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Section **B** :
Calcul des Probabilités et Statistique.

Iterates of a convolution on a non abelian group

by

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ABSTRACT. — Sufficient conditions are studied for $\| \nu * \mu^n \| \to 0$ when μ is a probability measure and ν is absolutely continuous with respect to the left Haar measure.

RÉSUMÉ. — On étudie des conditions suffisantes pour que $\| \nu * \mu^n \| \to 0$ lorsque μ est une probabilité et lorsque ν est une mesure absolument continue par rapport à la mesure de Haar à gauche.

NOTATION. — We shall use the notation of [*2*]: let G be a locally compact group, which is not necessarily Abelian. Let λ be its left Haar measure and M_a the two sided ideal of bounded measures which are absolutely continuous with respect to λ: see [*2*], Theorem 19.18. Note that M_a is identified with $L_1(\lambda)$ by the Radon Nikodym Theorem.

The convolution of two measures, τ and θ, is given by

$$(\tau * \theta)(A) = \int_G \theta(x^{-1}A)\tau(dx)$$

and if f is a bounded measurable function then

$$\langle \tau * \theta, f \rangle = \int f d(\tau * \theta) = \iint f(x \cdot y)\theta(dy)\tau(dx) = \langle \tau, \theta * f \rangle.$$

Let μ be a fixed probability measure. If $\nu \in M_a$ then $\nu * \mu \in M_a$ and

$\mu * \nu \in M_a$. Thus μ acts as an operator on M_a and on $L_\infty(\lambda) = M_a^*$. Note that if τ is a non negative measure then $\| \tau \| = \langle \tau, 1 \rangle$. Put $\mu^n = \mu * \dots * \mu$ (n times), $\mu^0 = \delta$ the identity.

THEOREM 1. — *Let μ and η be probability measures with $\mu * \eta = \eta * \mu$. Assume there is an integer h and a non negative measure ν, $\nu(G) > 0$, such that $\mu^h \geq \nu$ and $\mu^h \geq \nu * \eta$, then*

$$\| \mu^n - \mu^n * \eta \| \underset{n \to \infty}{\to} 0$$

Proof. — Put

$$\mu^h = \nu + \tau_1 = \nu * \eta + \tau_2 = \nu * \frac{1}{2}(\delta + \eta) + \theta_1$$

where ν and θ_1 are non negative. Thus

$$\begin{aligned}\mu^{2h} = \nu * \mu^h * \frac{1}{2}(\delta + \eta) + \theta_1 * \mu^h \\ &= \nu^2 * \left(\frac{1}{2}(\delta + \eta)\right)^2 + \nu * \theta_1 * \frac{1}{2}(\delta + \eta) + \theta_1 * \mu^h \\ &= \nu^2 * \left(\frac{1}{2}(\delta + \eta)\right)^2 + \theta_2.\end{aligned}$$

Continue the same computation to obtain

$$\mu^{nh} = \nu^n * \left(\frac{1}{2}(\delta + \eta)\right)^n + \theta_n \tag{1}$$

Now

$$\langle \theta_n, 1 \rangle = \langle \mu^{nh}, 1 \rangle - \left\langle \nu^n * \left(\frac{1}{2}(\delta + \eta)\right)^n, 1 \right\rangle = 1 - \langle \nu^n, 1 \rangle = 1 - \langle \nu, 1 \rangle^n < 1. \tag{2}$$

Let us use (1) now to obtain

$$\begin{aligned}\mu^{2nh} = \nu^n * \mu^{nh} * \left(\frac{1}{2}(\delta + \eta)\right)^n + \theta_n * \mu^{nh} \\ &= \nu^n * \mu^{nh} * \left(\frac{1}{2}(\delta + \eta)\right)^n + \theta_u * \nu^n * \left(\frac{1}{2}(\delta + \eta)\right)^n + \theta_n^2 \\ &= \rho_2 * \left(\frac{1}{2}(\delta + \eta)\right)^n + \theta_n^2\end{aligned}$$

and by an induction argument

$$\mu^{jnh} = \rho_j * \left(\frac{1}{2}(\delta + \eta)\right)^n + \theta_n^j \qquad (\rho_j \geq 0,\ \theta_n \geq 0,\ \langle \theta_n, 1 \rangle < 1). \tag{3}$$

Thus

$$\| \mu^{jnh} * (\delta - \eta) \| \leq \frac{1}{2^n} \left\| \sum_{i=0}^{n} \binom{n}{i} \eta^i - \sum_{i=0}^{n} \binom{n}{i} \eta^{i+1} \right\| + 2 \langle \theta_n, 1 \rangle^j.$$

Now the first sum is smaller then $\frac{\text{Const}}{\sqrt{n}}$ see [3], p. 1632, and $\langle \theta_n, 1 \rangle^j \underset{j \to \infty}{\to} 0$.

