Annales de l'I. H. P., section B

K. A. Astbury

A potential operator and some ergodic properties of a positive L_∞ contraction

Annales de l'I. H. P., section B, tome 12, n° 2 (1976), p. 151-162
<http://www.numdam.org/item?id=AIHPB_1976__12_2_151_0>

© Gauthier-Villars, 1976, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (http://www.elsevier.com/locate/anihpb) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré, Section B :
Vol. XII, n° 2, 1976, p. 151-162. *Calcul des Probabilités et Statistique.*

A Potential Operator and Some Ergodic Properties of a Positive L_∞ Contraction

by

K. A. ASTBURY
The Ohio State University, Columbus, Ohio 43210

SUMMARY. — A potential operator is introduced for a class of positive operators which includes the positive operators on L_∞. This potential operator is shown to have some of the familiar properties enjoyed by potential operators obtained from kernels (e. g. Domination Principle, Riesz Decomposition, and Balayage). Given a positive L_∞ contraction, its potential operator is used to obtain a Hopf decomposition into conservative and dissipative sets. We further study some of the Ergodic properties of a positive contraction of L_∞.

1. INTRODUCTION

Let $(X, \mathscr{A}, m)$ be a σ-finite measure space. Throughout this paper functions and sets will be considered equal if they are equal except for a set of measure zero.

The main result of this paper is that a positive linear contraction of $L_\infty(X, \mathscr{A}, m)$ determines a Hopf decomposition of the space X into conservative and dissipative regions. When the contraction is the adjoint of an $L_1(X, \mathscr{A}, m)$ operator, then the classical properties of the decomposition in terms of the L_1 operator are immediate.

In the « classical » case, it is well known that the conservative and dissipative regions can be characterized in terms of the finiteness of the sums $S(f) = \sum_{n=0}^{\infty} P^n f$. However, when P is not the adjoint of an L_1 operator, these sums do not have the crucial property that $P[S(f)] + f = S(f)$. The proper generalization of the sums is the potential operator defined in Section 2, which is devoted to its properties. For completeness, we extend the domain of the potential operator to the cone $\mathscr{F}$ of all positive extended real valued measurable functions on $(X, \mathscr{A}, m)$, and we allow P to be any positive linear operator on $\mathscr{F}$. Positive L_∞ operators extend to $\mathscr{F}$ in several natural ways (see Section 3) and the particular way the operator is extended does not affect the Hopf decomposition nor the related properties.

In Section 3 we define for a positive L_∞ operator the conservative and dissipative regions and study their properties. We give examples which illustrate properties which hold in the classical case but which fail to hold in the more general case.

We will use throughout this paper a well known property for $\mathscr{F}$, involving essential suprema, given by the following lemma:

LEMMA 1.1. — 1. *Let* $\{ f_\alpha \}_{\alpha\in\Delta} \subset \mathscr{F}$. *Then there exist two unique elements of* $\mathscr{F}$, *denoted by* $\sup_{\alpha\in\Delta} f_\alpha$ *and* $\inf_{\alpha\in\Delta} f_\alpha$, *such that for all* $f, g \in \mathscr{F}$:

(a) $f_\alpha \leq f$ *for all* $\alpha \in \Delta \Leftrightarrow \sup_{\alpha\in\Delta} f_\alpha \leq f$

(b) $f_\alpha \geq g$ *for all* $\alpha \in \Delta \Leftrightarrow \inf_{\alpha\in\Delta} f_\alpha \geq g$.

2. *Let* $\{ A_\alpha \}_{\alpha\in\Delta} \subset \mathscr{A}$. *Then there exists a unique element of* $\mathscr{A}$ *(up to sets of m-measure zero), denoted by* $\bigcup_{\alpha\in\Delta} A_\alpha$ *such that for all* $A \in \mathscr{A}$:

$$A_\alpha \subset A \quad \textit{for all} \quad \alpha \in \Delta \Leftrightarrow \bigcup_{\alpha\in\Delta} A_\alpha \subset A .$$

Proof. — See [*8*, Proposition II.4.1, p. 44-45].

LEMMA 1.2. — 1. $P \sup_{\alpha\in\Delta} f_\alpha \geq \sup_{\alpha\in\Delta} Pf_\alpha$.

2. $P \inf_{\alpha\in\Delta} f_\alpha \leq \inf_{\alpha\in\Delta} Pf_\alpha$.

