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Separabilities of a Gaussian Radon measure

by

Hiroshi SATO (*) and Yoshiaki OKAZAKI (**)

SUMMARY. — Let (X, Y) be a dual system of real linear spaces, $C(X, Y)$ the cylindrical σ -algebra of X , and $W(X, Y)$ the weak Borel field of X .

The main purposes of this paper are to prove the separability of $L^2(\mu)$ for a Gaussian measure on $(X, C(X, Y))$ under some assumptions and to prove for a Gaussian Radon measure on $(X, W(X, Y))$ the separability of $L^2(\mu)$ and the $\tau(X, Y)$ -separability of the support, where $\tau(X, Y)$ is the Mackey topology.

1. INTRODUCTION AND NOTATIONS

Let (X, Y) be a *pair* of real linear spaces X and Y with a bilinear form $\langle x, \xi \rangle$ on $X \times Y$, and let $C(X, Y)$ be the minimal σ -algebra of subsets of X that makes all functions $\{\langle \cdot, \xi \rangle; \xi \in Y\}$ measurable. Furthermore, if the bilinear form satisfies the separation axioms;

$$\begin{aligned} \langle x_0, \xi \rangle &= 0 \quad \text{for all } \xi \in Y \quad \text{implies} \quad x_0 = 0, \\ \langle x, \xi_0 \rangle &= 0 \quad \text{for all } x \in X \quad \text{implies} \quad \xi_0 = 0, \end{aligned}$$

we call (X, Y) a *dual system*. In this case, we denote the weak topology on X by $\sigma(X, Y)$ and the Mackey topology by $\tau(X, Y)$.

We say that a dual system (X, Y) is *topological* if X is a topological linear space such that all functions in Y are continuous, in other words, the

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topology of X is finer than $\sigma(X, Y)$, and we denote the Borel field of X by $B(X, Y)$. Any dual system is a topological dual system if X is equipped with $\sigma(X, Y)$ and we denote by $W(X, Y)$ the Borel field of X for $\sigma(X, Y)$. Evidently, $W(X, Y)$, *a fortiori*, $C(X, Y)$ is included in $B(X, Y)$ for any topological dual system.

Let U be a topological space and $B(U)$ be the Borel field of U . Then we say that a measure μ on $(U, B(U))$ is *Radon* if μ is a finite measure such that

$$\mu(A) = \sup \{ \mu(K) ; K \subset A, \text{ compact} \}$$

for every A in $B(U)$.

For a topological linear space X we denote the algebraic dual space of X by X^a and the topological dual space by X' .

Let (X, Y) be a pair of real linear spaces with a bilinear form $\langle x, \xi \rangle$. Then a *Gaussian measure* on $(X, C(X, Y))$ is a probability measure such that for every $\xi \in Y$, $\langle \cdot, \xi \rangle$ obeys a Gaussian law with mean $m(\xi)$ and variance $v(\xi)$. We call $m(\xi)$ the *mean functional* and $v(\xi)$ the *variance functional* of μ . In particular, if $m(\xi) \equiv 0$, we say the Gaussian measure is *centered*.

Let (X, Y) be a topological dual system. Then a *Gaussian Radon measure* on $(X, B(X, Y))$ is a Radon measure such that the restriction to $C(X, Y)$ is Gaussian.

In Section 2 of this paper we prove the separability of $L^2(\mu)$ for a Gaussian measure μ on $(X, C(X, Y))$ under the assumption of the existence of an *admissible metric* on Y where (X, Y) is a pair of linear spaces. In particular, we show that if X is a metrizable locally convex space and Y is a linear subspace of X' , then $L^2(\mu)$ is separable.

Let (X, Y) be a topological dual system. In Section 3, we remark the equivalent-singular dichotomy of two Gaussian Radon measures on $(X, B(X, Y))$; in Section 4, we prove the separability of $L^2(\mu)$ for a Gaussian Radon measure on $(X, B(X, Y))$.

