re } (1 - \Phi(y)) \leq 2$ for all $y \in E'$ we get,

$$\text{Re } (1 - \Phi(y)) \leq \frac{2}{\delta}(1 - \hat{\nu}(y)) + \mathscr{E} \qquad \text{for all} \quad y \in E' .$$

Thus

$$\begin{aligned} \int_{E'} \text{Re } [1 - \Phi(\pi_n'(\pi_m - \pi_k)'y)] P_\Psi(dy) &\leq \frac{2}{\delta} \int_{E'} [1 - \hat{\nu}(\pi_n'(\pi_m - \pi_k)'y)] P_\Psi(dy) + \mathscr{E} \\ &= \frac{2}{\delta} \int_{E'} \left(\int_E (1 - e^{i\langle \pi_n'(\pi_m - \pi_k)'y, x \rangle}) \nu(dx) \right) P_\Psi(dy) + \mathscr{E} \\ &= \frac{2}{\delta} \int_E (1 - \hat{P}_\Psi((\pi_m - \pi_k)\pi_n x)) \nu(dx) + \mathscr{E} . \end{aligned}$$

But $\| \pi_m x - x \| \to 0$ for all $x \in E$ and $\hat{P}_\Psi$ is continuous, thus

$$\lim_k \sup_n \lim_m \int_E [1 - \hat{P}_\Psi((\pi_m - \pi_k)\pi_n x)]\nu(dx) = 0\,,$$

giving

$$\lim_k \sup_n \lim_m \int_{E'} \operatorname{Re}\, [1 - \Phi(\pi_n'(\pi_m - \pi_k)'y)]P_\Psi(dy) \leqslant \mathscr{E}$$

but $\mathscr{E}$ is arbitrary and thus we get the result.

We observe that if a function on E′ is continuous in τ_{F_0}, then it is sequentially weak-star continuous. This and the above lemmas lead us to Theorem 2.4, for the proof of which we need the following concept related to the concept of tightness.

2.3. Definition ([1], p. 279). — A family $\{ \mu_\alpha : \alpha \in I \}$ of probability measures on (E, B(E)) is flatly concentrated if for every $\varepsilon > 0$ and $\delta > 0$ there exists a finite dimensional subspace M of E such that

$$\mu_\alpha \{ x \in E \mid \inf \{ \| x - Z \| : Z \in M \} \leq \mathscr{E} \} \geq 1 - \delta \qquad \text{for all} \quad \alpha \in I\,.$$

2.4. Theorem. — If E is embeddable in $L^0(\Omega, P)$ for a probability space (Ω, P), and has bounded approximation property, then every positive definite function Φ, with $\Phi(0) = 1$ and continuous in τ_{F_0}, is a ch. f. of a probability measure on E.

Proof. — We use the notations as defined above. We have noted that Φ is sequentially weak-star continuous. Hence its restriction to every finite dimensional subspace of E′ is continuous. Thus there exists a cylinder measure μ_0 associated with Φ. Let $\mu_n = \mu_0 \pi_n^{-1}$, then for each n, μ_n is a Borel probability measure on E having finite dimensional support $\pi_n(E)$ and $\hat{\mu}_n(y) = \Phi(\pi_n'(y))$. Using (2.1) we get the Lévy inequality, given $\mathscr{E} > 0$, there exists $h(\mathscr{E}) > 0$ such that

$$\mu_n \{ x \mid \| (\pi_m - \pi_k)x \| > \mathscr{E} \} \leq \frac{1}{h(\mathscr{E})} \int_{E'} [1 - \Phi(\pi_n'(\pi_m - \pi_k)'y]P_\Psi(dy)\,.$$

By Lemma (2.2) we have

$$\lim_k \sup_n \lim_m \int_{E'} [1 - \Phi(\pi_n'(\pi_m - \pi_k)'y]P_\Psi(dy) = 0$$

Therefore

$$\lim_k \sup_n \mu_n \{ x \mid \| (I - \pi_k)x \| > \mathscr{E} \} = \lim_k \sup_n \lim_m \mu_n \{ x \mid \| (\pi_m - \pi_k)x \| > \mathscr{E} \} = 0\,.$$

Thus for $\mathscr{E} > 0$, $\mathscr{E}' > 0$ there exists a finite-dimensional subspace $\pi_{n_0}(E)$ of E such that $\sup \mu_n \{ x \mid \| (I - \pi_{n_0})x \| > \mathscr{E} \} < \mathscr{E}'$ giving $\{ \mu_n \}_{n \in Z^+}$ is flatly concentrated.

