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On Choquet-Deny measures

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ABSTRACT. — Let μ be a probability measure on a group G; we give necessary and sufficient conditions for μ to be a Choquet-Deny measure (i. e. for μ to admit only constants as μ-harmonic functions). Most of the conditions are given in terms of the iterates of μ.

RÉSUMÉ. — Soit μ une probabilité sur un groupe G. Nous nous donnons des conditions nécessaires et suffisantes pour qu'elles n'admettent que les fonctions constantes comme fonctions harmoniques. Ces conditions sont essentiellement données à l'aide des itérés des μ.

1. INTRODUCTION

Let G be a locally compact topological group, λ a left Haar measure on G. Let M be the Banach algebra of bounded real measures on G with the total variation norm and the operation of convolution defined by

$$(\mu * \nu)(f) = \iint f(gh)d\mu(g)d\nu(h).$$

Here f is a continuous function on G which vanishes at infinity. Let M_a be the two sided ideal of M which consists of all absolutely continuous measures (with respect to λ), and let M_a^0 be the subideal of measures ν with the property $\nu(G) = 0$. For a measure $\mu \in M$ we let $\|\mu\|$ be its total variation.

A probability measure μ on G is called *aperiodic* if its support generates

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a dense subgroup of G. It is called *strictly aperiodic* if the support of μ is not contained in a coset of a proper closed normal subgroup of G. μ is *étalée* [1] if for some positive integer n, $\mu^n = \mu * \ldots * \mu$ is not singular with respect to λ. It is easy to see that this is equivalent to the existence of a k such that μ^k dominates a positive constant multiple of λ on a non-empty open subset of G. If μ is étalée we let S_μ be the open semigroup of elements $g \in G$ such that for some k, μ^k dominates a positive constant multiple of λ on a neighbourhood of g. A bounded real valued measurable function f on G is called *μ-harmonic* if for every $g \in G$

$$(f * \mu)(g) = \int f(gh)d\mu(h) = f(g).$$

If μ is étalée and $f \in L_\infty(\lambda)$ satisfies $f * \mu = f$ a. e. λ then it is easy to see that there exists a function f' which is in the equivalence class of f in $L_\infty(\lambda)$ and such that $f' * \mu = f'$ everywhere; i. e., f' is μ-harmonic. Moreover f' is necessarily continuous.

We say that μ is a *Choquet-Deny* (C. D.) *measure* if the only μ-harmonic functions are the constants. G is a C. D. *group* if every aperiodic probability measure on G is C. D. G is a *Liouville group* if every étalée aperiodic probability measure on G is C. D. [5].

We say that μ satisfies the *condition* (F) *with the positive integer k* if for some positive integer n the measures μ^n and μ^{n+k} are not mutually singular. The following result is due to S. R. Foguel [3].

Let μ be a probability measure on G *satisfying condition* (F) *with the positive integer k; then for $\nu \in M_a^0$*

$$\lim \| \nu * \mu^n \| = 0$$

iff $\langle \nu, f \rangle = \int f(g)d\nu(g) = 0$ *for every* $f \in L_\infty(\lambda)$ *satisfying* $f * \mu^k = f$ *a. e.* λ.

We will see that in many cases the assumption « μ satisfies condition (F) » is redundant. Let us write P(G) for the set of probability measures on G.

2. THE ITERATES OF μ ON M_a^0

PROPOSITION 1. — *If $\mu \in P(G)$ is strictly aperiodic étalée measure for which $S_\mu S_\mu^{-1} = G$, then μ satisfies condition* (F) *with* $k = 1$.

Proof. — Let S_l be the set of elements $g \in G$ such that μ^l dominates a positive constant multiple of λ on some neighbourhood of g; since μ is étalée there exists an l for which $S_l \neq \emptyset$. Let l_0 be the minimal l with this property. For $k, l \geqslant l_0$ we have $S_k S_l \subseteq S_{k+l}$. Put $S = \bigcup_{l \geqslant l_0} S_l$ then $S = S_\mu$ is an open subsemigroup of G. By our assumption $SS^{-1} = G$.