Proof. — P is positive and apply Lemma 1.1.

2. THE POTENTIAL OPERATOR

Let P be a positive linear operator on $\mathscr{F}$:

$$\begin{aligned} P(f+g) &= Pf + Pg \quad \text{for} \quad f, g \in \mathscr{F} \\ Paf &= aPf \qquad\quad \text{for} \quad a \in \mathrm{R}, 0 \leq a < \infty . \end{aligned}$$

(Multiplication in the extended reals for the finite case is as usual, and for the infinite case is defined by $a\cdot\infty=\infty\cdot a=\infty$ for $a>0$, and $0\cdot\infty=\infty\cdot 0=0$.)

DÉFINITION 2.1. — For any $f \in \mathscr{F}$, let $\mathscr{S} = \{ g \in \mathscr{F} \mid Pg + f \leq g \}$. Clearly the function $g = \infty \in \mathscr{S}$. Define the *potential* of f (with respect to P) by $E(f) = \inf_{g \in \mathscr{S}} g$. The map $E : \mathscr{F} \to \mathscr{F}$ is called the *potential operator*. It will be shown in the next theorem that E is linear so we will write Ef for $E(f)$. We will also write E(A) for $E(1_A)$.

THEOREM 2.1. — 1. $PEf + f = Ef$.

2. $g \in \mathscr{F}$ *and* $Pg + f \leq g \Rightarrow Ef \leq Pg + f$.

3. $E(f+g) = Ef + Eg$.
   $E(af) = aEf$ *for* $0 \leq a < \infty$.

4. $Ef = Eg \Rightarrow f = g$ *on* $\{ PEf < \infty \}$.

Proof. — 2. Let $Pg + f \leq g$. Then $P(Pg + f) + f \leq (Pg + f)$.

1. For any g which satisfies $Pg + f \leq g$, we have $PEf + f \leq Pg + f \leq g$, and $PEf + f$ is a lower bound for g. The other direction follows from 2.

3. From 1 we have $E(f+g) \leq Ef + Eg$ and $Eg \leq E(f+g)$. $h \in \mathscr{F}$ defined by

$$h = \begin{cases} \infty & \text{where} \quad E(f+g) = \infty \\ E(f+g) - Eg & \text{where} \quad E(f+g) < \infty \end{cases}$$

is the largest solution of $h + Eg = E(f+g)$. Another solution is $Ph + f$, hence $Ph + f \leq h$, and therefore $Ef \leq h$. The second part of 3, and 4 follow from 1.

The following characterization of potential is useful and is needed in the proof of Theorem 2.2.

DÉFINITION 2.2. — A subset $\mathscr{S}$ of $\mathscr{F}$ is called an E-*class* for $f \in \mathscr{F}$ if:

i) $f \in \mathscr{S}$

ii) $g \in \mathscr{S} \Rightarrow Pg + f \in \mathscr{S}$

iii) $\{ g_\alpha \}_{\alpha \in \Delta} \subset \mathscr{S} \Rightarrow \sup_{\alpha \in \Delta} g_\alpha \in \mathscr{S}$.

Note that $\mathscr{F}$ is an E-class for any $f \in \mathscr{F}$, and that arbitrary intersections of E-classes for f are again E-classes for f. Denote by $\mathscr{E}_f$ the smallest E-class for f, i. e., the intersection of all E-classes for f.

LEMMA 2.1. — $\mathrm{E}f = \sup_{g \in \mathscr{E}_f} g$.

Proof. — According to the definition of E-classes, $\mathrm{P}(\sup_{g \in \mathscr{E}_f} g) + f \in \mathscr{E}_f$, hence $\mathrm{P}(\sup_{g \in \mathscr{E}_f} g) + f \leq \sup_{g \in \mathscr{E}_f} g$. By the definition of E$f$, $\mathrm{E}f \leq \sup_{g \in \mathscr{E}_f} g$. To establish the other direction, notice that the set $\mathscr{S} = \{ g \in \mathscr{E}_f \mid g \leq \mathrm{E}f \}$ is an E-class for f. By the minimality of $\mathscr{E}_f$, we have $\mathscr{E}_f = \mathscr{S}$. Therefore $\sup_{g \in \mathscr{E}_f} g \leq \mathrm{E}f$.

COROLLARY 2.1. — $\sum_{n=0}^{\infty} \mathrm{P}^n f \leq \mathrm{E}f$. *Equality holds if* P *is monotonically continuous* ($f_n \nearrow f \Rightarrow \mathrm{P}f_n \nearrow \mathrm{P}f$).

