

# ANNALES DE L'I. H. P., SECTION B

DONALD GEMAN

JOSEPH HOROWITZ

## **Remarks on Palm Measures**

*Annales de l'I. H. P., section B*, tome 9, n° 3 (1973), p. 215-232

[http://www.numdam.org/item?id=AIHPB\\_1973\\_\\_9\\_3\\_215\\_0](http://www.numdam.org/item?id=AIHPB_1973__9_3_215_0)

© Gauthier-Villars, 1973, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (<http://www.elsevier.com/locate/anihpb>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## Remarks on Palm Measures

by

Donald GEMAN and Joseph HOROWITZ

University of Massachusetts

### 0. INTRODUCTION

This paper consists of several remarks on the theory of Palm measures and its application to Markov processes. In § 1, using Totoki's definition [17] of the Palm measure of an additive functional (AF), we derive a kind of « strong law of large numbers » which shows how Palm measures arise as limits of relative frequencies. Some examples due to Cramér and Leadbetter [5] and Kac and Slepian [10] then follow as special cases. The notion of Palm measure used here is more suited to the study of random processes than that of Mecke [14], as will be explained below.

Section 2 contains some results on Markov processes which are suggested by, but do not really depend, on those of § 1. Given the existence of « strictly recurrent points »—which is often easy to verify—we obtain a  $\sigma$ -finite invariant measure with an explicit representation in terms of local times. Other results are closely related to recent work of Revuz [15] and Azema, Duflo, and Revuz [1].

Finally, in § 3, we compute the invariant measure explicitly for a certain class of Markov processes. In so doing, we obtain a « concrete » representation of the stationary regenerative phenomena of Kingman [12].

#### NOTATION

$\mathbb{R}(\mathbb{R}_+)$  denotes the real line (positive half-line  $[0, \infty)$ ), always endowed

This work was partially supported by National Science Foundation Grant GP34485.

with Borel  $\sigma$ -field  $\mathcal{B}$  ( $\mathcal{B}_+$ ). If  $(E, \mathcal{E})$  is a measurable space,  $(\mathcal{E})$  denotes the family of extended real-valued  $\mathcal{E}$ -measurable functions on  $E$ . All other notation may be found in [3, Ch. 0].

## 1. PALM MEASURES

By a *flow* on a probability space  $(\Omega, \mathcal{F}, P)$  we mean a family  $\theta = (\theta_t)$ ,  $t \in \mathbb{R}$ , of measurable, measure-preserving transformations  $\theta_t : \Omega \rightarrow \Omega$  such that (1)  $\theta_0 = \text{identity on } \Omega$ ; (2)  $\theta_{t+s} = \theta_t \circ \theta_s$ ,  $s, t \in \mathbb{R}$ ; and (3) the mapping  $(t, \omega) \rightarrow \theta_t(\omega)$  is measurable. Each  $\theta_t$  is then invertible,  $\theta_t^{-1} = \theta_{-t}$ , and  $P(\theta_t^{-1}A) = P(\theta_t A) = P(A)$ ,  $t \in \mathbb{R}$ ,  $A \in \mathcal{F}$ . If  $\sigma : \Omega \rightarrow \mathbb{R}$  is measurable we write  $\theta_\sigma$  for the measurable function  $\omega \rightarrow \theta_{\sigma(\omega)}(\omega)$ . In general  $\theta_\sigma$  is not measure-preserving. A set  $A \in \mathcal{F}$  is *invariant* if  $\theta_t A = A$  for every  $t \in \mathbb{R}$ . The family of such sets is a  $\sigma$ -field  $\mathcal{A}$ . The flow is *ergodic* if  $A \in \mathcal{A} \Rightarrow P(A) = 0$  or 1.

An *additive functional* (AF) is a real-valued process  $\alpha = (\alpha(t, \omega))$  such that (a)  $\alpha(0) = 0$ ; (b) almost every trajectory is right-continuous, non-decreasing; and (c) for each  $s, t \in \mathbb{R}$ ,

$$(1.1) \quad \alpha(t + s, \omega) = \alpha(t, \omega) + \alpha(s, \theta_t \omega) \text{ a. s.}$$

If almost every trajectory  $\alpha(\cdot, \omega)$  is continuous, we call  $\alpha$  a *continuous AF* (CAF). We shall often construe  $\alpha(\cdot, \omega)$  as a measure on  $\mathbb{R}$  without special mention.

Before going on, let us briefly outline Mecke's approach [14] to Palm measures. Although his work is formulated in the context of an arbitrary locally compact Abelian group, we shall restrict our discussion to the real line  $\mathbb{R}$ . Denote by  $M$  the family of measures  $\mu$  on  $\mathbb{R}$  which are finite on compacts;  $M$  is endowed with the smallest  $\sigma$ -field  $\mathcal{M}$  which renders measurable all the mappings  $B \rightarrow \mu(B)$ ,  $B \in \mathcal{B}$ ,  $\mu \in M$ . A probability measure  $Q$  on  $\mathcal{M}$  is *stationary* if  $Q(T_t^{-1}D) = Q(D)$  for all  $D \in \mathcal{M}$ ,  $t \in \mathbb{R}$ , where  $T_t$  is translation by  $t$ :  $T_t \mu(B) = \mu(B + t)$ . For each such  $Q$ , Mecke defines the *Palm measure*  $Q^0$  by

$$(1.2) \quad Q^0(D) = \int_M \int_0^1 I_D(T_t \mu) \mu(dt) Q(d\mu), D \in \mathcal{M}.$$

Now in the analysis of random processes one is usually interested only in certain elements of  $M$ , namely the trajectories  $\alpha(\cdot, \omega)$  of some AF. Define  $f : \Omega \rightarrow M$  by  $f(\omega) = \alpha(\cdot, \omega)$  and take  $Q(D) = P \circ f^{-1}(D)$ ,