Let (X, Y) be a dual system. In Section 5, we prove the $\tau(X, Y)$ -separability of the support of a centered Gaussian Radon measure on $(X, W(X, Y))$. Furthermore, we prove the $\tau(X, Y)$ -separability of the support of a non-centered Gaussian Radon measure on $(X, W(X, Y))$ under the following assumption:

- (C.1) There exists an increasing sequence of
 $\sigma(X, Y)$ -compact absolutely convex sub-
 sets $\{F_n\}$ of X such that $\lim_n \mu(F_n) = 1$.

This is the case where X is $\tau(X, Y)$ -quasi-complete, in particular, X is a Fréchet space and $Y = X'$, or Y is a Fréchet space and $X = Y'$.

For a Gaussian measure on $(X, C(X, X'))$, where X is a separable or reflexive Banach space, the separability of the Hilbert space H_μ generated by the random variables $\langle \cdot, \xi \rangle$, $\xi \in X'$, is stated in H. Sato [7]. J. Kuelbs [4] has also stated the separability of H_μ for a centered Gaussian Radon measure on $(X, B(X, X'))$ where X is a complete locally convex Hausdorff space. But they have the same error since every pre-Hilbert space has not a complete orthonormal system (A. Badrikian and S. Chevet [1]). In this paper, we have corrected it and obtained more general results.

2. SEPARABILITY OF $L^2(\mu)$

Let (X, Y) be a pair of real linear spaces and let μ be a Gaussian measure on $(X, C(X, Y))$ with the mean functional $m(\xi)$ and the variance functional $v(\xi)$. We say a metric ρ on Y is *admissible* if it defines a locally convex topology on Y such that

$$\mu^*((Y, \rho)' \cap X) = 1$$

where $(Y, \rho)'$ is the topological dual space of Y with ρ -topology $(Y, \rho)' \cap X = \{x \in X; \langle x, \cdot \rangle \text{ is continuous in } (Y, \rho)\}$ and μ^* is the outer measure, that is, for every $A \subset X$,

$$\mu^*(A) = \inf \{ \mu(E) : E \in C(X, Y) \text{ and } E \supset A \}.$$

In this case, for every sequence $\{\xi_n\}$ convergent to ξ in (Y, ρ) , the random sequence $\{\langle x, \xi_n \rangle\}$ converges to $\langle x, \xi \rangle$ almost surely on the probability space $(X, C(X, Y), \mu)$. In fact, the set

$$A = \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \left\{ x \in X; |\xi_k(x) - \xi(x)| \leq \frac{1}{n} \right\}$$

belongs to $C(X, Y)$ and contains the set $(Y, \rho)' \cap X$, hence we have $\mu(A) = 1$, that is, $\langle x, \xi_n \rangle \rightarrow \langle x, \xi \rangle$ almost surely as $n \rightarrow \infty$. If a metric ρ on Y is admissible, there is a subset $E \subset X$ of outer measure one such that the topology ρ is finer than the pointwise convergence topology on E , that is, ρ is finer than the topology $\sigma(Y, E)$. Conversely, if a metrizable locally convex topology ρ on Y is finer than the one of pointwise convergence on a suitable subset E of outer measure one, ρ is admissible. In particular every metric on Y which defines a locally convex topology finer than the weak topology $\sigma(Y, X)$ is admissible.

LEMMA 2-1. — If ρ is an admissible metric on Y , then $m(\xi)$ and $v(\xi)$ are ρ -continuous.

Proof. — Since ρ is a metric, it is sufficient to show sequential continuity. Let $\{\xi_n\}$ be a sequence ρ -convergent to 0. Then the Gaussian random sequence $\{\langle \cdot, \xi_n \rangle\}$ converges to 0 almost surely and it is well-known that $m(\xi_n)$ and $v(\xi_n)$ also converge to 0.

This proves the lemma.

Define a linear transformation R_μ of Y into $L^2(\mu) = L^2(X, C(X, Y), \mu)$ by

$$R_\mu : \xi \in Y \rightarrow \langle \cdot, \xi \rangle \in L^2(\mu),$$

and H_μ the closure of $R_\mu Y$ in $L^2(\mu)$. Since we have

$$\|R_\mu \xi\|_{L^2(\mu)}^2 = m(\xi)^2 + v(\xi), \quad \xi \in Y,$$

it is easy to show the following lemma.