Proof. — $\sum_{n=0}^{\infty} \mathrm{P}^n f \in \mathscr{E}_f$. If P is monotonically continuous, then

$$\mathrm{P}\left(\sum_{n=0}^{\infty} \mathrm{P}^n f\right) + f = \sum_{n=0}^{\infty} \mathrm{P}^n f$$

and equality follows from the definition of Ef.

The following is an example where equality does not hold.

EXAMPLE 2.1. — Let $\mathrm{X} = \{0, 1, 2, \ldots\}$, let $\mathscr{A}$ be the family of all subsets of X, and let m be a probability measure having positive mass at each point of X. Let μ be a positive, purely finitely additive measure on $(\mathrm{X}, \mathscr{A})$ with $\mu(\mathrm{X}) = 1$. Define P by $\mathrm{P}g(x) = g(x-1)$ for $x > 0$, and $\mathrm{P}g(0) = \int g d\mu$. Let $f = 1_{\{\mathrm{N}\}}$ where $\mathrm{N} \in \mathrm{X}$. Then $\sum_{n=0}^{\infty} \mathrm{P}^n f = 1_{[\mathrm{N}, \infty)} \in \mathscr{E}_f$. After N applications of $\mathrm{P}(\cdot) + f$ we have $1 \in \mathscr{E}_f$, hence $1 \leq \mathrm{E}f$. Furthermore, $\mathrm{P}1 = 1$ and Theorem 2.3 will yield $\mathrm{E}f = \infty$.

THEOREM 2.2. — $\mathrm{PE}f = \mathrm{EP}f$.

Proof. — Clearly $\mathrm{EP}f \leq \mathrm{PE}f$. To establish the other direction, we wish to show that the set

$$\mathscr{S} = \{ g \in \mathscr{E}_f \mid \mathrm{P}g + f \geq g \text{ and } \mathrm{P}g \leq \mathrm{EP}f \}$$

is an E-class for f. Clearly *i*) and *ii*) of the definition hold. Let $\{ g_\alpha \}_{\alpha \in \Delta} \subset \mathscr{S}$.

Then

$$\mathrm{P}\,(\sup_{\alpha\in\Delta} g_\alpha) + f \geq \sup_{\alpha\in\Delta} \mathrm{P}g_\alpha + f = \sup_{\alpha\in\Delta} (\mathrm{P}g_\alpha + f) \geq \sup_{\alpha\in\Delta} g_\alpha\,.$$

We also have

$$\mathrm{P}\,(\sup_{\alpha\in\Delta} g_\alpha) \leq \mathrm{P}\sup_{\alpha\in\Delta} (\mathrm{P}g_\alpha + f) \leq \mathrm{P}\sup_{\alpha\in\Delta} (\mathrm{EP}f + f) = \mathrm{PEP}f + \mathrm{P}f = \mathrm{EP}f.$$

Hence *iii)* holds. By the minimality of $\mathscr{E}_f$, we have $\mathscr{S} = \mathscr{E}_f$ and, therefore, $\mathrm{PE}f \leq \mathrm{EP}f$.

THEOREM 2.3. — *Let* $\mathrm{P}g = g \leq \mathrm{E}f$. *Then* $g + \mathrm{E}f = \mathrm{E}f$.

Proof. — $h \in \mathscr{F}$ defined by

$$h = \begin{cases} \infty & \text{where} \quad \mathrm{E}f = \infty \\ \mathrm{E}f - g & \text{where} \quad \mathrm{E}f < \infty \end{cases}$$

is the largest solution of $h + g = \mathrm{E}f$. Another solution is $\mathrm{P}h + f$; hence $\mathrm{P}h + f \leq h$. Therefore $\mathrm{E}f \leq h$.

THEOREM 2.4 *(Domination Principle).* — *Let* $\mathrm{P}g \leq g$ *and let* $\mathrm{E}f \leq g$ *on* $\{ f > 0 \}$. *Then* $\mathrm{E}f \leq g$.

Proof. — Let $h = \min(\mathrm{E}f, g) = \min(\mathrm{E}f, g + f)$ by hypothesis. Furthermore $\mathrm{P}h + f \leq h$ and $\mathrm{E}f \leq h$.