$D \in \mathcal{M}$ , i. e.  $Q$  is the distribution of  $\alpha$  over  $M$ . Using (1.1) the stationarity of  $Q$  is clear. The Palm measure in this case is given by

$$(1.3) \quad Q^0(D) = E \int_0^1 I_{f^{-1}(D)} \circ \theta_t d\alpha(t), \quad D \in \mathcal{M}.$$

This suggests defining a measure  $P^0$  on  $f^{-1}(\mathcal{M}) \subset \mathcal{F}$  by the right member of (1.3). However, the  $\sigma$ -field  $f^{-1}(\mathcal{M})$  is too small to be of use for random processes: it contains only those events determined by the AF  $\alpha$ . (For example, if  $\alpha(t, \omega) \ll$  counts » the zeros during  $(0, t]$  of a stationary process,  $f^{-1}(\mathcal{M})$  cannot distinguish between two paths with the same zeros.) To overcome this difficulty we define (following Totoki [17]) the *Palm measure*  $\hat{P}_\alpha$  corresponding to  $\alpha$  by

$$(1.4) \quad \hat{P}_\alpha(A) = E \int_0^1 I_A \circ \theta_t d\alpha(t), \quad A \in \mathcal{F}.$$

Mecke's results, properly restated, now carry over to  $\hat{P}_\alpha$  largely as in [14]. In particular,  $\hat{P}_\alpha$  is always  $\sigma$ -finite [17], and obviously finite iff  $E\alpha(1) < \infty$  (we call  $\alpha$  *integrable* in this case). When no confusion results, we shall write  $\hat{P}$  for  $\hat{P}_\alpha$ .

The analogue of Satz 2.3 of [14] is

THEOREM 1. —

$$E \int_{\mathbb{R}} u(s, \theta_s \omega) d\alpha(s, \omega) = \hat{E} \int_{\mathbb{R}} u(s, \omega) ds, \quad u \in (\mathcal{B} \times \mathcal{F})_+$$

where  $\hat{E}$  is integration with  $\hat{P}$ .

For  $A \in \mathcal{F}$ , the process  $\int_0^t I_A \circ \theta_s(\omega) d\alpha(s, \omega)$  satisfies (1.1), so

$$(1.5) \quad t\hat{P}(A) = E \int_0^t I_A \circ \theta_s d\alpha(s), \quad t \in \mathbb{R}.$$

The theorem is now obvious if  $u$  is the indicator of a measurable rectangle, hence for all  $u$  as stated.

It is shown in [13] that when the flow is ergodic and an invariant set of measure zero is excluded from  $\Omega$ , the exceptional set in (1.1) may be chosen independently of  $s, t$ . The proof extends easily to the general case, so that

we may and do assume (1.1) holds for every  $s, t$ , and  $\omega$ . It will be helpful to exclude several other invariant null sets, as follows. By the ergodic theorem,  $\lim_{t \rightarrow \infty} \alpha(t, \omega)/t$  exists a. s.:

$$\int_0^{t-1} \alpha(1) \circ \theta_s ds \leq \alpha(t) \leq \int_{-1}^t \alpha(1) \circ \theta_s ds.$$

Since the exceptional set is invariant, we remove it if necessary and define  $z(\omega) = \lim_{t \rightarrow \infty} \alpha(t, \omega)/t$ ,  $\omega \in \Omega$ , and  $M = \{z > 0\}$ . Clearly,  $\alpha(t, \omega) \rightarrow \pm \infty$  as  $t \rightarrow \pm \infty$  on  $M$ ; moreover,  $z \in (\mathcal{A})_+$  is a version of  $E(\alpha(1) | \mathcal{A})$ , from which one verifies that  $\alpha(t, \omega) \equiv 0$  a. s. on  $M^c$ . Removing, finally, this exceptional invariant set we have:  $\Omega = M \cup \{\alpha(t, \omega) \equiv 0\}$ . To avoid trivialities, we assume  $M \neq \emptyset$ .

Let  $\hat{\alpha}(s)$  be the *time change* [3], [17] associated with  $\alpha$ :

$$\hat{\alpha}(s) = \inf \{t : \alpha(t) > s\}.$$

Then  $\hat{\alpha}(s)$  is finite on  $M$ . For  $\omega \in M$  put  $\hat{\theta}_s(\omega) = \theta_{\hat{\alpha}(s)}(\omega)$ , and  $\hat{\theta}_s(\omega) = \omega$  if  $\omega \notin M$ . If  $\alpha$  is a CAF, we have (i)  $\hat{\alpha}(t + s, \omega) = \hat{\alpha}(t, \omega) + \hat{\alpha}(s, \hat{\theta}_t \omega)$  and  $\hat{\theta}_{t+s} = \hat{\theta}_t \circ \hat{\theta}_s$  for every  $s, t \in \mathbb{R}$  and  $\omega \in \Omega$ , (ii)  $\hat{P}$  is preserved by  $\hat{\theta}$ , and (iii)  $\hat{P}$  is concentrated on  $\hat{\Omega} = \{\omega : \hat{\alpha}(0, \omega) = 0\}$ . Thus [17]  $\hat{\theta}$  is actually a flow (called the *time-changed* or *dual flow*) on  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ , where  $\hat{\mathcal{F}} = \Omega \cap \mathcal{F}$ , and  $\hat{\alpha}$  is an AF in the « dual » system  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \hat{\theta})$ . Proper choice of  $u$  in Theorem 1 now gives.