LEMMA 2-2. — If ρ is an admissible metric on Y , then R_μ is a ρ -continuous linear transformation of Y into H_μ .

Furthermore, by a slight modification of the proof of Proposition 3-4 of R. M. Dudley [2], we can prove the following key lemma.

LEMMA 2-3. — If ρ is an admissible metric on Y , then R_μ is a compact linear transformation of (Y, ρ) into H_μ and consequently the Hilbert space H_μ is separable.

Proof. — If $m(\xi)$ does not vanish, it is sufficient to show the compactness of the new transformation

$$R_0 \xi = R_\mu \xi - m(\xi)1, \quad \xi \in Y,$$

so that without loss of generality we may assume $m(\xi) \equiv 0$.

Since ρ defines a locally convex metric topology on Y , we can choose a countable increasing basis $\{p_n\}_{n=1}^\infty$ of continuous semi-norms in Y

$$p_1(\xi) \leq p_2(\xi) \leq \dots \leq p_n(\xi) \leq \dots$$

For every n , put

$$\begin{aligned} S_n &= \{ \xi \in Y ; p_n(\xi) \leq 1 \} \\ \Gamma_n &= R_\mu S_n \\ O_n &= \{ x \in X ; \sup_{\xi \in S_n} |\langle x, \xi \rangle| \leq n \}. \end{aligned}$$

By lemma 2-2 we may assume that Γ_n is bounded in H_μ for every n . In order to prove the compactness of R_μ , it is sufficient to show the precompactness of Γ_N for some N . Assume Γ_n is not precompact for every n . Then by the boundness of Γ_n there exist a positive number ε and an infinite sequence $\{\tilde{\xi}_j^n\}_{j=1}^\infty$ in Γ_n such that the distance of $\tilde{\xi}_{j+1}^n$ from the linear

span F_j^n of $\tilde{\xi}_1^n, \dots, \tilde{\xi}_j^n$ is at least ε for all j . Let $\{\xi_j^n\}_{j=1}^\infty$ be a sequence in S_n such that $\tilde{\xi}_j^n = R_\mu \xi_j^n$. Put

$$O'_n = \{x \in X; \sup_{j=1,2,\dots} |\langle x, \xi_j^n \rangle| \leq n\}.$$

It is easy to show $\bigcup_{n=1}^\infty O_n = (Y, \rho)' \cap X$ and $O_n \subset O'_n$ for every n , hence

$\bigcup_{n=1}^\infty O'_n \supset \bigcup_{n=1}^\infty O_n = (Y, \rho)' \cap X$. Since each O'_n belongs to $C(X, Y)$ and $\mu\left(\bigcup_{n=1}^\infty O'_n\right) \geq \mu^*((Y, \rho)' \cap X) = 1$, there exists a natural number N such that $\mu(O'_N) > 0$. Let

$$\tilde{\xi}_{j+1}^N = g_j^N + \sum_{k=1}^j a_{j,k} \tilde{\xi}_k^N$$

where $g_j^N \perp F_j^N$ and $\|g_j^N\|_H \geq \varepsilon$. Put

$$A_j^N = \{x \in X; \max_{1 \leq k \leq j} |\langle x, \xi_k^N \rangle| \leq N\}.$$

Then we have $O'_N \subset A_j^N$ for every j and hence

$$0 < \mu(O'_N) \leq \mu(A_j^N) \quad \text{for every } j.$$

On the other hand we have

$$\begin{aligned} & \mu(\{x \in X; |\langle x, \xi_{j+1}^N \rangle| \geq N\} \cap A_j^N) \\ & \geq \mu\left(\{x \in X; g_j^N(x) \geq N\} \cap \left\{x \in X; \left\langle x, \sum_{k=1}^j a_{j,k} \xi_k^N \right\rangle \geq 0\right\} \cap A_j^N\right) \\ & = \frac{1}{2} \mu(\{x \in X; g_j^N(x) \geq N\}) \mu(A_j^N). \end{aligned}$$

Now for some $\delta > 0$, we have for all j

$$\mu(\{x \in X; g_j^N(x) \geq N\}) \geq 2\delta$$

so

$$\mu(A_j^N) \leq (1 - \delta)^{j-1}$$

by induction. Hence we have

$$0 < \mu(O'_N) \leq \mu(A_j^N) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

This is a contradiction. Therefore Γ_n is precompact for some n , and this completes the proof.