Now as Φ is sequentially weak-star continuous and $\langle \pi'_n y, x \rangle \to \langle y, x \rangle$ for each $x \in E$, we get that $(\hat{\mu}_n(y) =)\Phi(\pi'_n y)$ converges pointwise to $\Phi(y)$. It also follows that restriction of Φ to each one-dimensional subspace of E′ is continuous.

Thus by ([*1*], Theorem 2.4, p. 280) we get the existence of a probability measure μ, such that $\hat{\mu}(y) = \Phi(y)$.

The above theorem includes a simpler proof of a result of Mustari ([*22*], Theorem 1 *c*)), which we restate in Corollary 2.5.

2.5. Corollary. — If E has bounded approximation property and imbeds in $L^0(\Omega, P)$, then E has Bochner Property I.

Proof. — In Theorem 2.4 we take the set F_0 to consist of ch. f. of all symmetric probability measures on E.

The continuity in τ_{F_0} of a ch. f. of any probability measure μ follows from the inequality $|1 - \hat{\mu}(y)|^2 \leq 2(1 - \operatorname{Re} \hat{\mu}(y))$ and the fact that $\operatorname{Re} \hat{\mu}(y)$ is the ch. f. of the symmetric probability measure ν given by

$$\nu(A) = \frac{1}{2}(\mu(A) + \mu(-A)).$$

We next show that for embeddable spaces with bounded approximation property and of Rademacher or of stable type *p*, we can get topologies of the form τ_F where F can be described explicitly.

2.6. Corollary. — Let E be of Rademacher type *p* embeddable in $L^0(\Omega, P)$ and have the bounded approximation property. Let G be a symmetric measure on E satisfying,

i) G is σ-finite on E with $G\{0\} = 0$ and finite outside every neighborhood of zero,

ii) $\displaystyle\int_{\|x\| \leq 1} \| x \|^p G(dx)$ is finite,

Let $F_p = \{ \psi : E' \to C \mid \psi(y) = \exp \int (\cos \langle y, x \rangle - 1) G(dx)$, for all G satisfying *i*) and *ii*) $\}$.

Then if a positive definite function Φ, with $\Phi(0) = 1$ is continuous in τ_{F_p}, it is a ch. f. of a probability measure on E.

Proof. — We note that F_p is the set of ch. f. 's of symmetric non-Gaussian infinitely divisible measures on E from ([*19*], Theorem 4.6), and also closed under convolution. The result now follows from Theorem 2.4.

Next we consider topologies associated with symmetric stable measures. We will denote by τ_p (for $p \in [1, 2]$) the topology τ_F obtained by taking $F = \left\{ \psi \mid \psi(y) = \exp\left[- \int_E |\langle y, x \rangle|^p \nu(dx) \right]\right.$, as ν varies through the set of measures for which $\left.\int_E \|x\|^p \nu(dx) < \infty \right\}$. Let $\tilde{\tau}_p$ be the topology τ_F obtained by taking F to be the set of ch. f. of all symmetric stable measures of index p on E. Then in general $\tilde{\tau}_p$ is weaker than τ_p [*30*], and coincides with τ_p if E is of stable type p by ([*19*], Theorem 4.7. Also [*2*]). From the same theorem it follows that if E has Bochner property I with respect to τ_p, then E is of stable type p. We also note that under τ_p, E′ is a topological vector space.

2.7. COROLLARY. — If E is of stable type p, embeddable in $L^0(\Omega, P)$ and has bounded approximation property, then E has Bochner property I with respect to τ_p, $1 \leq p \leq 2$.

Proof. — We note that the convolution of two symmetric stable measures is again a symmetric stable measure. Thus by the above comments and Theorem 2.4 it only remains it show that every ch. f. is continuous in τ_p.