COROLLARY 1. — a)

$$E(X\alpha(t)) = \hat{E} \int_0^t X \circ \theta_{-s} ds, \quad X \in (\mathcal{F})_+, \quad (u(s, \omega) = I_{[0, t[}(s) X \circ \theta_{-s}(\omega)).$$

If, moreover,  $\alpha$  is a *continuous* AF,

$$(b) \quad \hat{E}(X\hat{\alpha}(t)) = E \int_0^t X \circ \hat{\theta}_{-s} ds, \quad X \in (\hat{\mathcal{F}})_+, \quad (u(s, \omega) = I_{(\hat{\alpha}(-t, \theta_{-s}\omega), 0]}(s) X(\omega))$$

$$(c) \quad tE(X) = \hat{E} \int_0^{\hat{\alpha}(t)} X \circ \theta_s ds, \quad X \in (\mathcal{F})_+, \\ (u(s, \omega) = I_{(\hat{\alpha}(-t, \theta_{-s}\omega), 0]}(s) X \circ \theta_{-s}(\omega))$$

We omit the proofs, which utilize the identities  $\hat{\alpha}(t) \circ \theta_s = \hat{\alpha}(t + \alpha(s)) - s$  and  $\{\alpha(s) \leq t\} = \{\hat{\alpha}(t) \geq s\}$  for any CAF  $\alpha$ .

Part (a) is Ryll-Nardzewski's [16] defining equation for Palm probabilities for point processes; part (b), which is new, is the « dual » of (a)

in the system  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\theta})$ . Finally, the inversion formula (c) is well known for point processes (with appropriate modification); the assumed continuity of  $\alpha$  allows a slight improvement over Satz 2.4 of [14]. Observe that, with  $t = 1$  and  $X = I_A$ , (c) is the « dual » of (1.4) after a change of variable in the latter. But we cannot in general rewrite (c) as

$$(*) \quad EX = \hat{E} \int_0^1 X \circ \hat{\theta}_s d\hat{\alpha}(s),$$

which would be the exact dual of (1.4). Indeed the change of variable  $\hat{\alpha}(s) \rightarrow s$  leads from (\*) to (c) only if  $\hat{\alpha}(\alpha(t)) = t$  for almost every  $t$   $\hat{\mathbb{P}}$ -a. s., and this need not be so (e. g. Markov local times). Notice, however, that  $\hat{\alpha}(\alpha(t)) = t + \hat{\alpha}(0, \theta_t)$ . Thus,

$$\begin{aligned} \hat{E} \int_0^1 X \circ \hat{\theta}_s d\hat{\alpha}(s) &= \hat{E} \int_0^{\hat{\alpha}(1)} X \circ \theta_{\hat{\alpha}(s)} ds \\ &= \hat{E} \int_0^{\hat{\alpha}(1)} X \circ \theta_{s + \hat{\alpha}(0, \theta_s)} ds \\ &= \hat{E} \int_0^{\hat{\alpha}(1)} X \circ \theta_{\hat{\alpha}(0)} \circ \theta_s ds. \end{aligned}$$

Now (c) yields the dual Palm measure:

$$\hat{E} \int_0^1 X \circ \hat{\theta}_s d\hat{\alpha}(s) = E(X \circ \hat{\theta}_0).$$

If  $\alpha(\cdot, \omega)$  is strictly increasing  $\mathbb{P}$ -a. s.,  $\hat{\theta}_0 = \text{identity}$   $\mathbb{P}$ -a. s., (\*) holds, and we have perfect duality. There is no counterpart to  $\hat{\alpha}$  and  $\hat{\theta}$  in the set-up of [14].

Another consequence of Theorem 1 having no analogue in [14] is:

**THEOREM 2.** — a) Let  $\alpha$  and  $\beta$  be AFs. Then  $\hat{\mathbb{P}}_\alpha = \hat{\mathbb{P}}_\beta$  iff the trajectories  $\alpha(\cdot, \omega)$ ,  $\beta(\cdot, \omega)$  coincide for almost every  $\omega \in \Omega$ .

b) Let  $\mathbb{P}, \mathbb{Q}$  be stationary probabilities on  $(\Omega, \mathcal{F}, \theta)$ , and  $\alpha$  an AF relative to both. Then  $\hat{\mathbb{P}} = \hat{\mathbb{Q}}$  iff  $\mathbb{P}(\text{AM}) = \mathbb{Q}(\text{AM})$  for all  $A \in \mathcal{F}$ .

The proof of (a) appears in [8]. As for (b), from Corollary 1 (a), we have

$$\int_{\{\alpha(t) > 0\}} X d\mathbb{P} = \int_{\{\alpha(t) > 0\}} X d\mathbb{Q}$$

if  $\hat{\mathbb{P}} = \hat{\mathbb{Q}}$ , and the « only if » statement follows on making  $t \rightarrow \infty$ . The « if » statement is immediate from (1.4).

The following « strong law of large numbers » is analogous to the ergodic theorem for Markov AFs in [I, § 1.2).

THEOREM 3. — Let  $\alpha, \beta$  be AFs. Then:

$$(a) \lim_{t \rightarrow \infty} t^{-1} \beta(t) = E(\beta(1) | \mathcal{A}) \hat{P}_\alpha\text{-a. s.}$$

If, in addition,  $\alpha$  is an integrable CAF,

$$(b) \lim_{t \rightarrow \infty} t^{-1} \beta(\hat{\alpha}(t)) = (E\alpha(1))^{-1} \hat{E}_\alpha(\beta(\hat{\alpha}(1)) | \hat{\mathcal{A}}) \circ \hat{\theta}_0 \text{P-a. s. on } M, \text{ where } \hat{\theta}_t = \theta_{\hat{\alpha}(t)}, \text{ and } \hat{A} \text{ is the } \hat{\theta}\text{-invariant } \sigma\text{-field in } \hat{\Omega}.$$