This proves the lemma.

The above lemma can be extended.

LEMMA 2-4. — If Y is expressed as a union of at most countable numbers of linear subspaces $\{Y_n\}$ and in each Y_n an admissible metric is defined, then H_μ is separable.

Proof. — For every n , by Lemma 2-3, $R_\mu Y_n$ is a separable subspace of H_μ so that

$$H_\mu = \overline{R_\mu Y} = \overline{\bigcup_{n=1}^{\infty} R_\mu Y_n}$$

is separable, where the closure is taken in $L^2(\mu)$.

This proves the lemma.

On the other hand, we can prove the following result.

LEMMA 2-5. — If H_μ is separable, $L^2(\mu)$ is also separable.

Proof. — Since H_μ is separable, we can choose a countable subset $\mathcal{Z} = \{\zeta_n\}_{n=1}^{\infty}$ in Y for which $\{\langle \cdot, \zeta_n \rangle\}_{n=1}^{\infty}$ is dense in H_μ . In order to prove the lemma, it is sufficient to show that the closed linear subspace of $L^2(\mu)$ generated from

$$\{\cos \langle \cdot, \zeta_n \rangle, \sin \langle \cdot, \zeta_n \rangle; n = 1, 2, 3, \dots\}$$

is identical with $L^2(\mu)$.

Let $\phi(x)$ be a square summable function on $(X, C(X, Y), \mu)$ such that for every n ,

$$\int_X \phi(x) \cos \langle x, \zeta_n \rangle d\mu(x) = 0,$$

$$\int_X \phi(x) \sin \langle x, \zeta_n \rangle d\mu(x) = 0,$$

so that

$$\int_X \phi(x) e^{i\langle x, \zeta_n \rangle} d\mu(x) = 0.$$

Then we have only to show $\phi(x) = 0$, a. e. (μ) .

Denote the collection of all finite subsets of Y by Γ . Then Γ is a directed set with respect to the inclusion. For every $\gamma = \{\xi_1, \xi_2, \dots, \xi_k\}$ in Γ and every real numbers $t_1, t_2, \dots, t_k$, there exists a subsequence $\{\zeta_{n_j}\}$ of $\mathcal{Z}$ such that

$$\sum_{l=1}^k t_l \langle \cdot, \xi_l \rangle = \lim_{j \rightarrow +\infty} \langle \cdot, \zeta_{n_j} \rangle$$

in H_μ and we have

$$\int_X e^{i \sum_{l=1}^k t_l \langle x, \xi_l \rangle} \phi(x) d\mu(x) = \lim_{j \rightarrow +\infty} \int_X e^{i \langle x, \zeta_{n_j} \rangle} \phi(x) d\mu(x) = 0,$$

in other words, the conditional expectation

$$\phi_\gamma(x) = E[\phi(x) | \xi_1, \xi_2, \dots, \xi_k] = 0, \quad \text{a. e. } (\mu)$$

for every γ in Γ . By the convergence theorem of the filtered martingale of J. Neveu [5, Proposition V-1-2], ϕ_γ converges to ϕ in $L^2(\mu)$ so that we have $\phi(x) = 0$, a. e. (μ) .

This proves the lemma.

Summing up the above lemmas, we have the following theorem.

THEOREM 2-6. — Let (X, Y) be a pair of linear spaces X and Y with a bilinear form $\langle, \rangle$, μ be a Gaussian measure on $(X, C(X, Y))$ and assume that Y is expressed as a union of at most countable numbers of linear subspaces $\{Y_n\}$ and in each Y_n an admissible metric is defined. Then $L^2(\mu)$ is a separable Hilbert space.