Suppose $\hat{\mu}$ is the ch. f. of a probability measure μ on E. i. e.

$$\hat{\mu}(y) = \int_E e^{i\langle y,x \rangle} \mu(dx) .$$

Then

$$|1 - \hat{\mu}(y)|^2 \leq 2 \operatorname{Re}(1 - \hat{\mu}(y)) = 2\int_E (1 - \cos\langle y, x \rangle)\mu(dx) .$$

Since μ is a probability measure on E, given $\mathscr{E} > 0$, there exists a compact set $K \subset E$ such that $\mu(K) > 1 - \frac{\mathscr{E}}{2}$. Therefore

$$\begin{aligned} 2\int_E (1 - \cos\langle y, x \rangle)\mu(dx) &\leq 2\int_K (1 - \cos\langle y, x \rangle)\mu(dx) + \mathscr{E} \\ &\leq 4\int_K |\langle y, x \rangle|^p \mu(dx) + \mathscr{E} , \qquad 1 \leq p \leq 2 . \end{aligned}$$

Let ν be the measure on E such that for all $A \in B(E)$, $\nu(A) = \mu(K \cap A)$.

Thus
$$\int_E \| x \|^p \nu(dx) = \int_K \| x \|^p \nu(dx) \leq \sup_{x \in K} \| x \|^p \nu(K) < \infty ,$$
and
$$| 1 - \hat{\mu}(y) |^2 \leq 4 \int_E | \langle y, x \rangle |^p \nu(dx) + \mathscr{E} .$$
Hence $\hat{\mu}$ is continuous in τ_p.

It is well known that $\hat{\mu}$ is positive definite and $\hat{\mu}(0) = 1$.

2.8. Remark. — The fact that τ_p gives a necessary topology is true without any assumptions on E.

2.9. Corollary. — If E is of Rademacher type p, $1 < p \leq 2$, and has bounded approximation property and is embeddable in $L^0(\Omega, P)$, then E has Bochner Property I with respect to τ_{F_p}.

Proof. — By Corollary 2.6 it is enough to show that ch. f. of any probability measure μ, is continuous in τ_{F_p}.

From ([*21*], p. 79) E is of Rademacher type p implies it is of stable type q for all $q < p$. Therefore by Corollary 2.7 E has Bochner Property I with respect to τ_q. Thus given $\mathscr{E} > 0$, there exists a symmetric stable measure ν of index q and a $\delta > 0$, such that Re $(1 - \hat{\mu}(y)) < \mathscr{E}$ whenever $1 - \hat{\nu}(y) < \delta$.

We will show that $\hat{\nu}$ belongs to F_p. It is known [*30*] that
$$\nu(y) = \exp\left[- \int_T | \langle y, x \rangle |^q \lambda(dx) \right]$$
where λ is a finite measure on the boundary T of the unit ball of E. Since $\int (\cos ts - 1) \frac{Cq}{s^{1+q}} ds = - | t |^q$ for a constant Cq $(0 < Cq < \infty)$ ([*15*], p. 205), we get
$$\hat{\nu}(y) = \exp\left[\int_T \int_0^\infty \left(\cos \langle y, x \rangle s - 1 \frac{Cq}{s^{1+q}} ds \lambda(dx) \right) \right].$$
Identify $E = T \times [0, \infty)$ and defines measure on B(E) as in ([*19*], p. 323) by
$$G_1(A) = \int_T \int_0^\infty 1_A(x, s) Cq \frac{ds}{s^{1+q}} \lambda(dx)$$
and
$$G_2(A) = \int_T \int_{-\infty}^0 1_A(x, - s) \frac{ds}{| s |^{1+q}} \lambda(dx) .$$

Let $G = \frac{1}{2}(G_1 + G_2)$, then G is symmetric, finite outside every neighborhood of zero, and

$$\int_{||x|| \leq 1} || x ||^p G(dx) = \int_T \int_0^1 || u ||^p s^p \frac{ds}{s^{1+q}} \lambda(du)$$
$$= \lambda(T) \int_0^1 s^{p-q-1} ds .$$

which is finite since $q < p$.