The limit in (a) exists P-a. s. by the ergodic theorem. Let  $B \in \mathcal{A}$  be the set where it fails to exist,  $P(B) = 0$ . From (1.5),

$$t \hat{P}_\alpha(B) = \int_B \alpha(t) dP = 0.$$

To get (b), we first observe that  $\gamma(t) \equiv \beta(\hat{\alpha}(t))$  satisfies

$$\gamma(t + s, \omega) = \gamma(t, \omega) + \gamma(s, \hat{\theta}_t \omega)$$

on  $M$ . Let  $\tilde{\mathcal{A}}$  be the  $\sigma$ -field in  $\Omega$  of sets  $A \in \mathcal{F}$  for which  $\hat{\theta}_s^{-1} A = A, s \in \mathbb{R}$ . Since  $\hat{\theta}$  preserves  $\hat{P}_\alpha$  (as a measure on  $\mathcal{F}$ ), the ergodic theorem yields

$$t^{-1} \gamma(t) \rightarrow \hat{E}_\alpha(\gamma(1) | \tilde{\mathcal{A}}) / E\alpha(1), \quad \hat{P}_\alpha\text{-a. s.}$$

The set  $C$ , where this limit does not exist, belongs to  $\tilde{\mathcal{A}}$  and  $\hat{P}_\alpha(C) = 0$ , so  $P(C) = 0$  by Corollary 1 (b). Finally, one shows easily that, for  $Y \in (\mathcal{F})_+$ ,

$$\hat{E}_\alpha(Y | \tilde{\mathcal{A}}) = \hat{E}_\alpha(Y | \hat{\mathcal{A}}) \circ \hat{\theta}_0$$

except on a set in  $\tilde{\mathcal{A}}$  of  $\hat{P}_\alpha$ -measure zero, hence of P-measure zero.

Notes. — (1) The ergodic theorem shows that  $\hat{\alpha}(t)/t \rightarrow 1/z \hat{P}_\alpha\text{-a. s.}$  so that the limit in (b) may be rewritten as  $z_\beta/z$ , where  $z_\beta = \lim t^{-1} \beta(t)$ .

(2) An important special case of (b) is  $\beta(t) = \int_0^t X \circ \theta_s d\alpha(s), X \in (\mathcal{F})_+$  bounded. After a change of variable (b) reads

$$(1.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X \circ \hat{\theta}_s ds = (E\alpha(1))^{-1} \hat{E}_\alpha(X | \hat{\mathcal{A}}) \circ \hat{\theta}_0 \quad \text{P-a. s. on } M.$$

From (a), we obtain the « dual » of (1.6):

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t X \circ \theta_s ds = E(X | \mathcal{A}) \quad \hat{P}_\alpha\text{-a. s.}$$

(3) If  $M = \Omega$ ,  $P$  and  $\hat{P}_\alpha$  are actually equivalent (i. e. have the same sets of measure zero) on  $\mathcal{A}$ ,  $\tilde{\mathcal{A}}$ . For, if  $A \in \mathcal{A}$ , (1.5) gives

$$\hat{P}_\alpha(A) = \int_A \alpha(t)/t dP = \int_A z dP.$$

The « dual » argument works for  $\tilde{\mathcal{A}}$ : Corollary 1 (b) yields

$$P(A) = \int_A \hat{\alpha}(t)/t d\hat{P}_\alpha = \int_A \left(\frac{1}{z}\right) d\hat{P}_\alpha, \quad A \in \tilde{\mathcal{A}}.$$

(4) Suppose  $M = \Omega$  and  $\hat{P}$  is a  $\sigma$ -finite measure on  $(\Omega, \mathcal{F})$  which is preserved by  $\hat{\theta}$ , with  $\hat{E}\hat{\alpha}(1) < \infty$ . Define a finite measure  $P$  by setting  $t = 1$ ,  $X = I_A$  in the right member of Corollary 1 (c). Then  $P$  will be stationary for  $(\Omega, \mathcal{F}, \theta)$ , and will have Palm measure  $\hat{P}$ .

(5) Theorem 3 remains valid for integrable AFs  $\alpha$  which increase only by unit jumps (« counters ») if in (b) we let  $t$  be integer-valued and replace  $\hat{\alpha}(t)$  by  $R_k = \inf \{ t : \alpha(t) > k - 1 \}$ , and  $\hat{\theta}_t$  by  $\hat{\theta}_k = \theta_{R_k}$ . Now for counters it is known [16] that  $(\Omega, \mathcal{F}, P, \theta)$  is ergodic iff  $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \hat{\theta})$  is ergodic, and the same proof works for CAFs. In the ergodic case (1.6) for counters becomes

$$(1.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X \circ \hat{\theta}_k = \hat{E}(X)/E\alpha(1) \quad P\text{-a. s.}$$

We consider three examples.

*Example 1.* —  $x(t)$  is an ergodic, differentiable, stationary Gaussian process. Let  $R_1, R_2, \dots$  be the times of the successive zeros of  $x(t)$ . Then Kac and Slepian [10] show that the proportion of times  $R_1, \dots, R_n$  at which  $x'(R_i) \leq b$  converges a. s. to  $P \{ x'(0) \leq b \mid x(0) = 0 \}$ , where the « hw » (horizontal-window) probability is defined as

$$\lim_{\varepsilon \downarrow 0} P \{ x'(0) \leq b \mid x(s) = 0 \quad \text{for some } s \in (0, \varepsilon) \}.$$

Let  $v(t)$  count the zeros of  $x(\cdot)$  during time  $(0, t]$ . Then  $v$  is a « counter » (note (3) above), and taking  $X = I_{\{x'(0) \leq b\}}$  in (1.7) we find that the proportion of zeros at which  $x'(R_i) \leq b$  converges a. s. to  $\hat{P}_v \{ x'(0) \leq b \} / E v(1)$ . Using the ideas of [16] and Corollary 1 (a), it is easy to identify  $(E v(1))^{-1} \hat{P}_v$  as the hw probability, so that the Kac-Slepian result is a special case of (1.7).