As corollaries of the above theorem, we have the following theorems.

THEOREM 2-7. — Let (X, Y) be a pair of linear spaces X and Y with a bilinear form $\langle, \rangle$, μ be a Gaussian measure on $(X, C(X, Y))$ and assume that there exists a locally convex metrizable topology on Y finer than the weak topology $\sigma(Y, X)$. Then $L^2(\mu)$ is a separable Hilbert space.

THEOREM 2-8. — Let X be a metrizable locally convex topological linear space, Y be a linear subspace of X' and μ be a Gaussian measure on $(X, C(X, Y))$. Then $L^2(\mu)$ is a separable Hilbert space.

Proof. — Since X is a metrizable locally convex space, we may regard X as a dense subspace of the reduced projective limit $\varprojlim_n X_n$ of Banach spaces $\{X_n\}$ and we have $X' = \bigcup_n X'_n$ as a set. For every n , X'_n is equipped with a norm topology finer than the weak topology $\sigma(X', X)$. Therefore we have $Y = \bigcup_{n=1}^{\infty} (Y \cap X'_n)$ where $Y \cap X'_n$ has an admissible metric so that Theorem 2-6 is applicable.

This proves the theorem.

3. EQUIVALENCE OF GAUSSIAN RADON MEASURES

In this section we prove the equivalent-singular dichotomy for Gaussian Radon measures for later use.

To begin with, we prepare two lemmas. Let K be a compact Hausdorff space, $C(K)$ be the space of all continuous functions on K and $B_0(K)$ be

the Baire field, that is, the minimal σ -algebra of subsets of K that makes all functions in $C(K)$ measurable. We have the following lemma.

LEMMA 3-1. — Let K be a compact Hausdorff space and let μ_1 and μ_2 be Radon measures on $(K, B(K))$. Then for every A in $B(K)$ there exists A_0 in $B_0(K)$ such that

$$\mu_1(A \triangle A_0) = \mu_2(A \triangle A_0) = 0$$

where $A \triangle A_0 = A \cup A_0 - A \cap A_0$.

Proof. — Let A be in $B(K)$. Then, since μ_i ($i = 1, 2$) is Radon, there exists a decreasing sequence of open sets $\{O_n^i; n = 1, 2, 3, \dots\}$ such that

$$A \subset O_n^i, \\ \mu_i(O_n^i - A) < \frac{1}{n}, \quad n = 1, 2, 3, \dots, \quad i = 1, 2.$$

Put $O_n = O_n^1 \cap O_n^2$, $n = 1, 2, 3, \dots$. Then we have simultaneously

$$A \subset O_n, \\ \mu_i(O_n - A) < \frac{1}{n}, \quad n = 1, 2, 3, \dots, \quad i = 1, 2.$$

Since the indicator function χ_{O_n} is lower semi-continuous and μ_i ($i = 1, 2$) is Radon, there exists a sequence $\{f_n^i; n = 1, 2, 3, \dots\}$, $i = 1, 2$, of continuous functions on K such that

$$0 \leq f_n^i(x) \leq \chi_{O_n}(x), \quad x \in K, \quad i = 1, 2, \\ 0 \leq \mu_i(O_n) - \int_K f_n^i(x) d\mu_i(x) < \frac{1}{n}, \quad n = 1, 2, 3, \dots, \quad i = 1, 2.$$

For every n , let $A_n = \{x \in K; f_n^1(x) > 0\} \cup \{x \in K; f_n^2(x) > 0\} \in B_0(K)$

and let $A_0 = \bigcap_n \bigcup_{k=n}^{+\infty} A_k$. A_0 has the desired property.