COROLLARY 2.2. — *Let* $\mathrm{P}1 \leq 1$. *Then :*

1. *(Maximum Principle). If* $\mathrm{E}f \leq \mathrm{E}g + a$ *on* $\{ f > 0 \}$ *where* $0 \leq a \in \mathrm{R}$, *then* $\mathrm{E}f \leq \mathrm{E}g + a$.
2. *If* $\mathrm{E(A)} \leq a \in \mathrm{R}$ *on* A, *then* $\mathrm{E(A)} \leq a$.
3. *If* $\mathrm{E(A)} < \infty$, *then there exist sets* $\mathrm{A}_n \nearrow \mathrm{A}$ *such that* $\mathrm{E}(\mathrm{A}_n) \leq n$.

Proof. — Parts 1 and 2 are immediate. To establish 3, let $\mathrm{A}_n = \mathrm{A} \cap \{ \mathrm{E(A)} \leq n \}$. Then $\mathrm{E}(\mathrm{A}_n) \leq \mathrm{E(A)} \leq n$ on A_n. Apply part 2.

THEOREM 2.5 *(Riesz Decomposition).* — *If* $\mathrm{P}g \leq g$, *then there exist* f, $h \in \mathscr{F}$ *such that* $\mathrm{P}h = h$ *and* $g = \mathrm{E}f + h$. *Furthermore,* f *satisfies* $\mathrm{P}g + f = g$; *also, we can take for* f *any solution of* $\mathrm{P}g + f = g$.

Proof. — If $\mathrm{P}h = h$ and $g = \mathrm{E}f + h$, then $\mathrm{P}g + f = g$. Let f be any solution to $\mathrm{P}g + f = g$, e. g., $f = \infty$ where $g = \infty$ and $f = g - \mathrm{P}g$ where $g < \infty$. Clearly $\mathrm{E}f \leq g$. $k \in \mathscr{F}$ defined by

$$k = \begin{cases} 0 & \text{where} \quad \mathrm{E}f = \infty \\ g - \mathrm{E}f & \text{where} \quad \mathrm{E}f < \infty \end{cases}$$

is the smallest function satisfying $k + Ef = g$. Another solution is $Pk \geq k$. Notice that $k \leq g$. Let $\mathcal{S} = \{ j \in \mathcal{F} \mid j \leq Pj \text{ and } j \leq g \}$ and let $h = \sup_{j \in \mathcal{S}} j$. Clearly $h \geq k$ and $h + Ef \geq g$. Furthermore $Ph \geq h$ and $Ph \in \mathcal{S}$, hence $Ph = h$.

To establish that $h + Ef \leq g$, define $k_1 \in \mathcal{F}$ by

$$k_1 = \begin{cases} \infty & \text{where} \quad g = \infty \\ g - h & \text{where} \quad g < \infty \end{cases}.$$

Clearly k_1 is the largest function satisfying $k_1 + h = g$. Another solution $Pk_1 + f \leq k_1$ and therefore $Ef \leq k_1$.

COROLLARY 2.3. — *If* $Pg \leq g \leq Ef < \infty$, *then* $g = Eh$ *for some* $h \in \mathcal{F}$.

Proof. — $g = Eh + j$ where $Pj = j$. By Theorem 2.3 we have $Ef + j = Ef < \infty$ and, hence, $j = 0$.

COROLLARY 2.4. — *Let* $Pg \leq g$ *and let* $\lim_{n \to \infty} P^n g = 0$ *on* $\{ g < \infty \}$. *Then* $g = Ef$ *for some* $f \in \mathcal{F}$.

Proof. — Define f so that $Pg + f = g$ and $f = \infty$ where $g = \infty$. Then $g = Ef + h$ and $Ph = h$. $h = \lim_{n \to \infty} P^n h \leq \lim_{n \to \infty} P^n g = 0$ on $\{ g < \infty \}$. On $\{ g = \infty \}$ we have $Ef \geq f = \infty$.

THEOREM 2.6 *(Balayage)*. — *Let* $Ef < \infty$ *and* $A \in \mathcal{A}$. *Then there exists a unique* $g \in \mathcal{F}$ *such that* $Eg \leq Ef$, g *is supported on* A, *and* $Eg = Ef$ *on* A.