Finally since G is symmetric we get

$$\hat{v}(y) = \exp \int_E (\cos \langle y, x \rangle - 1) G(dx) ,$$

giving $\hat{v} \in F_p$. Thus $\hat{\mu}$ is continuous in τ_{F_p}.

3. BOCHNER PROPERTY I WITH RESPECT TO τ_p AND τ_{F_p}

In this section we prove the converse of Corollaries 2.7 and 2.9 (without bounded approximation property), thereby obtaining a relation between the structure of Banach spaces and explicit topological solution to Problem I.

3.1. LEMMA. — If a real separable Banach space E is not embeddable in L^p then there exist sequences $\{ Ui \}$ and $\{ Vi \}$ in E such that

$$\sum_{i=1}^{\infty} || Ui ||^p < \infty , \quad \sum_{i=1}^{\infty} || Vi ||^p = \infty$$

(3.2)

$$\sum_i | \langle y, Ui \rangle |^p \geq \sum_i | \langle y, Vi \rangle |^p \quad \text{for each } y \in E' .$$

Proof. — A separable (closed) subspace of a $L^p(\Omega, \mu)$ for an arbitrary measure μ is embeddable in L^p, since by ([*6*], Lemma 5, p. 168 and [*11*], Theorem C, p. 173) it embeds in $L^p \oplus l^p$, which embeds in L^p by ([*16*], p. 133). Thus E not embeddable in L^p implies it can not be embeddable in $L^p(\Omega, \mu)$. Then by a result of Lindenstrauss and Pelezynski ([*15*], Theorem 7.3 ([2])) we get that for each k, there exist finite sequences $\{ Ui^k \}_{i=1}^{N_k}$ and $\{ Vj^k \}_{j=1}^{M_k}$ in E such that for all $y \in E'$

$$\sum_{i=1}^{N_k} |\langle y, Ui^k \rangle|^p \geq \sum_{j=1}^{M_k} |\langle y, Vj^k \rangle|^p,$$

but

$$2^{kp} \sum_{i=1}^{N_k} \| Ui^k \|^p < \sum_{j=1}^{M_k} \| Vj^k \|^p .$$

Without loss of generality we can assume that $\sum_{j=1}^{M_k} \| Vj^k \|^p = 1$.

We note that $\sum_{k=1}^{\infty} \sum_{i=1}^{N_k} \| Ui^k \|^p < \sum_{k=1}^{\infty} \frac{1}{2^{kp}} < \infty$, and $\sum_{k=1}^{\infty} \sum_{j=1}^{M_k} \| Vj^k \|^p = \infty$.

Thus we can find two sequences $\{ Ui \}_{i=1}^{\infty}$ and $\{ Vi \}_{i=1}^{\infty}$ such that (3.2) holds.

3.3. THEOREM. — If E has Bochner Property I with respect to τ_p, then E is embeddable in L^p.

Proof. — Let $1 \leq p < 2$. Suppose E has Bochner Property I with respect to τ_p, but is not embeddable in L^p. Then by the above lemma there exist sequences $\{ Ui \}$ and $\{ Vi \}$ in E satisfying (3.2).

We have already noted that since E has Bochner Property I with respect to τ_p it is of stable type p. Thus by definition of stable type p, $\Sigma \| Ui \|^p$ is finite implies $\Sigma U_i \eta_i$ converges a. e., for $\{ \eta_i \}$ i. i. d. symmetric stable random variables of index p. Thus there exists a probability measure μ on E such that $\hat{\mu}(y) = e^{-\sum_{i=1}^{\infty} |\langle y, Ui \rangle|^p}$ for each $y \in E'$. Moreover μ is symmetric p-stable measure on E. Using this and the second inequality in (3.2) we can show that the positive definite function $\Phi(y) = e^{-\sum_{i=1}^{\infty} |\langle y, Vi \rangle|^p}$ is continuous in τ_p. Also $\Phi(0) = 1$. Now by the hypothesis, there exists a probability measure v on E, such that

$$\hat{v}(y) = \Phi(y) = e^{-\sum_{i=1}^{\infty} |\langle y, Vi \rangle|^p}$$

By a result of Ito-Nisvo ([*12*], Theorem 3.1 and 4.1) we get that $\Sigma V_i \eta_i$ converges a. e. for $\{ \eta_i \}$ i. i. d. symmetric p-stable random variables, but this implies $\Sigma \| Vi \|^p < \infty$, contradicting (3.2). This gives the result.