If, for some positive integers n and k, $S_n \cap S_{n+k} \neq \emptyset$ then μ^n and μ^{n+k} are not mutually singular and μ satisfies condition (F) with a positive integer less than or equal to k. Thus, if μ does not satisfy condition (F) with $k = 1$ then one of the following cases occurs.

Case I : $S_l \cap S_k = \emptyset$ whenever $l \neq k$.

Case II : There exist positive integers n_0 and $k_0 > 1$ such that $S_{n_0+k_0} \cap S_{n_0} \neq \emptyset$ and whenever $n, m \geqslant l_0$, $0 < |n - m| < k_0$ then $S_n \cap S_m = \emptyset$.

In the first case we let, for $i \in Z$ (= integers)

$$T_i = \cup \{ S_k S_l^{-1} : k - l = i \,;\, k, l \geqslant l_0 \}$$
$$T'_i = \cup \{ S_l^{-1} S_k : k - l = i \,;\, k, l \geqslant l_0 \}$$

Next we show that (a) $T_i^{-1} = T_{-i}$, (b) $T'_i \subseteq T_i$, (c) $T_i T_j \subseteq T_{i+j}$, ($d$) $i \neq j$ implies $T_i \cap T_j = \emptyset$ and (e) $G = \cup \{ T_i : i \in Z \}$.

(a) Is clear. To show (b) let $s_l \in S_l$ and $s_k \in S_k$ where $k - l = i$. Then since $S_l^{-1} S_k \subseteq S\,S^{-1} = G$, there are p and q and $s_p \in S_p$, $s_q \in S_q$ such that $s_l^{-1} s_k = s_p s_q^{-1}$. This implies $s_k s_q = s_l s_p$ and $S_{q+k} \cap S_{p+l} \neq \emptyset$. Hence $q + k = p + l$ or $i = k - l = p - q$. Thus $T'_i \subseteq T_i$. (c) If $a \in T_i$ and $b \in T_j$ then $a = s_k s_l^{-1}$, $b = s_p s_q^{-1}$ where $k - l = i$ and $p - q = j$. Since $T'_{p-l} \subseteq T_{p-l}$ we have $s_l^{-1} s_p = s_u s_v^{-1}$ for some u and v such that $u - v = p - l$. Now

$$ab = s_k s_l^{-1} s_p s_q^{-1} = s_k s_u s_v^{-1} s_q^{-1} \in S_{k+u} S_{v+q}^{-1} \subseteq T_{k+u-(v+q)} = T_{k+p-l-q} = T_{i+j} .$$

Thus $T_i T_j \subseteq T_{i+j}$. ($d$) is proved similarly and (e) follows from the equality $SS^{-1} = G$.

In the second case, for $p = l_0, l_0 + 1, \ldots, l_0 + k_0 - 1$, let

$$R_p = \cup \{ S_{p+nk_0} : n \text{ a non negative integer.} \}$$

We claim that for $p \neq q$ $R_p \cap R_q = \emptyset$. Indeed, if $R_p \cap R_q \neq \emptyset$ then $S_u \cap S_v \neq \emptyset$ for some $u > v \geqslant l_0$ such that $u - v \not\equiv 0 \pmod{k_0}$. Denote $V = S_{n_0+k_0} \cap S_{n_0}$ then for every $m > 1$

$$V^m \subseteq S_{m(n_0+k_0)} \cap S_{mn_0} .$$

By choosing an appropriate m we can have

$$0 < (u + mn_0) - (v + m(n_0 + k_0)) = u - (v + mk_0) < k_0$$

Since $V^m(S_u \cap S_v) \subseteq S_{u+mn_0} \cap S_{v+m(n_0+k_0)}$, this is a contradiction to the definition of k_0. We now define for $i \in \{0, 1, \ldots, k_0 - 1\}$

$$T_i = \cup \{ R_p R_q^{-1} : p - q = i \}$$
$$T_i' = \cup \{ R_q^{-1} R_p : p - q = i \}$$

If we let $Z_{k_0} = \{0, 1, \ldots, k_0 - 1\}$ be the cyclic group of order k_0 and consider i and j as elements of this group then statements (*a*)-(*e*) above (where in (*e*) Z should be replaced by Z_{k_0}) still hold and the proofs are very similar.

