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On a characterization of Gaussian measures in a Hilbert space

by

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SOMMAIRE. — Dans cet article nous offrons une condition nécessaire et suffisante pour qu'une mesure μ donnée sur un espace hilbertien soit Gaussienne. En même temps nous dérivons une équation fonctionnelle qui est satisfaite par les densités (les dérivées de Radon-Nikodym). Il en apparait que cette équation rappelle les équations fonctionnelles satisfaites par les fonctionnelles caractéristiques

ABSTRACT. — In this article we give a necessary and sufficient condition in order that a measure μ on a Hilbert space be Gaussian. We also derive a functional equation satisfied by the densities (Radon-Nikodym derivatives), and this equation is similar to the functional equations satisfied by the characteristic functionals.

1. INTRODUCTION

The study of the characterization of Gaussian distribution was initiated by M. Kac (and S. Bernstein, cf. Kac [5]). Since 1939 several authors (Darmois, Linnik, Marcinkiwicz, Prokhorov, Skitovich to name few) have characterized normal distributions through independent linear forms.

Kac's proof depended upon a functional equation satisfied by the characteristic functions of the variables. This has become a standard method (cf. Kac [5], Feller [2]). Using extensively a functional equation satisfied by the Fourier-Stieltjes transforms Corwin [1] has given a characterization of Gaussian measures in R^n. In this article we give a necessary and sufficient condition in order that a measure μ on a Hilbert space $\mathscr{H}$ be Gaussian. This extends Corwin's result. In establishing the sufficiency we borrow some basic steps of the proof from Corwin (cf. also Kac [5]), but in proving these steps we differ from Corwin [1] mostly. It is well known that two Gaussian measures on $\mathscr{H}$ are either equivalent or orthogonal to each other. We also present here a functional equation satisfied by the densities (Radon-Nikodym derivatives).

Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{B}$ be the σ-algebra of Borel sets of $\mathscr{H}$. Recall that a probability measure μ on $(\mathscr{H}, \mathscr{B})$ is called *Gaussian* if every bounded linear functional $l_z(x) = \langle z, x \rangle$ is a Gaussian random variable. We need the following well known characterization of Gaussian measures on $\mathscr{H}$ (cf. Parthasarathy [6]). *A distribution μ on $(\mathscr{H}, \mathscr{B})$ is a (centered) Gaussian distribution if and only if its characteristic functional $\varphi(x)$ admits the representation*

$$\varphi(x) = \exp\left\{ -\frac{1}{2}\langle \mathrm{K}x, x \rangle \right\}, \tag{1.1}$$

where K *is a nuclear operator called the covariance operator of* μ. Also recall that, for a positive definite Hermitian operator K, the function $\varphi(x) = \exp\left[-\frac{1}{2}\langle \mathrm{K}x, x \rangle \right]$ is the characteristic functional of a probability measure on $\mathscr{H}$ if and only if K is a nuclear operator.

2. A CHARACTERIZATION OF GAUSSIAN MEASURES

Let $\mathscr{H}$ be a separable Hilbert space. All the measures that we consider on the Borel σ-algebra $\mathscr{B} = \mathscr{B}(\mathscr{H})$ are second order measures, that is, $\int \| x \|^2 d\mu(x) < \infty$. Let S, T and K be any three mutually commuting self-adjoint bounded linear operators on $\mathscr{H}$ such that S, T and $(\mathrm{S}^2 + \mathrm{T}^2)$ are invertible and K is a nuclear operator. Define a map $f : \mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$ by

$$f(x, y) = (\mathrm{S}x + \mathrm{T}y, \mathrm{T}x - \mathrm{S}y). \tag{2.1}$$

Before stating our main theorem we remark that, even though we work with probability measures for simplicity sake, one can see that the method works for more general measures than probability measures.