*Example 2.* — Consider an ergodic process of « calls » [16],  $v(t)$  = number of calls during  $(0, t]$  and  $v_x(t)$  = number of calls during  $(0, t]$  for which

no call occurs within a further time  $x$ . Taking  $X = I_{\{R_1 > x\}}$  in (1.7) one obtains  $v_x(t)/t \rightarrow \hat{P}_v(R_1 > x)$  a. s. Since  $v(t)/t \rightarrow E v(1)$  a. s., we have  $v_x(t)/v(t) \rightarrow \hat{P}_v(R_1 > x)/E v(1)$ . Now  $E v(1)$  is the « intensity » of the stream of calls, and  $(E v(1))^{-1} \hat{P}_v$  is the Palm probability of [16]. This result may be applied to the stationary stream of crossings of a fixed level  $u$  by a sufficiently « nice » stationary process (see [5]). Regarding each crossing as a « call », we obtain  $N_x(t)/N(t) \xrightarrow{a.s.} P \{ R_1 > x | X(0) = uhw \}$ , where  $N(t)$  is the number of  $u$ -crossings in  $(0, t]$ ,  $N_x(t)$  is the number of crossings followed by a  $u$ -free interval of length at least  $x$ , and  $R_1$  is the time of the first crossing. (The identification of  $(E v(1))^{-1} \hat{P}_v$  and the  $hw$  probability is as in Example 1.)

*Example 3.* — Consider an ergodic stationary Gaussian process  $x(t, \omega)$ , which is not differentiable in quadratic mean. Berman [2] has obtained conditions on the covariance under which there is an « occupation-time density »—a process  $\alpha_t(x, \omega)$ ,  $x \in \mathbb{R}$ ,  $t \geq 0$ , such that a. s.

$$(1.8) \quad \int_{\Gamma} \alpha_t(x, \omega) P(x(0) \in dx) = \int_0^t I_{\Gamma}(x(s, \omega)) ds, \quad t \geq 0, \quad \Gamma \in \mathcal{B}.$$

In addition,  $\alpha_t(x, \omega)$  may be chosen a. s. right-continuous and non-decreasing in  $t$  for every  $x$ . Assuming  $\mathcal{F}$  is separable and integrating (1.8) over  $A \in \mathcal{F}$ , we have, for almost every  $x$ ,

$$\int_0^t P^x(\theta_s A) ds = \int_A \alpha_t(x, \omega) P(d\omega) \quad \text{for all } A \in \mathcal{F}, \quad t \geq 0,$$

where  $\{P^x, x \in \mathbb{R}\}$ , is any regular version of the conditional probabilities  $P(\cdot | x(0) = x)$ . Stationarity now yields a version of  $\alpha$  which is an AF for almost every  $x$ , and it is easy to check (use Corollary 1 (a)) that  $\hat{P}_{\alpha(x)} = P^x$

for such  $x$ . Taking  $\beta(t) = \int_0^t \phi \circ \theta_s d\alpha_s(x)$ ,  $\phi \in (\mathcal{F})_+$  bounded, in Theorem 3 (a) we have

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^t \phi \circ \theta_s d\alpha_s(x) = E(\phi | x(0) = x) \text{ a. s. } P \text{ or } P^x, \text{ a. e. } [x].$$

Thus  $E(\phi | X(0) = x)$  arises as the limit of relative frequencies « sampled » along the support of  $\alpha_t(x)$ , a subset of  $\{t : x(t) = x\}$  if  $x(t)$  is continuous.

## 2. APPLICATION TO MARKOV PROCESSES

In this section we consider the analogues of the results of § 1 in the context of Markov processes. It should be noticed that the work of Azema,

Duflo, and Revuz [1], and especially Revuz [15] becomes very natural from the viewpoint of Palm measures. On the other hand, in some cases the theorems of § 1 are trivial in the Markov situation, being simple applications of the strong Markov property. The terminology of [3] will be used without further explanation.

Let  $X = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, \theta_t, P^x)$  be a standard process. The translation operators  $\theta_t$  [3, Ch. I] no longer constitute a flow in the sense of § 1, though we may and do assume properties (1), (3) of § 1, with  $s, t$  in (2) restricted to  $\mathbb{R}_+$ . Denote by  $M_1, M, M_\sigma$  the families of probability, finite,  $\sigma$ -finite measures, respectively, on the Borel  $\sigma$ -field  $\mathcal{E}$  of the state space  $E$ . For  $\mu \in M_\sigma$  we define a  $\sigma$ -finite measure

$$(2.1) \quad P^\mu(\Lambda) = \int_E \mu(dx) P^x(\Lambda), \quad \Lambda \in \mathcal{F},$$

which will be finite (probability) if  $\mu \in M(M_1)$ . Under  $P^\mu$ ,  $X$  is a strong Markov process (we still refer to « processes » even when the underlying measure  $P^\mu$  is not a probability) having  $\mu$  as *initial distribution*:  $\mu(\Gamma) = P^\mu \{x_0 \in \Gamma\}$ . Write  $\mu P_t(\Gamma) = P^\mu \{x_t \in \Gamma\}$ ,  $t \in \mathbb{R}_+$ ,  $\Gamma \in \mathcal{E}$ . If  $\mu = \mu P_t, \forall t$ , it is *invariant*, in which case  $X$  is strictly stationary under  $P^\mu$ .

We shall use the notation  $\langle \mu, f \rangle$  interchangeably with  $\int f d\mu$  below as convenience dictates.