Proof. — Let $\mathcal{S} = \{ j \in \mathcal{F} \mid j = Ef \text{ on A, and } Pj \leq j \}$. Clearly $Ef \in \mathcal{S}$. Let $h = \inf_{j \in \mathcal{S}} j$. Clearly $Ph \leq h \leq Ef$ and $h = Ef$ on A. By Corollary 2.3, we have $h = Eg$ for some $g \in \mathcal{F}$. Let $g_1 = 1_A g$, $g_2 = 1_{A^c} g$ and let $g_3 = g_1 + Pg_2$. Then, using Theorem 2.2,

$$g_2 + Eg_3 = g_2 + Eg_1 + EPg_2 = g_2 + PEg_2 + Eg_1 = Eg_2 + Eg_1 = Eg = h.$$

Therefore $Eg_3 = Eg = Ef$ on A and, by minimality of h, $Eg \leq Eg_3$. Hence $g_2 + Eg \leq g_2 + Eg_3 = Eg < \infty$ and $g_2 = 0$, i. e. g is supported on A. To establish uniqueness, let g' also be supported on A with $Eg' = Ef$ on A. By Theorem 2.4, $Eg' = Eg < \infty$ and $g = g'$ follows from Theorem 2.1(4).

Remark. — Previous abstract studies of potential operators assumed P to be monotonically continuous. See [*1*] and [7, Chapter IX].

3. THE ERGODIC DECOMPOSITION

Throughout this section we will assume that P is a positive linear $L_\infty(X, \mathscr{A}, m)$ contraction. We wish to extend P to $\mathscr{F}$ in order to apply the potential operator E. One natural extension is given by $Pf = \sup_{f \geq g \in L_\infty} Pg$. This is the smallest possible extension and has the sometimes useful property that $P \infty 1_A = \infty P 1_A$. The largest possible extension is given by $Pf = \infty 1_X$ for all $f \in \mathscr{F} - L_\infty$. Additional extensions can be generated from any pair of extensions P_1 and P_2 by $1_A P_1 + 1_{A^c} P_2$ for any $A \in \mathscr{A}$. We will assume for the remainder of this section that P has been extended to $\mathscr{F}$, but we will make no restriction concerning the particular choice of the extension.

DÉFINITION 3.1. — *The dissipative part of* X *is* $D = \cup \{ A \in \mathscr{A} \mid E(A) < \infty \}$. *The conservative part of* X *is* $C = X - D$.

An immediate application of Corollary 2.2 gives us $D = \bigcup_{E(A) \in L_\infty} A$. Hence, for any positive linear contraction of L_∞, the dissipative part does not depend on the particular extension of the operator to $\mathscr{F}$.

THEOREM 3.1. — *Let* $\{ f > 0 \} \subset C$. *Then* Ef *takes only the values* 0 *or* ∞.

Proof. — First we wish to show that $Ef = \infty$ on $\{ f > 0 \}$. Let $a \in R$, and, for each $n \in Z^+$, let $A_n = \{ Ef \leq a \} \cap \left\{ f \geq \frac{1}{n} \right\}$. Then $1_{A_n} \leq nf$ and $E(A_n) \leq E(nf) \leq na$ on A_n. By Corollary 2.2 we have $E(A_n) \leq na$. However, $A_n \subset C$ by hypothesis. Hence $A_n = \emptyset$. Also

$$\{ Ef \leq a \} \cap \{ f > 0 \} = \bigcup_{n=1}^{\infty} A_n = \emptyset$$

and $Ef = \infty$ on $\{ f > 0 \}$.

Let

$$g = \begin{cases} \infty & \text{where} \quad Ef = \infty \\ 0 & \text{elsewhere} \end{cases}.$$

Clearly $g \leq Ef$. Also $Pg \leq PEf \leq Ef$. Notice that Pg can take only the values 0 or ∞ since, for any $0 \leq b \in R$, $bPg = Pbg = Pg$. Hence $Pg \leq g$. Furthermore, $Pg + f \leq g$ because $Ef = \infty$ on $\{ f > 0 \}$. Therefore $Ef \leq g$ and $Ef = g$.

The following corollary gives us the classical Hopf ergodic decomposition.

COROLLARY 3.1. — *Let* P *be generated by the dual of a positive* L_1 *contraction* T, *and let* $u \in L_1^+$. *Then :*

$$\sum_{n=0}^{\infty} T^n u = 0 \text{ or } \infty \qquad \text{on } C$$

$$\sum_{n=0}^{\infty} T^n u < \infty \qquad \text{on } D .$$

Proof. —

$$\int_A \sum_{n=0}^{\infty} T^n u dm = \int_X \left(\sum_{n=0}^{\infty} T^n u\right) 1_A dm = \int_X \left(\sum_{n=0}^{\infty} P^n 1_A\right) u dm = \int_X E(A) u dm .$$

This is 0 or ∞ for all $A \subset C$, and bounded above by $n \int_X udm$ for $A = A_n$ of Corollary 2.2.