THEOREM 2.1. — *A measure μ on $(\mathcal{H}, \mathcal{B})$ is Gaussian with covariance operator* K *if and only if there is a measure ν on $(\mathcal{H}, \mathcal{B})$ such that, for* $B \in \mathcal{B} \otimes \mathcal{B}$,

$$(\mu \times \mu)(B) = (\nu \times \nu)(f(B)). \tag{2.2}$$

Proof. — *Necessity.* Let μ be a Gaussian measure on $\mathcal{H}$ with covariance operator K and characteristic functional $\varphi(x)$. Then,

$$\varphi(x) = \exp\left[-\frac{1}{2}\langle Kx, x \rangle \right].$$

Define

$$\psi(x) = \varphi(Sx)\varphi(Tx). \tag{2.3}$$

Then,

$$\begin{aligned}\psi(x) &= \varphi(Sx)\varphi(Tx)\\ &= \exp\left[-\frac{1}{2}\{ \langle KSx, Sx \rangle + \langle KTx, Tx \rangle \} \right]\\ &= \exp\left[-\frac{1}{2}\langle K(S^2 + T^2)x, x \rangle \right].\end{aligned}$$

Clearly $\psi(x)$ is the characteristic functional of a measure ν. First we claim that

$$\psi(x)\psi(y) = \varphi(Sx + Ty)\varphi(Tx - Sy), \tag{2.4}$$

and hence that

$$\varphi(x)\varphi(y) = \psi(U(Sx + Ty))\psi(U(Tx - Sy)), \tag{2.5}$$

where $U = (S^2 + T^2)^{-1}$. Indeed,

$$\begin{aligned}&\varphi(Sx + Ty)\varphi(Tx - Sy)\\ &= \exp\left[-\frac{1}{2}\{ \langle K(Sx + Ty), Sx + Ty \rangle + \langle K(Tx - Sy), Tx - Sy \rangle \} \right]\\ &= \exp\left[-\frac{1}{2}\{ \langle K(S^2 + T^2)x, x \rangle + \langle K(S^2 + T^2)y, y \rangle \} \right]\\ &= \psi(x)\psi(y).\end{aligned}$$

Now,

$$\begin{aligned}&\varphi(x)\varphi(y)\\ &= \iint \exp[i\{\langle u, x \rangle + \langle v, y \rangle\}]d\mu(u)d\mu(v)\\ &= \iint \exp[i\{\langle Su+Tv, U(Sx+Ty)\rangle + \langle Tu-Sv, U(Tx-Sy)\rangle\}]d(\mu\times\mu)(u, v),\end{aligned}$$

and

$$\psi(\mathrm{U}(\mathrm{S}x + \mathrm{T}y))\psi(\mathrm{U}(\mathrm{T}x - \mathrm{S}y)) = \iint \exp\left[i\{\langle \mathrm{S}u + \mathrm{T}v, \mathrm{U}(\mathrm{S}x + \mathrm{T}y)\rangle + \langle \mathrm{T}u - \mathrm{S}v, \mathrm{U}(\mathrm{T}x - \mathrm{S}y)\rangle\}\right] d(\nu \times \nu)(\mathrm{S}u + \mathrm{T}v, \mathrm{T}u - \mathrm{S}v).$$

Noting (2.5), from the analytical properties of Fourier-Stieltjes transforms, we get

$$(\mu \times \mu)(\mathrm{B}) = (\nu \times \nu)(f(\mathrm{B})),$$

for $\mathrm{B} \in \mathscr{B} \otimes \mathscr{B}$.

Sufficiency. — Let φ and ψ be the characteristic functionals of μ and ν respectively. From the proof of the necessity one would obviously expect that φ and ψ satisfy the functional equations (2.4) and (2.5). φ and ψ do satisfy these equations; for,

$$\begin{aligned}
&\varphi(x)\varphi(y)\\
&= \iint \exp[i\{\langle u, x\rangle + \langle v, y\rangle\}]d\mu(u)d\mu(v)\\
&= \iint \exp[i\{\langle \mathrm{S}u+\mathrm{T}v, \mathrm{U}(\mathrm{S}x+\mathrm{T}y)\rangle + \langle \mathrm{T}u-\mathrm{S}v, \mathrm{U}(\mathrm{T}x-\mathrm{S}y)\rangle\}]d(\mu\times\mu)(u, v)\\
&= \iint \exp[i\{\langle \mathrm{S}u+\mathrm{T}v, \mathrm{U}(\mathrm{S}x+\mathrm{T}y)\rangle + \langle \mathrm{T}u-\mathrm{S}v, \mathrm{U}(\mathrm{T}x-\mathrm{S})\rangle\}]\, d(\nu\times\nu)(\mathrm{S}u+\mathrm{T}v, \mathrm{T}u-\mathrm{S}v)\\
&= \iint \exp[i\{\langle u, \mathrm{U}(\mathrm{S}x+\mathrm{T}y)\rangle + \langle v, \mathrm{U}(\mathrm{T}x-\mathrm{S}y)\rangle\}]d(\nu\times\nu)(u, v)\\
&= \psi(\mathrm{U}(\mathrm{S}x+\mathrm{T}y))\psi(\mathrm{U}(\mathrm{T}x-\mathrm{S}y)).
\end{aligned}$$