In both cases, using (*a*) and (*c*) with $i = j = 0$, we see that T_0 is a subgroup of G. Moreover, for every $x \in G$ there exists an i such that $x \in T_i$, and therefore

$$x T_0 x^{-1} \subseteq T_i T_0 T_{-i} \subseteq T_0$$

Thus T_0 is an open and closed, normal subgroup of G.

To complete the proof we let, for $k \geqslant l_0$, $\mu^k = \eta^{(k)} + \theta^{(k)}$ where $\eta^{(k)}$ is absolutely continuous and $\theta^{(k)}$ is singular with respect to λ. Clearly, $\eta^{(k)}(S_k) > 0$ and moreover, if $\eta^{(k)}(S_l) > 0$ then $(\eta^{(k)})^2$ and hence also μ^{2k} dominate a positive constant multiple of λ on an open non-empty subset of $S_{k+l} \supseteq S_k S_l$. This implies $S_{k+l} \cap S_{2k} \neq \emptyset$ and we conclude that $k = l$ in case I and that $k \equiv l \pmod{k_0}$ in case II. Thus $\eta^{(k)}$ is supported by S_k in the first case and by R_p, where p is the unique integer for which $S_k \subseteq R_p$, in the second case.

Since, in the first case, $S_k \subseteq S_k T_0 = T_k$ and in the second $R_p \subseteq R_p T_0 = T_{\bar{k}}$ (where $\bar{k} \in \{0, 1, \ldots, k_0 - 1\}$ is the residue of p, and hence also of k, modulo k_0) we can deduce that for every k, $\eta^{(k)}$ is supported by T_k ($T_{\bar{k}}$ respectively).

Now

$$\mu^{2k} = (\eta^{(k)} + \theta^{(k)})^2 = (\eta^{(k)})^2 + \eta^{(k)} * \theta^{(k)} + \theta^{(k)} * \eta^{(k)} + (\theta^{(k)})^2$$

and

$$\eta^{(2k)} \geqslant (\eta^{(k)})^2 + \eta^{(k)} * \theta^{(k)} + \theta^{(k)} * \eta^{(k)} .$$

If $\theta^{(k)}(T_j) > 0$ (where $j \in Z$ in case I and $j \in Z_{k_0}$ in case II) then

$$\theta^{(k)} * \eta^{(k)}(T_{j+k}) > 0$$

($\theta^{(k)} * \eta^{(k)}(T_{j+\bar{k}}) > 0$ respectively). But this implies $j = k$ ($\overline{2k} = j + \bar{k}$ and hence $j = \bar{k}$ respectively). We conclude that $\theta^{(k)}$ and therefore also μ^k are supported by T_k ($T_{\bar{k}}$ respectively). Since the latter is a coset of T_0 in G this contradicts the strict aperiodicity of μ. The proof is completed.

Remarks. — (1) The assumption « μ is strictly aperiodic » can be dropped in proposition 1.1, if G is a connected group, or more generally, if G does not admit a nontrivial cyclic group as a factor.

(2) If μ is étalée and C. D. then by [*1*, prop. IV.3, p. 83] $S_\mu S_\mu^{-1} = G$.

THEOREM 2. — *Let* G *be a locally compact topological group. A strictly aperiodic étalée measure* μ *in* P(G) *is* C. D. *iff*

$$\lim \| \nu * \mu^n \| = 0 \qquad \forall \nu \in M_a^0 \tag{1}$$

In particular, G *is Liouville iff* (1) *is satisfied by every strictly aperiodic étalée measure.*

Proof. — If (1) is satisfied by a probability measure μ and f is μ-harmonic then

$$\langle \nu, f \rangle = \langle \nu, f * \mu^n \rangle = \langle \nu * \mu^n, f \rangle \to 0$$

Therefore, $\langle \nu, f \rangle = 0$ for every $\nu \in M_a^0$ and f is a constant. Conversely, if μ is strictly aperiodic étalée and C. D. then $S_\mu S_\mu^{-1} = G$ and by proposition 1, μ satisfies condition (F) with $k = 1$. Now (1) follows from Foguel's theorem.