COROLLARY 3.2. — *If* $Pg \leq g$ *then* $Pg = g$ *on* C.

Proof. — Let $A_n = C \cap \left\{ Pg + \frac{1}{n} < g \right\}$. Then $Png + 1_{A_n} \leq ng$, and, by Theorem 2.1(2), $E(A_n) \leq Png + 1_{A_n}$, which is strictly less than ∞ on A_n by the definition of A_n. According to Theorem 3.1, $A_n = \emptyset$. Hence

$$C \cap \{ Pg < g \} = \bigcup_{n=0}^{\infty} A_n = \emptyset .$$

THEOREM 3.2. — Ef *can take only the values* 0 *or* ∞ *on* C.

Proof. — Let $\mathcal{S} = \{ g \in \mathcal{F} \mid g = Ef \text{ on C, and } Pg \leq g \}$. Clearly $Ef \in \mathcal{S}$. Let $h = \inf_{g \in \mathcal{S}} g$. Clearly $h \in \mathcal{S}$. Also $P(Ph) \leq Ph$, and $Ph = h = Ef$ on C by Corollary 3.2; hence $Ph \in \mathcal{S}$. By minimality of h we have $Ph = h \leq Ef$. Theorem 2.3 gives us $Ef + h = Ef$, and $2Ef = Ef$ on C.

The following theorem is a summary of the previous theorems.

THEOREM 3.3. — *The following are equivalent:*

1. $A \subset C$.
2. $Pg \leq g \in \mathcal{F} \Rightarrow Pg = g$ *on* A.

3. *$f \in \mathscr{F}$ and $\{ f > 0 \} \subset A \Rightarrow$ Ef takes only the values 0 or ∞.*
4. *For $f \in \mathscr{F}$, Ef can take only the values 0 or ∞ on A.*

Furthermore, the theorem remains true if $\mathscr{F}$ is replaced by L_∞ in 2, and if $\mathscr{F}$ is replaced by L_∞^+ in 3 and 4.

Proof. — 1 $\Rightarrow$ 2, 3 and 4 by previous theorems.

2 $\Rightarrow$ 2(L_∞). Let $Pg \leq g \in L_\infty$. Then $P(g + \| g \|) \leq g + \| g \| \in \mathscr{F}$. By 2, $P(g + \| g \|) = g + \| g \|$ on A, and also $P1 = 1$ on A. Hence $Pg = g$ on A.

3 $\Rightarrow$ 3(L_∞^+) and 4 $\Rightarrow$ 4(L_∞^+). Obvious.

$\sim 1 \Rightarrow \sim 2(L_\infty)$, $\sim 3(L_\infty^+)$, and $\sim 4(L_\infty^+)$. Let $A \cap D \neq \emptyset$. By Corollary 2.2, there exists a set B such that $\phi \neq B \subset A \cap D$ and $E(B) \in L_\infty$. Let $g = E(B)$ and $f = 1_B$.

COROLLARY 3.3. — *For $n \in Z^+$, P and P^n have the same ergodic decompositions.*

Proof. — Let C and C_n be the conservative parts of P and P^n respectively, and apply part 2(L_∞) of Theorem 3.3. If $Pg \leq g \in L_\infty$, then $P^n g \leq g$, $P^n g = g$ on C_n, and $Pg = g$ on C_n; hence $C_n \subset C$. If $P^n g \leq g \in L_\infty$, then $P\left(\sum_{i=0}^{n-1} P^i g\right) = \sum_{i=1}^{n} P^i g \leq \sum_{i=0}^{n-1} P^i g$, equality holds on C, and $P^n g = g$ on C by cancellation; hence $C \subset C_n$.

Remark. — A similar decomposition for a transition probability on a topological space is given in [*4*] [*5*] and [*6*].

We will now give some examples which are not treated by the classical theory, and which will illustrate some of the unexpected things that can occur. In each case (X, $\mathscr{A}$, *m*) and μ will be the same as in Example 2.1, and $P1 = 1$.