Consider an AF  $A = (A_t)$  of  $X$ , as defined in [3, Ch. IV], having finite  $\lambda$ -potential, where  $\lambda > 0$  is fixed. For  $\mu \in M_\sigma$  we define a « Palm measure »

$$(2.2) \quad \hat{P}_A^\mu(\Lambda) = \lambda E^\mu \int_0^\infty e^{-\lambda t} I_{A_t} \circ \theta_t dA_t, \quad \Lambda \in \mathcal{F}.$$

This is really a Laplace-transformed version of (1.5); if  $\mu$  is invariant,  $\hat{P}_A^\mu$  is a Palm measure as in § 1. The present definition is better suited to Markov processes. Now let  $\Lambda \in \mathcal{F}$  and  $\hat{A}_t$  be the functional inverse of  $A_t$  (see § 1 and [3]). Changing variables:

$$\begin{aligned} \hat{P}_A^\mu(\Lambda) &= \lambda \int_E \int_0^\infty E^x(e^{-\lambda \hat{A}_t} I_{\Lambda_t} \circ \theta_{\hat{A}_t}) dt \mu(dx) \\ &= \lambda \int_E \int_0^\infty E^x(e^{-\lambda \hat{A}_t} P^{x \hat{A}_t}(\Lambda)) dt \mu(dx) \quad (\text{strong Markov property}) \\ &= \lambda \int_E E^x \int_0^\infty e^{-\lambda t} P^{x_t}(\Lambda) dA_t \mu(dx). \end{aligned}$$

Consequently,

$$(2.3) \quad \hat{P}_A^\mu = \lambda P^{\mu U_A^\lambda} \quad \text{on } \mathcal{F},$$

where  $U_A^\lambda f$  denotes the  $\lambda$ -potential of  $f \in (\mathcal{E})_+$  relative to  $A_t$ , i. e.,

$$E^x \int_0^\infty e^{-\lambda t} f(x_t) d_{A_t}.$$

Let  $A_t^x$  be the local time at  $x \in E$ : if  $x$  is regular (for  $\{x\}$ ),  $A_t^x$  is the CAF in [3, V. 3]; if  $x$  is irregular nonpolar,  $A_t^x$  counts the visits to  $x$  during  $(0, t]$ . For  $f \in (\mathcal{E})_+$ ,  $\langle \mu U_x^\lambda f \rangle = f(x) \langle \mu, u_x^\lambda \rangle$  ( $u_x^\lambda = U_x^\lambda 1$ ), so

$$(2.4) \quad \hat{P}_{A_x^\lambda}^\mu = \lambda \langle \mu, u_x^\lambda \rangle P^x,$$

where  $U_x^\lambda = U_{A_x^\lambda}$ , etc. Equation (2.3) becomes

$$(2.5) \quad \hat{P}_A^\mu = \int_E \mu U_A^\lambda(dx) \langle \mu, u_x^\lambda \rangle^{-1} \hat{P}_x^\mu(\cdot),$$

which suggests (cf. [3, VI, (4.21)])

$$(2.6) \quad A_t = \int_E A_t^x \langle \mu, u_x^\lambda \rangle^{-1} \mu U_A^\lambda(dx).$$

The following analogue of Theorem 2 (a) is a consequence of Motoo's theorem [3, V. (2.8)]; cf. [15, § 11]. By a reference measure on  $\mathcal{E}$  we mean a countable sum of finite measures on  $\mathcal{E}$  such that sets of  $\mu$ -measure zero are exactly those of potential 0.

LEMMA 1. — Let  $\mu$  be a reference measure, and  $A_t, B_t$  be CAFs. Then  $\hat{P}_A^\mu = \hat{P}_B^\mu$  iff  $A = B$  (in the sense of equivalence of AFs).

THEOREM 4. — Let  $A_t$  be a CAF,  $\mu$  a reference measure such that  $\mu U_A^\lambda$  is finite; suppose further that every  $x \in E$  is regular, and  $T_x < \infty$  a. s., where  $T_x$  is the hitting time of  $x$ . Then (2.6) holds a. s.

Since both members of (2.6) are CAFs, the result is immediate from Lemma 1. Notice that  $\langle \mu, u_x^\lambda \rangle \neq 0$  for all  $x$ .

Let us call  $x \in E$  *recurrent* (*strictly recurrent*) if the set

$$M_x(\omega) = \{t \in \mathbb{R}_+ : x_t(\omega) = x\}$$

is unbounded  $P^x$ -a. s. ( $P^y$ -a. s.  $\forall y \in E$ ). We observe that if  $x$  is strictly recurrent,  $A_t^x \rightarrow +\infty$  as  $t \rightarrow +\infty$   $P^y$ -a. s.  $\forall y \in E$ .

LEMMA 2. — If  $E$  contains a strictly recurrent point  $x$ , there is at most one  $\sigma$ -finite invariant measure (up to constant multiples).

Suppose  $\mu, \nu \in M_\sigma$  are invariant. By (2.4),  $\hat{P}_x^\mu = c\hat{P}_x^\nu$ . Now Theorem 2 (b) remains valid for  $\sigma$ -finite measures and clearly can be made to apply to the stationary measures  $P^\mu, P^\nu$ ; hence  $P^\mu = cP^\nu$ , so  $\mu = c\nu$ .

The next result, suggested by Corollary 1 (c), generalizes a well-known technique for obtaining invariant measures for Markov chains [4, Ch. 7, 15], [6, XIV.9], and is related to the method used in [9] for certain diffusions.