To complete the proof we have to show that if (1) holds for every étalée strictly aperiodic μ then G is Liouville. Indeed if μ is étalée aperiodic and f is μ-harmonic then $\mu' = \Sigma(1/2^n)\mu^n$ is étalée, strictly aperiodic and f is also μ'-harmonic. Thus by our assumption f must be a constant and G is Liouville; the proof is completed.

Remarks. — (1) Let us observe that if $\Delta = (a, b)$ is an open interval of the real line, then there always are n and k, positive integers, such that $n\Delta \cap (n + k)\Delta \neq \emptyset$ ($l\Delta = \Delta + \ldots + \Delta$, l times). Indeed, we have to consider only the case $a > 0$ and in that case we can choose $k = 1$ and n such that $n(b - a) > a$. For then $na < (n + 1)a < nb$. It follows that whenever S_μ intersects a one parameter subgroup of G then μ satisfies condition (F). For example, if G is a simply connected solvable Lie group, then the image of the exponential map is dense in G (see [*2*], theorem 2) and we can conclude that every étalée probability measure on a connected solvable Lie group satisfies condition (F) with $k = 1$.

(2) Let G be the free group on two generators a and b; then it is easy to see that the probability measure $\mu(a) = \mu(b) = \mu(ab^{-1}a^2) = \frac{1}{3}$ is strictly aperiodic, étalée and does not satisfy condition (F).

(3) Let $\mu \in P(G)$ be a strictly aperiodic, étalée and C. D. and let n be a positive integer. Let V be the space of μ^n-harmonic functions and denote by P the operator $Pf = f * \mu$. If $Q = I + P + \ldots + P^{n-1}$ and we put $W = V + iV$, the complexification of V, then QW is the one-dimensional

space of constant functions. If W is more than one-dimensional then there exists a non-constant function $f \in W$ such that $Pf = \alpha f$ for α an n^{th} root of unity. By remark (2) above and proposition 1, μ satisfies condition (F) with $k = 1$ and as was observed in [*4*] this implies $\alpha = 1$, a contradiction to the fact that μ is C. D. Thus μ^n is C. D. for each positive integer n (Actually it can be shown that this conclusion holds without the assumption that μ is étalée).

3. THE ITERATES OF μ ON SPACES OF CONTINUOUS FUNCTIONS

We let C be the space of all bounded continuous functions on G. For $f \in C$ and $g \in G$ we define the functions $l_g(f) = {}_gf$ and $r_g(f) = f_g$ as follows:

$$ {}_gf(h) = f(gh) \qquad \text{and} \qquad f_g(h) = f(hg) \qquad (h \in G). $$

The function f is *left uniformly continuous* (l. u. c.) if whenever $g_i \to e$ is a convergent net in G then $\| {}_{g_i}f - f \|_\infty \to 0$, where $\| f \|_\infty = \sup_{g \in G} | f(g) |$.

Wet Let L be the Banach algebra of all l. u. c. functions. The space R of all *right uniformly continuous* functions is defined similarly; we denote $U = R \cap L$. C, R and L are invariant under both r_g and $l_g (g \in G)$. Write $C_l^0 (C_r^0)$ for the closed subspace of C which consists of all functions which vanishes under alleft (right) invariant means on C. $L_l^0, L_r^0, R_l^0, R_r^0$ are defined similarly. When G is non-amenable these subspaces coincide with the whole space. Let $| U |$ stand for the maximal ideal space of U.