([2]) I thank Professor Weron for bringing this result to my attention in this context.

For $p = 2 : \tau_2$ is generated by symmetric Bilinear forms, hence E is isomorphic to a Hilbert space ([*23*]), thus E imbeds in L^2.

3.4. REMARK. — It is known that L^p for $1 \leq p \leq 2$ is embeddable in $L^0(\Omega, P)$ for some probability space (Ω, P) [*29*].

We thus obtain the following result.

3.5. THEOREM. — Let E have bounded approximation property. Then E is of stable type p and embeddable in $L^0(\Omega, P)$ iff E has Bochner Property I will respect to τ_p.

3.6. REMARK. — We note that if E is of stable type p and has Bochner Property I with respect to some topology τ, then $\tau_p \equiv \tilde{\tau}_p$ is weaker than τ. This together with Remark 2.8 implies that E has Bochner Property I with respect to τ_p.

It follows from ([*19*], Theorem 4.6) that if E has Bochner Property I with respect to τ_{F_p}, then E is of Rademacher type p, therefore of stable type q for $q < p$. Using this and the above Remark we get the converse of Corollary 2.9. Thus we get,

3.7. THEOREM. — Let E have the bounded approximation property. Then E is of Rademacher type p and embeddable in $L^0(\Omega, P)$ iff E has Bochner Property I with respect to τ_{F_p}.

3.8. REMARK. — Combining a result of Maurey ([*20*], Theorem 9.8) with the above result we get that for real separable Banach spaces with bounded approximation property the following are equivalent.

(1) E has Bochner Property I with respect to τ_p.

(2) E is strongly embeddable in $L^p(\Omega, \mu)$ for some probability space (Ω, μ).

4. AN EXTENSION OF A RESULT OF SAZANOV

In [*25*] [*28*], it is shown that a sufficient condition for the tightness of a family of probability measures on a real separable Hilbert space is the equicontinuity in the S-topology of the corresponding ch. f's (i. e. in the topology generated by ch. f's of all symmetric Gaussian probability measures). In this section we show that the largest class for which the above result is valid is the class of cotype 2 spaces, which includes the class of Hilbert spaces [*14*] and l_p spaces $(0 < p \leq 2)$ (See e. g. [*21*]).

We first make a few definitions.

4.1. DEFINITION. — We will say that a Banach space E has Bochner Property II, if there exists a topology τ on E′, such that given a family

$\{ \mu_\alpha, \alpha \in I \}$ of probability measures on E, the equi-continuity in τ of their ch. f. is a sufficient condition for tightness.

4.2. DEFINITION ([*24*], p. 154). — An S-operator on a Hilbert space H is a linear, symmetric, non-negative, compact operator having finite trace.

Given an E-valued Gaussian random variable X defined on some probability space (Ω, P). Let $\mu = P \circ X^{-1}$. Then by Fernique's Theorem, [*8*], we know that $\int_E \| x \|^2 \mu(dx)$ is finite. Define an operator A from E′ into E by

$$Ay = \int_E \langle y, x \rangle x\mu(dx)$$

where the integral is in sense of Bochner. Then A is called the covariance operator of X and the ch. f. of μ is $\exp\left(-\frac{1}{2}\langle Ay, y \rangle \right)$. Thus $\tilde{\tau}_2$ coincides with the topology for which a basis of neighborhoods of zero is given by the system of sets $\{ y \in E' \mid \langle Ay, y \rangle < 1 \}$, where A runs through the set of Gaussian covariance operators. We note that if E is a separable Hilbert space then $\tilde{\tau}_2$ is the same as the S-topology and A is an S-operator [*24*].