Consider a point, call it 0, in E with local time  $A_t^0$ . We write T for the hitting time of 0. If 0 is regular, let  $\tau(t)$  be the inverse local time [3, p. 217] (so  $\tau(0) = T$ ); otherwise,  $T_n = T_{n-1} + T \circ \theta_{T_{n-1}}$  denotes the  $n^{\text{th}}$  hitting time.  $\tau(t)$  (respectively  $T_n$ ) plays the role here of  $\hat{\alpha}(t)$  ( $R_n$ ) in § 1; in particular,  $\theta_{\tau(t)}(\theta_{T_n})$  preserves  $P^0$ -measure. Observe that if 0 is recurrent,  $\tau(t)$  ( $T_n$ ) is finite  $P^0$ -a. s.

THEOREM 5. — a) Let 0 be regular and recurrent; then

$$(2.7) \quad \pi_0(\Gamma) = E^0 \int_0^{\tau(1)} I_\Gamma(x_s) ds, \quad \Gamma \in \mathcal{E},$$

defines an invariant measure (for  $x$  irregular, replace  $\tau(1)$  by T).

b) Suppose further: (i) all 1-excessive functions are lower semi-continuous and (ii)  $P^x(T < \infty) > 0 \forall x \in E$ . Then  $\pi_0$  is  $\sigma$ -finite.

If 0 is *strictly* recurrent (which occurs iff 0 is recurrent and  $P^x(T < \infty) \equiv 1$ )  $\pi_0$  will be the only  $\sigma$ -finite invariant measure. The existence of a recurrent point is not necessary for that of an invariant measure, nor is (ii) needed for  $\sigma$ -finiteness, as shown by the process of uniform motion with velocity 1 (Lebesgue measure invariant).

Moreover,  $\pi_0$  may be trivial: for a constant process  $X_t \equiv X_0$ ,  $\pi_0 = \delta_0 =$  unit mass on 0. On the other hand, the present conditions are often easy to verify, e. g. for Brownian motion, Ornstein-Uhlenbeck (OU) process, many diffusions, etc. Finally,  $\pi_0$  is finite iff  $E^0(\tau(1)) < \infty (E^0(T) < \infty$  in the irregular case). Thus Brownian motion has no finite invariant measure because  $\tau(t)$  is the stable subordinator of index 1/2 [3, p. 227] (of course Lebesgue measure is invariant). And, since the normal distribution is invariant for the OU process,  $E^0(\tau(1)) < \infty$ . The conditions in Theorem 5 imply condition (H) of [1].

We shall deal only with the regular case; the irregular case is the same except that  $T_n$  is used instead of  $\tau(t)$ . Since each  $\tau(t)$  is a stopping time, the Markov property gives, for  $\Lambda \in \mathcal{F}$ ,

$$E^0 \int_0^{\tau(1)} I_\Lambda \circ \theta_r dr = E^0 \int_0^{\tau(1)} E^{x_r}(I_\Lambda) dr = P^{\pi_0}(\Lambda).$$

Next, since  $\theta_{\tau(t)}$  preserves  $P^0$ , the relation  $g(t + s) = g(t) + g(s)$ ,

$$g(t) = E^0 \int_0^{\tau(t)} Y \circ \theta_r dr, \quad Y \in (\mathcal{F})_+,$$

follows from the identity  $\tau(t + s, \omega) = \tau(t, \omega) + \tau(s, \theta_{\tau(t)}\omega)$ ,  $t, s \geq 0$  (see [3, p. 217]). As a result,

$$(2.8) \quad tE^{\pi_0}(Y) = E^0 \int_0^{\tau(t)} Y \circ \theta_r dr, \quad t \in \mathbb{R}_+.$$

Now to check that  $\pi_0$  is invariant, fix  $s \in \mathbb{R}_+$ ,  $Y \in (\mathcal{F})$ ,  $0 \leq Y \leq 1$ . If  $E^{\pi_0}(Y) = \infty$  then  $E^{\pi_0}(Y \circ \theta_s) = \infty$ . Otherwise, by (2.8),

$$\begin{aligned} |E^{\pi_0}(Y \circ \theta_s) - E^{\pi_0}(Y)| &= t^{-1} \left| E^0 \int_0^{\tau(t)} Y \circ \theta_{s+r} dr - E^0 \int_0^{\tau(t)} Y \circ \theta_r dr \right| \\ &\leq 2st^{-1} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

As for (b), we have, from (2.8),

$$(2.9) \quad t\pi_0(\Gamma) = E^0 \int_0^{\tau(t)} I_\Gamma(x_r) dr, \quad t \in \mathbb{R}_+, \quad \Gamma \in \mathcal{E}.$$

Taking Laplace transforms we obtain

$$(2.10) \quad \pi_0(\Gamma) = E^0 \int_0^\infty e^{-\Lambda_s^\Gamma} I_\Gamma(x_s) ds, \quad \Gamma \in \mathcal{E},$$

or  $\pi_0(\Gamma) = V_0(0, \Gamma)$ , where  $V_0$  is the potential kernel of the subprocess corresponding to the multiplicative functional  $e^{-\Lambda_s^\Gamma}$  [3, Ch. III].

Define

$$\phi(x) = E^x \int_0^\infty e^{-t - \Lambda_t^\Gamma} dt.$$

A computation gives  $\phi(x) = 1 - E^x(e^{-T})(1 - \phi(0))$ . Let  $J_\beta = \{x : \phi(x) > \beta\}$ ,  $\beta \in [0, 1]$ . Clearly  $\phi(x) \geq \phi(0)$  and  $0 < \phi(0) < 1$ , so  $J_\beta = E$  for  $\beta < \phi(0)$ . Thus every open set having compact closure is « special » [3, p. 133].