These equations are basic in the proof. At this point we pause for a moment to make a remark about the theorem. In the theorem one can say that there exist measures ν_1 and ν_2 such that $(\mu \times \mu)(\mathrm{B}) = (\nu_1 \times \nu_2)(f(\mathrm{B}))$ holds in place of (2.2). We omit the details of the extension to this case except that we content ourselves by showing that the basic functional equation holds true in this case also. To see this, let ψ_i be the characteristic functional of ν_i, $i = 1, 2$. Then, clearly,

$$\psi_1(x)\psi_2(y) = \varphi(\mathrm{S}x + \mathrm{T}y)\varphi(\mathrm{T}x - \mathrm{S}y), \tag{2.6}$$

and hence (noting $\varphi(0) = 1 = \psi_i(0)$),

$$\psi_1(x) = \varphi(\mathrm{S}x)\varphi(\mathrm{T}x), \qquad \psi_2(y) = \varphi(\mathrm{T}y)\varphi(-\mathrm{S}y). \tag{2.7}$$

Now define

$$\xi(x) = \frac{\psi_2(x)}{\psi_1(-x)} = \frac{\varphi(\mathrm{T}x)}{\varphi(-\mathrm{T}x)} \tag{2.8}$$

(It is easy to see that φ and ψ_i do not vanish.) From (2.7) and (2.8)

$$\frac{\psi_1(\mathrm{T}x)}{\psi_2(\mathrm{T}x)} = \frac{\varphi(\mathrm{ST}x)}{\varphi(-\mathrm{ST}x)} = \xi(\mathrm{S}x),$$

and hence

$$\xi(x) = \frac{\psi_1(\mathrm{TS}^{-1}x)}{\psi_2(\mathrm{TS}^{-1}x)} \tag{2.9}$$

From (2.8) and (2.9)

$$\begin{aligned}\xi(x)\xi(y) &= \frac{\psi_2(x)}{\psi_1(-x)}\frac{\psi_1(\mathrm{TS}^{-1}y)}{\psi_2(\mathrm{TS}^{-1}y)} \\ &= \frac{\varphi(\mathrm{T}y + \mathrm{T}x)}{\varphi(-\mathrm{T}x - \mathrm{T}y)} \\ &= \xi(x+y)\end{aligned}$$

Set

$$\begin{aligned}h(x) &= \xi\left(-\frac{x}{2}\right) \\ \Phi(x) &= \varphi(x)h(\mathrm{T}^{-1}x) \\ \Psi(x) &= \psi_2(x)h(x)h(-\mathrm{T}^{-1}\mathrm{A}x).\end{aligned}$$

Then,

$$\begin{aligned}\Phi(\mathrm{S}x + \mathrm{T}y)\Phi(\mathrm{T}x - \mathrm{S}y) &= \psi_1(x)\psi_2(y)h(\mathrm{T}^{-1}\mathrm{S}x + y)h(x - \mathrm{T}^{-1}\mathrm{S}y) \\ &= \Psi(y)\psi_1(x)h(\mathrm{T}^{-1}\mathrm{S}x)h(x) \\ &= \Psi(y)\psi_1(x)h(\mathrm{T}^{-1}\mathrm{S}x)\frac{\Psi(x)}{\psi_2(x)}h(\mathrm{T}^{-1}\mathrm{S}x) \\ &= \Psi(y)\Psi(x)\frac{\psi_1(x)}{\psi_2(x)}\xi(-\mathrm{T}^{-1}\mathrm{S}x) \\ &= \Psi(y)\Psi(x).\end{aligned}$$

Thus one can find a homomorphism h such that Φ and Ψ satisfy the basic functional equations.