If $\mu \in P(G)$ and $f \in C$ we define

$$ (\mu * f)(g) = \int f(g'g) d\mu(g') $$

$$ (f * \mu)(g) = \int f(gg') d\mu(g') $$

One can check that each of the spaces C, R, L and U is invariant under both right and left convolution with μ. Notice that if $\nu \in P(G)$ and $f \in C$ then $(\mu * \nu) * f = \nu * (\mu * f)$.

If we write $\tilde{f}(g) = f(g^{-1})$ then the map $f \to \tilde{f}$ is an isometric involutive isomorphism of R onto L and

$$ \mu * f = \tilde{f} * \tilde{\mu} \qquad \text{where} \qquad \int f(g) d\tilde{\mu}(g) = \int f(g^{-1}) d\mu(g). $$

For $\mu \in P(G)$ we let

$$ J_\mu = \{ f \in C : \| \mu^n * f \|_\infty \to 0 \}, $$
$$ K_\mu = \{ f \in C : \| f * \mu^n \|_\infty \to 0 \}. $$

It was shown in [4] that if G is abelian and μ is strictly aperiodic then $U_l^0 = U_r^0 = K_\mu \cap U$. Next we shall see how this theorem can be extended to the non-abelian case when μ is étalée.

LEMMA 3. — *Let* $\mu \in P(G)$, *if* $L_l^0 \subseteq K_\mu$ *then* μ *is* C. D.

Proof. — Suppose $f \in L$ is μ-harmonic, then so is ${}_gf$. Hence $({}_gf - f) * \mu = {}_gf - f$. Now by our assumption $\| ({}_gf - f) * \mu^n \|_\infty \to 0$ thus ${}_gf - f = 0$ and f is a constant. This implies that μ is C. D.

LEMMA 4. — *Let* $\mu \in P(G)$ *be étalée strictly aperiodic,* C. D. *measure then for every* $g \in G$

$$\| (\delta_e - \delta_g) * \mu^n \| \to 0 .$$

Proof. — By theorem 2 $\| \nu * \mu^n \| \to 0$ $\forall \nu \in M_a^0$. Write $\mu^n = \eta^{(n)} + \theta^{(n)}$ where $\eta^{(n)}$ is absolutely continuous and $\theta^{(n)}$ is singular with respect to λ. Then, for large n, $\| \theta^{(n)} \|$ is small. We notice that $(\delta_e - \delta_g) * \eta^{(n)} \in M_a^0$ and write

$$\begin{aligned} \| (\delta_e - \delta_g) * \mu^{n+k} \| &= \| [(\delta_e - \delta_g) * \eta^{(n)} + (\delta_e - \delta_g) * \theta^{(n)}] * \mu^k \| \\ &\leqslant \| (\delta_e - \delta_g) * \eta^{(n)} * \mu^k \| + \| (\delta_e - \delta_g) * \theta^{(n)} * \mu^k \| \\ &\leqslant \| (\delta_e - \delta_g) * \eta^{(n)} * \mu^k \| + 2 \| \theta^{(n)} \| . \end{aligned}$$

Letting k tend to infinity we conclude that $\lim \| (\delta_e - \delta_g) * \mu^n \| = 0$.

LEMMA 5. — *For* μ *as in Lemma 4.*

$$(1) \qquad \| \mu^n * f \|_\infty \to 0 \qquad \forall f \in C_l^0$$

$$(2) \qquad \| f * \tilde{\mu}^n \|_\infty \to 0 \qquad \forall f \in C_r^0 .$$

Proof. — Let $f \in C$ then

$$\begin{aligned} \| \mu^n * (f - {}_gf) \|_\infty &= \| \mu^n * ((\delta_e - \delta_g) * f) \|_\infty \\ &= \| ((\delta_e - \delta_g) * \mu^n) * f \|_\infty \leqslant \| (\delta_e - \delta_g) * \mu^n \| . \| f \|_\infty \to 0 . \end{aligned}$$

By the Hahn-Banach theorem

$$C_l^0 = \overline{\bigcup_{g \in G} (L - l_g) C} ,$$

and (1) follows. To see (2) we observe that $\widetilde{\mu^n * f} = \tilde{f} * \tilde{\mu}^n$ and that $\widetilde{C_l^0} = C_r^0$.