EXAMPLE 3.1. — Define P by $Pf(x) = f(0)$ for $x \geq 1$, and $Pf(0) = \int f d\mu$. It is easy to show that $C = \{ 0 \}$ and $D = \{ 1, 2, 3, \ldots \}$. Therefore $P1_C = 1_D$ and $P1_D = 1_C$. Furthermore the averages $\frac{1}{N} \sum_{n=0}^{N} P^n f$ converge uniformly for all $f \in L_\infty$, but there is no finite invariant (countably additive) measure.

EXAMPLE 3.2. — Same as Example 3.1 except replace μ by $\frac{1}{2}\mu + \frac{1}{2}m$.

Then $X = C$, the averages $\frac{1}{N}\sum_{n=0}^{N} P^n f$ converge uniformly for all $f \in L_\infty$, for non-negative $f \neq 0$ the limit is not 0, but still there is no finite invariant measure.

EXAMPLE 3.3. — Define P by $Pf(x) = f(x-1)$ for $x \geq 2$, $Pf(1) = f(1)$ and $Pf(0) = \int f d\mu$. In this example $\frac{1}{N}\sum_{n=0}^{N} P^n 1_{\{1\}}$ converges pointwise to $1_{\{1,2,\ldots\}}$ which is not invariant ($P1_{\{1,2,\ldots\}} \neq 1_{\{1,2,\ldots\}}$).

EXAMPLE 3.4. — Define P by $Pf(x) = f(x)$ for $x \geq 1$, and $Pf(0) = \int f d\mu$. Let $A_n = [1, n]$, so that $A_n \nearrow A = [1, \infty)$. Then $P1_{A_n} = 1_{A_n}$, but $P1_A \neq 1_A$.

Example 3.4 shows that the invariant sets $\Sigma_i = \{ A \in \mathscr{A} \mid P1_A = 1_A \}$, which form a ring (see [2, p. 8]), do not necessarily form a σ-ring, even if they form an algebra. However, in the conservative case, the invariant sets are more conventional.

THEOREM 3.4. — *Let* $X = C$. *Then:*

1. Σ_i *is a* σ*-algebra.*
2. $Pf = f \in \mathscr{F} \Rightarrow f$ *is* Σ_i *measurable.*
3. $f \in L_\infty$ *is* Σ_i *measurable* $\Rightarrow Pf = f$.
4. $f \in \mathscr{F}$ *is* Σ_i *measurable, and* $P \infty 1_A = \infty P1_A$ *for all* $A \in \mathscr{A} \Rightarrow Pf = f$.

Proof. — 1. $P1 = 1$ by conservativity. Let $A_n \in \Sigma_i$, $A_n \searrow A$. Then $\inf_n 1_{A_n} = 1_A$ and $P1_A \leq 1_A$. Equality holds by conservativity and $A \in \Sigma_i$.

2. Let $Pf = f \in \mathscr{F}$. Notice that if $Pf_\alpha = f_\alpha$, $\alpha \in \Delta$, then $P \inf_{\alpha\in\Delta} f_\alpha \leq \inf_{\alpha\in\Delta} f_\alpha$ and equality holds by conservativity. For $a \in R^+$, let $h = \min(f, a) \in L_\infty$ and let $g = f - h \in \mathscr{F}$. $Ph = h$ and $Pg = g$. Clearly $\{ g > 0 \} = \{ f > a \}$. Let $g_n = \min(1, ng)$. Then $g_n \nearrow 1_{\{f > a\}}$ and $(1 - g_n) \searrow 1_{\{f \leq a\}}$ with $P(1 - g_n) = 1 - g_n$. Therefore $\{ f \leq a \} \subset \Sigma_i$.

3. This is clear by approximation.

4. Let $f \in \mathscr{F}$ be Σ_i measurable and let $P \infty 1_A = \infty P1_A$ for all $A \in \mathscr{A}$. Then $P(\infty 1_{\{f>n\}}) = \infty P1_{\{f>n\}} = \infty 1_{\{f>n\}}$. By 3, $\min(f, n)$ is invariant under P. Therefore $Pf_n = f_n$ where $f_n = \min(f, n) + \infty 1_{\{f>n\}}$. Finally $f = \inf_n f_n$, hence $Pf = f$.