Next, returning back to the original φ and ψ, define

$$l(x, y) = \frac{\varphi(x+y)}{\varphi(x)\varphi(y)}. \tag{2.10}$$

We claim that

$$l(x_1 + x_2, y) = l(x_1, y)l(x_2, y). \tag{2.11}$$

Because of the symmetry of l in x and y, relation corresponding to (2.11) holds in the y variable also. In this article we shall call such mappings « BLM-functionals » (B for bi, L for linearly, M for multiplicative !). From (2.4) and (2.5) one can easily see that φ and ψ are even functions. From (2.4)

$$\varphi(x)\varphi(\mathrm{V}x)\varphi(\mathrm{V}y)\varphi(y) = \varphi(x + \mathrm{V}y)\varphi(\mathrm{V}x - y),$$

where $V = TS^{-1}$. Thus

$$l(x, Vy)l(Vx - y) = 1\,. \tag{2.12}$$

Using (2.12)

$$\begin{aligned}
l(x + Vy, Vz) &= \frac{\varphi(x + Vy + Vz)}{\varphi(x + Vy)\varphi(Vz)} \\
&= \frac{\varphi(x + Vy + Vz)}{\varphi(x)\varphi(Vy + Vz)} \frac{\varphi(Vy + Vz)}{\varphi(Vz)\varphi(Vy)} \frac{\varphi(Vy)\varphi(x)}{\varphi(x + Vy)} \\
&= l(x, Vy + Vz)\frac{l(Vy, Vz)}{l(x, Vy)} \\
&= \frac{l(Vx, -y)l(Vy, Vz)}{l(Vx, -y - z)} \\
&= \frac{\varphi(Vx - y)}{\varphi(Vx)\varphi(-y)} \frac{\varphi(Vx)\varphi(y + z)}{\varphi(Vx - y - z)} l(Vy, Vz) \\
&= \frac{\varphi(Vx - y)\varphi(-z)}{\varphi(Vx - y - z)} \frac{\varphi(y + z)}{\varphi(y)\varphi(z)} l(Vy, Vz) \\
&= \frac{l(y, z)l(Vy, Vz)}{l(Vx - y, -z)}
\end{aligned} \tag{2.13}$$

Setting $x = -Vy$, and recalling that φ is even, we get

$$l(V^2y + y, z) = l(y, z)l(Vy, Vz)\,. \tag{2.14}$$

Changing x into $(x - Vy)$ in (2.13) and using (2.14),

$$l(x, Vz)l(Vx - V^2y - y, z) = l(V^2y + y, z)\,.$$

Now from (2.12) we get

$$l(Vx - u, -z) = l(-u, -z)l(Vx, -z)\,,$$

where $u = (V^2 + I)y$. Thus we see that $l(.\,,.)$ is a « BLM-functional ».
Let $\{e_i\}$ be an orthonormal basis for $\mathscr{H}$. Define l_{ij} on $R \times R$ by

$$l_{ij}(r, s) = \frac{\varphi(re_i + se_j)}{\varphi(re_i)\varphi(se_j)}$$

Since l_{ij} is BLM, there is a scalar k_{ij} such that

$$l_{ij}(r, s) = \exp\left[-\frac{1}{2}k_{ij}rs\right],$$

and in particular

$$l_{ij}(s, s) = \exp\left[-\frac{1}{2}k_{ij}s^2\right].$$

Since $\frac{l(x, x)}{l(-x, x)} = \varphi(2x)$, it is clear that for any one dimensional subspace we get $\varphi(x) = \exp\left[-\frac{1}{2}\langle Kx, x \rangle\right]$. It is not hard to see, by induction, that this holds on any n-dimensional subspace, $n \geq 1$. Clearly $k_{ij} = k_{ji}$. Since φ is positive definite, bounded and nonvanishing K is symmetric and positive definite. Now we pass to the limit $(\mathscr{H})$ in a standard way to obtain the representation (1.1). We simply point this out: let $\lambda_i's$ be the eigenvalues, and $a_i's$ be reals. Then,

$$\int \exp\left\{i\sum_{k=1}^{n} a_k \langle x, e_k \rangle\right\} d\mu = \exp\left[-\frac{1}{2}\sum_{1}^{n} \lambda_k a_k^2\right]$$
$$= \prod_{1}^{n} \exp\left[-\frac{1}{2}\lambda_k a_k^2\right] = \prod_{1}^{n} \int \exp\{ia_k \langle x, e_k \rangle\} d\mu .$$