LEMMA 3-2. — Let (X, Y) be a topological dual system and let μ_1 and μ_2 are Radon measures on $(X, B(X, Y))$. Then, for every A in $B(X, Y)$, there exists A_0 in $C(X, Y)$ such that

$$\mu_1(A \triangle A_0) = \mu_2(A \triangle A_0) = 0.$$

Proof. — Since μ_1 and μ_2 are Radon measures, there exists an increasing sequence of compact subsets $\{K_n\}$ of X such that

$$\mu_i(X - K_n) < \frac{1}{n}, \quad n = 1, 2, 3, \dots, \quad i = 1, 2.$$

Let A be a set in $B(X, Y)$. Then, by Lemma 3-1, for every n there exists A_n^0 in $B_0(K_n)$ such that

$$\mu_i((A \cap K_n) \triangle A_n^0) = 0, \quad i = 1, 2.$$

On the other hand, using the Stone-Weierstrass theorem, we can easily prove that

$$B_0(K_n) = C(X, Y) \cap K_n, \quad n = 1, 2, 3, \dots$$

Therefore, for every n , there exists A_n in $C(X, Y)$ such that $A_n \cap K_n = A_n^0$

and put $A_0 = \bigcup_n \bigcap_{k=n}^{+\infty} A_k$. It is easy to see that A_0 has the desired property.

Utilising the above lemma and the well-known results concerning the equivalent-singular dichotomy of Gaussian measures on $(X, C(X, Y))$, we have the following theorem without difficulty (Ju. A. Rozanov [6]).

THEOREM 3-3. — Let (X, Y) be a topological dual system of real linear spaces X and Y , $B(X, Y)$ be the Borel field of X , and let μ and μ' be Gaussian Radon measures on $(X, B(X, Y))$. Then μ and μ' are either equivalent (mutually absolutely continuous) or singular and they are equivalent if and only if their restrictions to $C(X, Y)$ are equivalent.

The above theorem derives the same results as shown in [6] in our terminology. Let (X, Y) be a topological dual system and μ be a Gaussian Radon measure on $(X, B(X, Y))$. Furthermore let R_μ be a linear transformation of Y into $L^2(\mu) = L^2(X, B(X, Y), \mu)$ defined by

$$R_\mu : \xi \in Y \rightarrow \langle \cdot, \xi \rangle \in L^2(\mu),$$

let H_μ be the closure of the range of R_μ in $L^2(\mu)$, and let R_μ^* be the algebraic adjoint transformation of H_μ^a into Y^a . As usual, we identify the topological dual space of H_μ with H_μ .

THEOREM 3-4. — Let (X, Y) be a topological dual system, μ a centered Gaussian Radon measure on $(X, B(X, Y))$, and $\bar{B}(X, Y)$ the μ -completion of $B(X, Y)$. Moreover, assume that $R_\mu^* H_\mu \subset X$ and that H_μ is separable. Then we have:

(1) For $x \in X$ let μ_x be a Gaussian Radon measure on $(X, B(X, Y))$ defined by

$$\mu_x(A) = \mu(A + x), \quad A \in B(X, Y).$$

Then μ and μ_x are equivalent if and only if $x \in R_\mu^* H_\mu$.

(2) Let X_0 be a $B(X, Y)$ -measurable linear subspace of X such that $\mu(X_0) = 1$. Then we have $R_\mu^* H_\mu \subset X_0$.

The proof of the above theorem is the same as those shown in [6]. Out of completeness, we give the proof of (2).

Let X_0 be a $\bar{B}(X, Y)$ -measurable linear subspace of X such that $\mu(X_0) = 1$, and assume that $R_\mu^* H_\mu$ is not included in X_0 . Then there exists an element x_0 in $R_\mu^* H_\mu$ such that $x_0 \notin X_0$. Since X_0 is a linear subspace of X , X_0 and $X_0 + x_0$ are disjoint and, by (1) we have $\mu(X_0) = \mu(X_0 + x_0) = 1$. Consequently we have

$$1 \geq \mu(X_0 \cup \{X_0 + x_0\}) = \mu(X_0) + \mu(X_0 + x_0) = 1 + 1 = 2.$$

This is a contradiction.

4. SEPARABILITY OF $L^2(\mu)$

In this section we prove the separability of the Hilbert space $L^2(\mu)$ for a Gaussian Radon measure μ .