4.3. THEOREM. — E has Bochner Property II with respect to the $\tilde{\tau}_2$ topology iff E is of cotype 2.

Proof. — Suppose E is of cotype 2, and $\{ \mu_\alpha \}_{\alpha \in I}$ a family of probability measures such that their ch. f. $\hat{\mu}_\alpha$ are equicontinuous at 0 in $\tilde{\tau}_2$. Then given $\mathscr{E} > 0$, there exists a Gaussian covariance operator $A_{\mathscr{E}}$, such that

$$(4.4) \quad \langle A_{\mathscr{E}} y, y \rangle \leq 1 \quad \text{implies} \quad 1 - \operatorname{Re} \hat{\mu}_\alpha(y) \leq | 1 - \hat{\mu}_\alpha(y) | < \mathscr{E}, \qquad \alpha \in I.$$

Then as in [*27*] we can choose a single covariance operator A of a symmetric Gaussian E-valued r. v. X, such that for every $\mathscr{E} > 0$, there exists a $\delta > 0$ satisfying

$$(4.5) \qquad | 1 - \hat{\mu}_\alpha(y) | < \mathscr{E}, \qquad \text{whenever} \qquad \langle Ay, y \rangle < \delta.$$

Let λ be the Gaussian measure corresponding to X (i. e. the distribution of X). Now since E is of cotype 2, there exists a Hilbert space H, a continuous linear operator U from H into E and a Gaussian measure λ_1 on H such that $\lambda = \lambda_1 \circ U^{-1}$ ([*9*], Theorem 4). Without loss of generality we can (and we do !) assume that U is one-one. Let T be a covariance operator of λ_1, then T is an S-operator and $\hat{\lambda}_1(h) = e^{-\frac{1}{2}(\mathrm{T}h,h)}$. Therefore $\hat{\lambda}(y) = e^{-\frac{1}{2}\langle \mathrm{UTU}^*y,y\rangle}$, where U* denotes the adjoint. By uniqueness of ch. f. we get $\mathrm{A} = \mathrm{UTU}^*$. Define $\hat{\nu}_\alpha$ on U*(E') by $\hat{\nu}_\alpha(\mathrm{U}^*y) = \hat{\mu}_\alpha(y)$. If $\mathrm{U}^*y_1 = \mathrm{U}^*y_2$ then $\mathrm{UTU}^*y_1 = \mathrm{UTU}^*y_2$ i. e., $\mathrm{A}y_1 = \mathrm{A}y_2$. Hence by (4.5) and positive-definiteness of $\hat{\mu}_\alpha$ we get $\hat{\mu}_\alpha(y_1) = \hat{\mu}_\alpha(y_2)$. Hence $\hat{\nu}_\alpha$ is well defined on the range of U* and

$$|1 - \hat{\nu}_\alpha(\mathrm{U}^*y)| < \mathscr{E} \quad \text{if} \quad (\mathrm{TU}^*y, \mathrm{U}^*y)_\mathrm{H} < \delta .$$

But the range of U* is dense in H and $\hat{\nu}_\alpha$ is uniformly continuous on the range of U* giving

$$(4.6) \qquad |1 - \hat{\nu}_\alpha(h)| < \mathscr{E} \quad \text{whenever} \quad (\mathrm{T}h, h)_\mathrm{H} < \delta .$$

In other words, $\{ \hat{\nu}_\alpha, \alpha \in \mathrm{I} \}$ is equicontinuous in S-topology in the sense of ([*24*], p. 155). By [*28*] $\{ \nu_\alpha, \alpha \in \mathrm{I} \}$ is tight. Since U is continuous, $\{ \nu_\alpha \circ \mathrm{U}^{-1}, \alpha \in \mathrm{I} \}$ is tight on E' ([*4*], p. 30). But $\nu_\alpha \circ \mathrm{U}^{-1} = \mu_\alpha$, completing the sufficiency part.

The proof of the necessity part is along the lines of the proof of ([*19*], Theorem 2.3). Suppose that E has Bochner Property II with respect to $\tilde{\tau}_2$ but is not of cotype 2. Then there exists a sequence $\{ \chi_k \}_{k=1}^{\infty}$ such that for γ_k independent standard Gaussian random variables, $\Sigma \chi_k \gamma_k$ converges a. e., but $\Sigma \| \chi_k \| = \infty$. Let $\mathrm{A}_k = \left(\sum_{i=1}^{k} \| \chi_i \|^2 \right)^{1/2}$, then $\mathrm{A}_k^2 \to \infty$ and $\sum_{k=1}^{n} \frac{\| \chi_k \|^2}{\mathrm{A}_k^2} \to \infty$.