Thus, for any measurable function $f_1, \ldots, f_n$ on the reals, we get

$$\int \prod_{1}^{n} f_i(\langle x, e_i \rangle) d\mu = \prod_{1}^{n} \int f_i(\langle x, e_i \rangle) d\mu ,$$

provided the integrals exist. Passage to the limit under the integral sign is clearly valid. Recalling that φ is continuous in norm, positive definite and nonvanishing (in our case) we get the theorem.

3. FUNCTIONAL EQUATION FOR DENSITIES

Let μ_1 and μ_2 be two Gaussian measures on $\mathscr{H}$ such that $\mu_2 \ll \mu_1$. It is known that the density $\xi(x) = \frac{d\mu_2}{d\mu_1}(x)$ is positive everywhere and hence $\mu_2 \ll \mu_1$ actually implies that μ_1 and μ_2 are equivalent. Violation of absolute continuity means that μ_1 and μ_2 are orthogonal to each other. Hence two Gaussian measures are either equivalent or orthogonal. For further properties and representations of the densities we refer to Gikhman and Skorokhod [4]. In this section we merely derive a functional equation satisfied by the densities.

Let μ_i, ν_i, λ_i, $i = 1, 2$, be probability measures on $(\mathscr{H}, \mathscr{B})$ such that

$$\mu_2 \ll \mu_1 ,$$
$$(\mu_1 \times \mu_1)(B) = (\nu_1 \times \nu_1)(f(B)) ,$$
$$(\mu_2 \times \mu_2)(B) = (\lambda_1 \times \lambda_2)(f(B)) ,$$

for $B \in \mathscr{B} \otimes \mathscr{B}$. Then, μ_1 and μ_2 are two equivalent Gaussian measures. Let $\xi(x) = \frac{d\mu_2}{d\mu_1}(x)$. If B is a null set of $\nu_1 \times \nu_2$, then $f^{-1}(B)$ is a null set of $\mu_1 \times \mu_1$ and hence a null set of $\mu_2 \times \mu_2$ which implies that B is a null set of $\lambda_1 \times \lambda_2$. Thus $\lambda_1 \times \lambda_2 \ll \nu_1 \times \nu_2$ and $\frac{d(\lambda_1 \times \lambda_2)}{d(\nu_1 \times \nu_2)}(x, y)$ exists. Clearly $\lambda_1 \ll \nu_1$ and $\lambda_2 \ll \nu_2$. Set $\eta_1(x) = \frac{d\lambda_1}{d\nu_1}(x)$, $\eta_2(x) = \frac{d\lambda_2}{d\nu_2}(x)$. Then,

$$\begin{aligned}
\eta_1(x)\eta_2(y) &= \frac{d(\lambda_1 \times \lambda_2)}{d(\nu_1 \times \nu_2)}(x, y) \\
&= \frac{d(\mu_2 \times \mu_2)}{d(\mu_1 \times \mu_1)}(\mathrm{U}(\mathrm{S}x + \mathrm{T}y), \mathrm{U}(\mathrm{T}x - \mathrm{S}y)) \\
&= \frac{d\mu_2}{d\mu_1}(\mathrm{U}(\mathrm{S}x + \mathrm{T}y))\frac{d\mu_2}{d\mu_1}(\mathrm{U}(\mathrm{T}x - \mathrm{S}y)) \\
&= \xi(\mathrm{U}(\mathrm{S}x - \mathrm{T}y))\xi(\mathrm{U}(\mathrm{T}x - \mathrm{S}y))
\end{aligned}$$

Thus the densities ξ and η_i, $i = 1, 2$, satisfy the functional equation

$$\eta_1(x)\eta_2(y) = \xi(\mathrm{U}(\mathrm{S}x + \mathrm{T}y))\xi(\mathrm{U}(\mathrm{T}x - \mathrm{S}y)).$$

(Note that this equation is similar to the functional equation satisfied by the characteristic functionals.)

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