LEMMA 6. — *Let* $\mu \in P(G)$ *and suppose that*

$$\| \mu^n * f \|_\infty \to 0 \qquad \forall f \in C_l^0$$

then for every $f \in C_l^0$, $f * \mu^n \to 0$ *point-wise on* G, *and* μ *is* C. D.

Proof. — Our assumption implies that for every $f \in \mathrm{C}$ and $g \in \mathrm{G}$

$$(\mu^n * ({}_gf - f))(e) = \int (f - {}_gf)\, d\mu^n \rightarrow 0$$

Now let $h \in \mathrm{G}$ then ${}_h({}_gf - f) = {}_{hg^{-1}h}({}_hf) - {}_hf$, therefore

$$[({}_gf - f) * \mu^n](h) = \int ({}_gf - f)(hg')\, d\mu^n(g')$$

$$= \int {}_h({}_gf - f)(g')\, d\mu^n(g') = \int [{}_{hg^{-1}h}({}_hf) - {}_hf]\, d\mu^n \rightarrow 0$$

Now this convergence is pointwise and not necessarily uniform, however if $f \in \mathrm{C}$ is μ-harmonic then so is ${}_gf$ and it follows that $({}_gf - f) * \mu^n = {}_gf - f = 0$. Thus f is a constant and μ is C. D.

THEOREM 7. — *Let* $\mu \in \mathrm{P(G)}$ *then*

(1) $\mathrm{L}_l^0 \subseteq \mathrm{K}_\mu \Rightarrow \mu$ *is* C. D.
In general, this implication cannot be reversed.

(2) *If* μ *is strictly aperiodic étalée, then*

$$\mu \text{ is C. D.} \Leftrightarrow \mathrm{C}_r^0 \subseteq \mathrm{K}_{\tilde{\mu}} \Leftrightarrow \mathrm{C}_l^0 \subseteq \mathrm{J}_\mu$$

(3) *If* μ *is strictly aperiodic étalée then* μ *is* C. D. *iff* $f * \mu^n \rightarrow 0$ *pointwise* $\forall f \in \mathrm{C}_l^0$.

Proof. — Statement (1) is just lemma 3. Statements (2) and (3) follow from lemmas 5 and 6.

Let G be a group with equivalent uniform structures and suppose $\mu \in \mathrm{P(G)}$ is étalée, strictly aperiodic, symmetric and C. D. Since $\mu = \tilde{\mu}$ we have by (2) $\mathrm{U}_r^0 \subseteq \mathrm{K}_\mu \cap \mathrm{U}$. If ν is a right invariant mean on U and $f \in \mathrm{U}$ then one can check that $\nu(f * \mu) = \nu(f)$. Hence $\| f * \mu^n \|_\infty \rightarrow 0$ implies $\nu(f) = 0$ and we conclude that $\mathrm{U}_r^0 = \mathrm{K}_\mu \cap \mathrm{U}$.

Suppose that the converse of the implication of (1) is true; then we also have $\mathrm{U}_l^0 \subseteq \mathrm{K}_\mu$ and thus $\mathrm{U}_l^0 \subseteq \mathrm{U}_r^0$. By symmetry $\mathrm{U}_l^0 = \mathrm{U}_r^0$.

In particular, every left invariant mean on G must also be right invariant.

Now it is shown in ([*8*], p. 239) that the group $\mathrm{G} = \mathrm{Z}_2 * \mathrm{Z}_2$ (free product) has a left invariant mean which is not right invariant. Since G is discrete every measure on it is étalée and the uniform structures on G are equivalent. Moreover, G is a Z_2 extension of Z and it is hence easy to see that G is a C. D. group. Therefore, choosing the symmetric measure on G which assignés mass 1/2 to each of the two free generators, we have a measure for which the converse of the implication of (1) fails. This completes the proof.