Let (X, Y) be a topological dual system and μ be a Gaussian Radon measure on $(X, B(X, Y))$. Then, since the topology $\sigma(X, Y)$ is coarser than the topology of X , μ is also a Gaussian Radon measure on $(X, W(X, Y))$. Consequently, there exists the minimal $\sigma(X, Y)$ -closed linear subspace X_μ of X such that $\mu(X_\mu) = 1$. We call X_μ the *topological linear support* of μ .

Let X_μ^0 be the polar set of X_μ in Y . Then we have the following lemma.

LEMMA 4-1. — For ξ in Y , ξ is in X_μ^0 if and only if $\langle \cdot, \xi \rangle = 0$, a. e. (μ) .

Proof. — Since $\mu(X_\mu) = 1$, the necessity is obvious.

Assume that $\langle \cdot, \xi \rangle = 0$, a. e. (μ) . Since $X^\xi = \{x \in X; \langle x, \xi \rangle = 0\}$ is a $\sigma(X, Y)$ -closed linear subspace of X and $\mu(X^\xi) = 1$, we have $X_\mu \subset X^\xi$ by the minimality of X_μ , in other words, $\xi \in X_\mu^0$.

This proves the lemma.

Define the linear transformation R_μ of Y into $L^2(\mu) = L(X, B(X, Y), \mu)$ and H_μ as in the previous section. Then, using the above lemma, we can easily prove that X_μ^0 is the kernel of R_μ .

Put $Y_\mu = Y/X_\mu^0$. Then we have $X_\mu \cap W(X, Y) = W(X_\mu, Y_\mu)$ and $X_\mu \cap C(X, Y) = C(X_\mu, Y_\mu)$. Furthermore, since the induced topology of $\sigma(X, Y)$ on X_μ is identical with $\sigma(X_\mu, Y_\mu)$, a subset of X_μ is $\sigma(X, Y)$ -closed if and only if $\sigma(X_\mu, Y_\mu)$ -closed and the induced topology of $\tau(X, Y)$ on X_μ is coarser than $\tau(X_\mu, Y_\mu)$ (H. H. Schaefer [8], Chap. IV, § 4).

On the other hand, by Lemma 3-2 we have

$$L^2(X, B(X, Y), \mu) = L^2(X, C(X, Y), \mu) = L^2(X_\mu, C(X_\mu, Y_\mu), \mu).$$

Therefore, in order to prove either the separability of

$$L^2(\mu) = L^2(X, B(X, Y), \mu)$$

or the $\tau(X, Y)$ -separability of X_μ , we have only to prove it in the case of

$$(C.II) \quad X = X_\mu.$$

In this case we have $Y = Y_\mu$ and by Lemma 4-1 we can easily show that for ξ in Y

$$(C.II') \quad \langle \cdot, \xi \rangle = 0, \quad \text{a. e. } (\mu) \quad \text{if and only if } \xi = 0,$$

and that R_μ is an injection of Y into H_μ .

THEOREM 4-2. — Let (X, Y) be a topological dual system and μ be a Gaussian Radon measure on $(X, B(X, Y))$. Then $L^2(\mu) = L^2(X, B(X, Y), \mu)$ is a separable Hilbert space.

Proof. — We may assume (C.II) without loss of generality.

Since μ is also a Gaussian Radon measure on $(X, W(X, Y))$, there exists an increasing sequence of $\sigma(X, Y)$ -compact subsets $\{K_n\}$ of X such that

$$\lim_n \mu(K_n) = 1.$$

Let Z be the linear hull of $\bigcup_n K_n$ and denote by ρ the topology on Y of uniform convergence on all K_n . Then, using (C.II'), we can easily show that ρ is locally convex metrizable and finer than $\sigma(Y, Z)$. On the other hand, we have $Z \subset (Y, \rho)' \cap X$ and $\mu^*(Z) = 1$. Therefore ρ is an admissible metric on Y and consequently, by Theorem 2-6 and Lemma 3-2, $L^2(X, B(X, Y), \mu) = L^2(X, C(X, Y), \mu)$ is separable.

This proves the theorem.

5. SEPARABILITY OF THE SUPPORT OF A GAUSSIAN RADON MEASURE

Using the preceding results, we prove the following theorem.