Define E-valued independent symmetric random variables $\{ \mathrm{Y}_k \}_{k=1}^{\infty}$ by

$$\mathrm{P}\left(\mathrm{Y}_k = \frac{\mathrm{A}_k \chi_k}{\| \chi_k \|} \right) = \mathrm{P}\left(\mathrm{Y}_k = - \frac{\mathrm{A}_k \chi_k}{\| \chi_k \|} \right) = \frac{1}{4} \frac{\| \chi_k \|^2}{\mathrm{A}_k^2}$$

and

$$\mathrm{P}(\mathrm{Y}_k = 0) = 1 - \frac{1}{2} \frac{\| \chi_k \|^2}{\mathrm{A}_k^2} .$$

Since $\sum_k \mathrm{P}\{ \| \mathrm{Y}_k \| > \mathscr{E} \} = \frac{1}{2} \sum_k \frac{\| \chi_k \|^2}{\mathrm{A}_k^2} = \infty$, by Borel-Cantelli Lemma we get that $\mathrm{Y}_k \nrightarrow 0$ a. e. And hence $\Sigma \mathrm{Y}_k$ diverges a. e.

Let Φ_k denote the ch. f. of the Gaussian E-valued random variable $\sum_{n=1}^{k} \chi_n \gamma_n$ and Φ the ch. f. of the Gaussian E-valued random variable $\sum_{n=1}^{\infty} \chi_n \gamma_n$. Then $\Phi_k(y)$ and $\Phi(y)$ are continuous in $\tilde{\tau}_2$.

Moreover

$$(4.7) \quad 1 - \Phi_k(y) \leq 1 - \Phi(y) \qquad \text{for all } k .$$

$$1 - \Phi(y) < \mathscr{E} \qquad \text{implies} \qquad \frac{1}{2} \sum_{k=1}^{\infty} \langle y, \chi_k \rangle^2 < - \log (1 - \mathscr{E})$$

which tends to « 0 » as $\mathscr{E} \to 0$. Thus can take $\mathscr{E}$ small such that $1 - \Phi(y) < \mathscr{E}$ implies $| \langle y, \chi_k \rangle | < 1$ for all k, then

$$(4.8) \qquad e^{-\frac{1}{2}\langle y, \chi_k \rangle^2 \leq 1 - \frac{1}{4}\langle y, \chi_k \rangle^2} .$$

Note that

$$1 - \cos \left\langle y, \frac{A_k}{\| \chi_k \|} \chi_k \right\rangle \leq \frac{A_k^2}{\| \chi_k \|^2} \frac{\langle y, \chi_k \rangle^2}{2}$$

therefore

$$1 - \frac{1}{4} \langle y, \chi_k \rangle^2 \leq 1 - \frac{1}{2} \frac{\| \chi_k \|^2}{A_k^2} \left(1 - \cos \left\langle y, \frac{A_k \chi_k}{\| \chi_k \|} \right\rangle \right), \qquad \text{for all} \quad k$$

Thus if $1 - \Phi(y) < \mathscr{E}$ for small $\mathscr{E}$ then using (4.8) we get

$$(4.9) \quad \Phi_k(y)$$

$$\leq \prod_{n=1}^{k} \left(1 - \frac{1}{4} \langle y, \chi_n \rangle^2 \leq \prod_{n=1}^{k} \left[1 - \frac{1}{2} \frac{\| \chi_n \|^2}{A_n^2} \left(1 - \cos \left\langle y, \frac{A_n \chi_n}{\| \chi_n \|} \right\rangle \right) \right] . \right.$$

Let v_k denote the distribution of $\sum_{k=1}^{n} Y_k$, then from (4.9), $\Phi_k(y) \leq \hat{v}_k(y)$ for all k, therefore

$$1 - \hat{v}_k(y) \leq 1 - \Phi_k(y) \leq 1 - \Phi(y) \qquad \text{for all} \quad k .$$

Thus the family $\{ \hat{v}_k \}_{k=1}^{\infty}$ is equicontinuous in $\tilde{\tau}_2$. Hence by hypothesis the family $\{ v_k \}_{k=1}^{\infty}$ is tight. $\{ Y_k \}$ are symmetric, hence by ([*12*], Theorem 3.1 and 4.1) we get that ΣY_k converges a. e., contradicting ΣY_k diverges. Hence E is of cotype 2 and this completes the proof.