COROLLARY 3.4. — *Let* $P \infty 1_A = \infty P1_A$ *for all* $A \in \mathscr{A}$ *and let* $Ef = \infty$ *for all* $f \in \mathscr{F}$, $f \neq 0$. *Then* $Pf = f \Rightarrow f = $ *constant.*

Proof. — Clearly $X = C$. Let $A \in \Sigma_i$. Then $P \infty 1_A + 1_A = \infty 1_A$ and $E(A) \leq \infty 1_A$. By hypothesis, $A = X$ or $\emptyset$. Therefore Σ_i is trivial, and, by Theorem 3.4, f is Σ_i measurable.

As an application we obtain a result on eigenvalues. P can be extended to a contraction of the complex $L_\infty : P(g_1 + ig_2) = Pg_1 + iPg_2$. By identifying L_∞ as a C(K) space it is known that if $Pf_j = \lambda_j f_j$, $|\lambda_j| = 1$ and $|f_j| \equiv 1$ for $j = 1, 2$, then $P(f_1 f_2) = \lambda_1\lambda_2 f_1 f_2$ (see [9]).

THEOREM 3.5. — *Let* $X = C$. *If* $Pf_j = \lambda_j f_j$, $|\lambda_j| = 1$ *for* $f_j \in L_\infty, j = 1, 2$, *then* $P(f_1 f_2) = \lambda_1\lambda_2 f_1 f_2$.

Proof. — Let $Pf_j = \lambda_j f_j$, $|\lambda_j| = 1$ and $f_j \in L_\infty$ for $j = 1, 2$.

$$P|f_j| \geq |Pf_j| = |\lambda_j f_j| = |f_j|.$$

By conservativity $P|f_j| = |f_j|$ (consider $\|f_j\|_\infty - |f_j|$) and $|f_j|$ is Σ_i measurable. Notice that for $A \in \Sigma_i$ and $g \in L_\infty$ we have $P1_A g = 1_A Pg$. On $A = \{ |f_1||f_2| = 0 \} \in \Sigma_i$ we clearly have $P(f_1 f_2) = 0 = \lambda_1\lambda_2 f_1 f_2$. We will restrict our attention to the invariant set $A = \{ |f_1||f_2| > 0 \}$, so we may assume $|f_1||f_2| > 0$. a. e. We will make use of another property of Σ_i, obtained by approximation: if $g_1, g_2 \in L_\infty$ and g_1 is Σ_i measurable, then $P(g_1 g_2) = g_1 Pg_2$. Setting $g_1 = |f_j|$ and $g_2 = \dfrac{f_j}{|f_j|}$ yields that $\dfrac{f_j}{|f_j|}$ is a unimodular eigenfunction for P with eigenvalue λ_j. Let $g_1 = |f_1||f_2|$ and $g_2 = \dfrac{f_1}{|f_1|}\dfrac{f_2}{|f_2|}$ and apply the result of [9] to complete the proof.

ACKNOWLEDGMENT

The author wishes to thank Professor Michael Lin for his very helpful suggestions and for his advice during the preparation of the manuscript.

REFERENCES

[1] J. DENY, Les noyaux élémentaires, Séminaire de Théorie du Potential (directed by M. BRELOT, G. CHOQUET, AND J. DENY), *Institut Henri Poincaré*, Paris, 4e année, 1959-1960.

[2] S. R. FOGUEL, *The Ergodic Theory of Markov Processes*. New York, Van Nostrand Reinhold, 1969.

[3] S. R. FOGUEL, More on « The Ergodic Theory of Markov Processes ». *University of British Columbia Lecture Notes*, Vancouver, 1973.

[4] S. R. FOGUEL, Ergodic Decomposition of a Topological Space. *Israel J. Math.*, t. **7**, 1969, p. 164-167.

[5] S. HOROWITZ, Markov Processes on a Locally Compact Space. *Israel J. Math.*, t. **7**, 1969, p. 311-324.
[6] M. LIN, Conservative Markov Processes on a Topological Space. *Israel J. Math.*, t. **8**, 1970, p. 165-186.
[7] P. A. MEYER, *Probability and Potentials*. Waltham, Massachusetts, Blaisdell Publishing Company, 1966.
[8] J. NEVEU, *Mathematical Foundations of the Calculus of Probability*. San Francisco, Holden-Day, 1965.
[9] H. H. SCHAEFER, Invariant Ideals of Positive Operators in C(X), I. *Illinois J. Math.*, t. **11**, 1967, p. 701-715.

(Manuscrit reçu le 31 octobre 1975)