THEOREM 5-1. — Let (X, Y) be a dual system and μ be a centered Gaussian Radon measure on $(X, W(X, Y))$. Then the topological linear support X_μ of μ is $\tau(X, Y)$ -separable.

Proof. — As stated in § 4, we may assume the condition (C.II), that is, $X = X_\mu$. The measure μ can be extended to a centered Gaussian Radon

measure on $(Y^a, W(Y^a, Y))$. Then X is $\overline{W}(Y^a, Y)$ -measurable, hence by theorem 3-4 (2), we have $R_\mu^* H_\mu \subset X$. It is known $\mu(\overline{R_\mu^* H_\mu}^{\tau(X, Y)}) = 1$, where $-\tau(X, Y)$ means the closure for $\tau(X, Y)$ in X , remark that $\overline{R_\mu^* H_\mu}^{\tau(X, Y)} = R_\mu^* H_\mu^{00}$ (the bipolar) $= \{x \in X; \langle x, \xi \rangle = 0 \text{ for all } \xi \in R_\mu^* H_\mu^0\}$ and for every $\xi \in R_\mu^* H_\mu^0$ $\mu(\{X \in X | \langle x, \xi \rangle = 0\}) = 1$. The minimality of $X = X_\mu$ implies that $R_\mu^* H_\mu$ is $\tau(X, Y)$ -dense in X . By theorem 4-2, H_μ is separable and $X = X_\mu$ is $\tau(X, Y)$ -separable.

This proves the theorem.

For non-centered Gaussian Radon measures we have the following result.

THEOREM 5-2. — Let (X, Y) be a dual system and μ be a Gaussian Radon measure on $(X, W(X, Y))$ satisfying the condition (C. I). Then the topological linear support X_μ of X is $\tau(X, Y)$ -separable.

Proof. — As stated in Section 4, we have only to prove in the case $X = X_\mu$.

Let $\{F_n\}$ be the increasing sequence of $\sigma(X, Y)$ -compact absolutely convex subsets of X given in (C. I). Then the topology ρ on Y of the uniform convergence on all F_n is finer than $\sigma(Y, X)$ and coarser than $\tau(Y, X)$ and therefore an admissible metric on Y . By Lemma 2-3, R_μ is a compact linear transformation of (Y, ρ) , *a fortiori*, of $(Y, \tau(Y, X))$ into H_μ and H_μ is separable. Consequently R_μ^* is also a compact linear transformation of H_μ into $(X, \tau(X, Y))$ with dense range and this proves the theorem.

REFERENCES

- [1] A. BADRIKIAN and S. CHEVET, Mesures cylindriques, espaces de Wiener et fonctions aléatoires gaussiennes. *Lecture notes in mathematics*, Vol. 379, Springer-Verlag, Berlin, 1974.
- [2] R. M. DUDLEY, The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. of Func. Anal.*, Vol. 1, 1967, p. 290-330.
- [3] R. M. DUDLEY, J. FELDMAN and L. LECAM, On semi-norms and probabilities, and abstract Wiener measures. *Ann. of Math.*, Vol. 93, 1971, p. 390-408.
- [4] J. KUELBS, Some results for probability measures on linear topological vector spaces with an application to Strassen's loglog law. *J. of Func. Anal.*, Vol. 14, 1973, p. 28-43.
- [5] J. NEVEU, *Martingales à temps discret*. Masson et Cie, Paris, 1972.
- [6] Ju. A. ROZANOV, Infinite-dimensional Gaussian distributions. *Proc. Steklov Inst. Math.*, Vol. 108, 1968, A. M. S. (English translation).
- [7] H. SATO, Gaussian measures on a Banach space and abstract Wiener measure. *Nagoya Math. J.*, Vol. 36, 1969, p. 65-81.
- [8] H. H. SCHAEFER, *Topological vector spaces*. MacMillan, New York, 1966.
- [9] L. SCHWARTZ, Radon measures on arbitrary topological vector spaces (*to appear*).
- [10] N. N. VAKHANIA, On some questions of the theory of probability measures on Banach spaces. *Lecture note of Nagoya University*, 1973.

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