Let τ_p be the topology as before. It is easy to show that if a Banach space has Bochner Property II with respect to τ_2 then it is of type 2, thus from ([19], Theorem 3.6) $\tau_2 \equiv \tilde{\tau}_2$, and in view of the above Theorem it is of cotype 2, therefore is isomorphic to a Hilbert space [14]. We next state some results connected with Bochner Property II with respect to τ_p, $1 \leq p < 2$. The proofs of these are similar to those of Theorem 3.3 and Theorem 2.4 respectively.

4.10. THEOREM. — If E has Bochner Property II with respect to τ_p then E is of stable type p and embeddable in $L^p(\mu)$. (In view of a comment above this theorem is valid for $p = 2$.)

4.11. THEOREM. — If E has the b. a. p., is of stable type p and embeddable in $L^0(\Omega, P)$ then E has Bochner Property II with respect to τ_p.

4.12. REMARK. — For a real separable Banach space E having b. a. p. the following are equivalent.

i) E has Bochner Property I with respect to τ_p.
ii) E has Bochner Property II with respect to τ_p.
iii) E is strongly embeddable in $L^p(\Omega, \mu)$ for a probability space (Ω, μ).

5. FINAL REMARKS

(1) Since spaces having accessible norm (ch. § 1) are embeddable in $L^0(\Omega, P)$, they are of cotype 2 ([22], Theorem 1 (A)). Thus Theorem 4.3 includes a result of Kuelbs ([13], Theorem 6.3).

(2) Theorem 4.3 does not have the assumption of the b. a. p. In this context, however, the support of measures involved does have the property. This fact allows us to circumvent the problem.

(3) Is the support of a stable measure on a stable type p space, $L^{p'}$, $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$? Answering the above question will enable us to remove the assumption of approximation property. For spaces having Bochner Property I with respect to τ_p it is known (3) that the support is $L_{p'}$.

(4) If the Lindenstrauss-Pelczynski result ([15], Proposition 7.1) can be generalized to L^p $(0 < p < 1)$ (4), it would lead to the answer of the question whether embeddability in $L^0(\Omega, P)$ is also a necessary condition for real separable Banach spaces to have Bochner Property I (at least for spaces with b. a. p.).

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(3) Personal Communication by Professors V. Mandrekar and A. Weron.
(4) See addendum.

(5) In ([22], Theorem 1 (A), (B)) Mustari has shown that,

a) spaces embeddable in $L^0(\Omega, P)$ are of cotype 2.

b) spaces having Bochner Property I are of cotype 2.

Using a result of Rosenthal we are able to find examples which show that the converse of the above two statements is not true. These will appear elsewhere.

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ADDENDUM

After this manuscript was typed the author learnt about Maurey's [*31*] generalization of the Lindenstrauss-Pelczynski result ([*15*], Theorem 7.3) to $L^p(\Omega, \mu)$ (with the usual topology) for $0 < p < 1$. This together with the fact that L^p, l^p $(0 < p < 1)$ are embeddable in $L^0(\Omega, P)$ from [*29*], enable us to extend our Theorems 3.3 and 3.5 to include the case $0 < p < 1$.

We also note that Corollary 2.7 and Remark 3.6 are valid for all p, $0 < p \leq 2$. This and the fact that every Banach space is of stable type p $(0 < p < 1)$, gives.

THEOREM. — A real separable Banach space with b. a. p. is embeddable in a $L^0(\Omega, P)$ if and only if it has Bochner Property I. Moreover the corresponding topology is given by τ_q for any q in (0, 1).

This gives a converse of a result of Mustari [*22*].