The following Corollary is included in [1], Theorem V.1. We present it here since the proof is very different. Let H be the center of G: if $y \in H$, $x \in G$ then $xy = yx$. Denote by δ_x the Dirac measure at x. Thus $\delta = \delta_e$ where $ex = xe = x$ for all $x \in G$. Let H_0 be the set of points in H such that μ^n and $\mu^n * \delta_x$ are not mutually singular for all n. If $x \in H_0$ then $x^{-1} \in H_0$ too.

COROLLARY 1. — *Let the closed group generated by* H_0 *be* H. *Let* ν *be a measure on* H *which is absolutely continuous with respect to the left Haar measure on* H. *Then* $\| \nu * \mu^n \| \to 0$ *provided* $\nu(H) = 0$.

Proof. — By the Hahn Banach Theorem it is enough to show that if f is a bounded measurable function on H such that $\langle \nu, f \rangle = 0$ whenever $\| \nu * \mu^n \| \to 0$ then $f =$ Const. a. e. with respect to the left Haar measure on H.

Now if θ is a measure on H and is absolutely continuous with respect to the left Haar measure on H then, by Theorem 1, if $x \in H_0$ then

$$\| (\theta - \theta * \delta_x) * \mu^n \| = \| \theta * (\mu^n - \mu^n * \delta_x) \| \to 0.$$

Thus

$$\langle \theta, f \rangle = \langle \theta * \delta_x, f \rangle \quad \text{for} \quad \text{all} \quad x \in H_0.$$

Now, this equality remains true for the closed subgroup generated by H_0 namely all of H. Thus $f =$ Const. a. e. with respect to the left Haar measure on H.

Throughout the rest of the paper we shall assume.

ASSUMPTION 1. — *The measures* μ^n *are not pairwise mutually singular.*

Thus there exists two integers n and k and a non negative measure ν with $\nu(G) > 0$ such that $\mu^n \geq \nu$, $\mu^{n+k} \geq \nu$.

Therefore

$$\mu^{n+k} \geq \nu \quad \text{and} \quad \mu^{n+k} \geq \nu * \mu^k$$

and we may use Theorem 1 with $\eta = \mu^k$:

COROLLARY 2. — *If* μ^n *and* μ^{n+k} *are not mutually singular then*

$$\lim_{m \to \infty} \| \mu^m - \mu^{m+k} \| = 0.$$

Remark. — If $k = 1$ it follows that either $\| \mu^n - \mu^{n+1} \| = 2$ for all n or $\| \mu^n - \mu^{n+1} \| \to 0$. This « zero-two » law, was proved in [3], Theorem 3.1, for general Markov operators that are ergodic and conservative. In our case the measure μ induces a random walk which need not be neither ergodic nor conservative.

Now if $\| \mu^n - \mu^{n+1} \| \to 0$ then the spectrum of μ intersects the circumference of the unit circle only at the point 1: Let φ be a homomorphism of the commutative Banach Algebra generated by μ. Then

$$| \varphi(\mu) |^n | 1 - \varphi(\mu) | \to 0 \qquad \text{hence either} \qquad | \varphi(\mu) | < 1 \text{ or } \varphi(\mu) = 1.$$

By Gelfand's Theory the spectrum of μ, even in the smaller algebra, touches the circumference of the unit circle only at 1. In particular: if $\mu * f = \alpha f$ where $| \alpha | = 1$ then $\alpha = 1$.

Let us conclude with:

THEOREM 2. — *Let* $\| \mu^n - \mu^{n+k} \| < 2$. *Let* $\nu \in M_a$ *then* $\| \nu * \mu^m \| \to 0$ *if and only if* $\langle \nu, f \rangle = 0$ *for all* $f \in L_\infty(\lambda)$ *with* $\mu^k * f = f$ a. e. λ.

Proof. — If $\mu^k * f = f$ a. e. λ then

$$| \langle \nu, f \rangle | = | \langle \nu, \mu^{jk} * f \rangle | = | \langle \nu * \mu^{jk}, f \rangle | \leq \| \nu * \mu^{jk} \| \, \| f \|.$$

Thus if $\| \nu * \mu^m \| \to 0$ and $\mu^k * f = f$ then $\langle \nu, f \rangle = 0$.

Conversely, put $A = \{ \nu : \nu \in M_a \text{ and } \| \nu * \mu^m \| \to 0 \}$. If $\langle \nu, f \rangle = 0$ for all $\nu \in A$ then by Corollary 2 $\langle \theta * (\delta - \mu^k), f \rangle = 0$ for all $\theta \in M_a$. Thus $f = \mu^k * f$ a. e. λ and our result follows from the Hahn Banach Theorem.

Remark. — If $k = 1$ and the Choquet-Deny Theorem holds namely: $\mu * f = f$ a. e. λ implies $f =$ Const. a. e. λ then $\nu \in A$ if and only if $\nu(G) = 0$.

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