Remark. — Let $G = Z_2 * Z_2$ be the free product generated by a and b with the relations $a^2 = b^2 = e$. We give an alternative proof to that of [*6*] that $U = U(G)$ has a left invariant mean which is not right invariant. Let A be the subset of G of all words of the form *a*, *ba*, *aba*, *baba*, ... i. e., words which end with *a*. If we take $\bar{A}$ in $|U|$ then it is clear that $\bar{A}$ is a closed left-invariant subset of the left G-space $|U|$. Since G is amenable, there exists a left G-invariant probability measure on $\bar{A}$. Now $|U|$ is also a right G-space and clearly $\bar{A}b \cap \bar{A} = \emptyset$ (take a function on G which is zero on A and one on Ab). Thus ν is not right invariant.

Remark. — There is a group G and a measure $\mu \in P(G)$ such that μ is C. D. while $\tilde{\mu}$ is not. Indeed, it was shown by Azencott ([*1*], p. 121) that an étalée probability measure μ on the group of matrices of the form $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where a, b are real and $a > 0$ is C. D. if

$$(i) \qquad 0 < \int \log |a| \, d\mu(g) \leqslant \infty$$

and it is not C. D. if

$$(ii) \qquad -\infty \leqslant \int \log |a| \, d\mu(g) < 0, \qquad \int |b| \, d\mu(g) \leqslant \infty$$

and

$$\int |b|^2 d\mu(g) < \infty$$

Thus, if μ is étalée and satisfies the conditions (*ii*) then μ is not C. D. while $\tilde{\mu}$ which then satisfies (*i*) is C. D. For this μ we have $C_r^0 \subseteq K_\mu$ yet μ is not C. D.

We conclude with the following

THEOREM 8. — *Let* G *be a connected locally compact topological group on which the right and left uniform structures are equivalent then* G *is Liouville.*

Proof. — Let S be an open sub-semigroup of G; we show that $SS^{-1} = G$. Let U be an open neighbourhood of the identity of G such that for some $g \in G$, $gU \subseteq S$. We can assume that $U^{-1} = U$ and we let

$$V = \cap \{ g^n U g^{-n} : n \in Z \}.$$

Since the uniform structures on G are equivalent $V = V^{-1}$ is a neighbourhood of the identity and $gVg^{-1} = V$.

Let T be the semigroup generated by gV then clearly $T = \cup \{ g^n V^n : n \geqslant 1 \}$ and

$$TT^{-1} = \cup \{ g^{n-m} V^{n+m} : n, m \geqslant 1 \}.$$

The latter is an open subgroup of G. Since G is connected $TT^{-1} = G$ and *a fortiori* $SS^{-1} = G$.

Let μ be an étalée probability measure on G then it follows that $S_\mu S_\mu^{-1} = G$. We let W be a neighbourhood of the identity such that $\bar{W}$ is compact and $gWg^{-1} \subseteq W$ for every $g \in G$. Theorem IV.1 of [*1*] implies now that for every μ-harmonic function f and every $g \in G$ and $h \in W$, $f(gh) = f(g)$. Since G is connected, this equality holds for every $h \in G$; i. e., f is a constant. This completes the proof.

REFERENCES

[*1*] R. AZENCOTT, Espace de Poisson des groupes localement compacts. *Lecture Notes*, n° 148, Springer-Verlag, 1970.

[*2*] J. DIXMIER, L'application exponentielle dans les groupes de Lie résolubles. *Bull. Soc. Math. France*, t. **85**, 1957, p. 113-121.

[*3*] S. R. FOGUEL, Iterates of a convolution on a non-abelian group. *Ann. Inst. Henri Poincaré*, Vol. XI, n° 2, 1975, p. 199-202.

[*4*] S. R. FOGUEL, Convergence of the iterates of a convolution. *To appear.*

[*5*] Y. GUIVARCH, Croissance polynomiale et période des fonctions harmoniques. *Bull. Soc. Math. France*, t. **101**, 1973, p. 333-379.

[*6*] E. HEWITT and K. A. ROSS, *Abstract Harmonic analysis.* Vol. I, Springer-Verlag, Berlin, 1963.

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