Let  $K$  be compact in  $E$ . Then  $f(x) = E^x(e^{-T})$  is 1-excessive, hence, by (i) and (ii),  $f(x) \geq \varepsilon$ ,  $x \in K$ , for some  $\varepsilon > 0$ ; thus  $\phi(x) \leq 1 - \varepsilon(1 - \phi(0)) < 1$  on  $K$ , so that  $K$  is disjoint from some  $J_\beta$ . By [3, p. 135],  $V_0(x, \Gamma)$  is bounded in  $x$  for each special set  $\Gamma$ ; in particular  $\pi_0(\Gamma) < \infty$ . Since  $E$  has a countable basis consisting of relatively compact open sets,  $\pi_0$  is  $\sigma$ -finite, and Theorem 5 is proven.

If every  $x \in E$  is strictly recurrent and (i) holds, we can define, for each  $x$ ,

$$\pi_x(\Gamma) = E^x \int_0^{\tau_x(1)} I_\Gamma(x_s) ds$$

( $\tau_x(t)$  is inverse local time at  $x$  for  $x$  regular; we omit the similar discussion of the irregular case). The measures  $\pi_x$  are all  $\sigma$ -finite invariant, and differ from each other by constants:

$$\pi_x = \frac{\langle \pi_x, u_0^\lambda \rangle}{\langle \pi_0, u_0^\lambda \rangle} \pi_0 \quad (\text{use (2.4)}).$$

LEMMA 3. — Each  $\pi_x$  is a reference measure.

Suppose  $\Gamma$  has potentiel zero:

$$U(x, \Gamma) = E^x \int_0^\infty I_\Gamma(x_s) ds = 0 \quad \forall x \in E.$$

Then  $V_0(0, \Gamma) = 0$  (see (2.10)), so  $\pi_0(\Gamma) = 0$ . Conversely, if  $\pi_0(\Gamma) = 0$ , (2.9) or (2.10) shows  $U(0, \Gamma) = 0$ ; likewise  $U(x, \Gamma) = 0$  for all  $x$ .

Let  $\mu \in M_\sigma$  be an invariant reference measure. Since Theorem 2 (a) carries over to the present situation, we can strengthen Theorem 4 slightly as follows.

THEOREM 4'. — If  $A_t$  is an AF with  $\mu U_A^\lambda$  finite, (2.6) holds a. s.

Denote by  $B_t$  the AF on the right side of (2.6). Since  $\hat{P}_A^\mu = \hat{P}_B^\mu$ , we have  $A_t = B_t \forall t \in \mathbb{R}_+$   $P^\mu$ -a. s. Now  $f(x) = 1 - P^x \{ A_t = B_t \forall t \in \mathbb{R}_+ \}$  is excessive, hence  $f(x) \equiv 0$ .

Finally we describe the limiting behavior of the transition function  $P_t(x, \Gamma)$ ,  $\Gamma \in \mathcal{E}$ , as  $t \rightarrow \infty$ , when  $x$  is a recurrent point. Similar results are known for so-called semilinear Markov processes; the oldest such results are due to Doob for renewal processes; see [7]. We consider, as usual, only the case in which  $x$  is regular, and use a renewal-theoretic approach (cf. [6, Ch. XI]). For convenience take  $x = 0$ .

For the next theorem we assume: (a) 0 is regular, recurrent, (b)  $E^0(\tau(1)) < \infty$ , (c)  $f \in \mathcal{E}$  is bounded and  $P_t f(0)$  is continuous in  $t$ , (d)  $\tau(t)$  has a continuous distribution.

THEOREM 6. — Let  $x \in E$  satisfy  $P^x(T < \infty) = 1$ . Then

$$(2.11) \quad \lim_{t \rightarrow \infty} P_t f(x) = \langle \pi_0, f \rangle / E^0(\tau(1)).$$

By the strong Markov property at time  $T$  (= hitting time of 0),

$$P_t f(x) = E^x(f(x_t); t < T) + \int_0^t P_{t-s} f(0) P^x(T \in ds),$$

from which it is obvious that we need only consider  $x = 0$  in (2.11). Clearly

we can restrict  $f$  to be non-negative as well. Since  $\tau(1)$  is a finite stopping time we have

$$(2.12) \quad P_t f(0) = E^0(f(x_t); t < \tau(1)) + \int_0^t P_{t-s} f(0) dF(s),$$

where  $F(s) = P^0(\tau(1) \leq s)$ . This is a standard type of renewal equation, and the existence of the limit in (2.11) (with  $x = 0$ ) will follow from [6, p. 349] as soon as we prove that  $z(t) \equiv E^0(f(x_t); t < \tau(1))$  is « directly integrable ». By (2.12),  $z(t)$  is continuous, and  $0 \leq z(t) \leq \|f\| (1 - F(t))$ . The direct integrability now follows from (b), since

$$\int_0^\infty (1 - F(t)) dt < \infty.$$

By [6, p. 349] again, we can identify the limit as

$$\left( \int_0^\infty (1 - F(t)) dt \right)^{-1} \int_0^\infty E^0(f(x_t); t < \tau(1)) dt = \langle \pi_0, f \rangle / E^0(\tau(1)) \text{ by (2.7).}$$

*Note.* — Hypotheses (c) and (d) can be weakened somewhat: all that is needed is that  $z(t)$  be directly integrable, and that  $\tau(1)$  not have an « arithmetic » distribution, i. e. concentrated on a set of the form  $\{0, \pm a, \pm 2a, \dots\}$ . The following proposition is therefore of some interest.

**PROPOSITION.** — If 0 is regular and recurrent,  $\tau(1)$  does not have an arithmetic distribution.

Let  $\phi_t(u) = E^0(e^{iut(t)})$  be the characteristic function of  $\tau(t)$ . Since  $\{\tau(t, \omega), P^0\}$  has stationary, independent increments,  $\phi_t(u) = (\phi_1(u))^t$ ,  $t \geq 0$ . If  $\tau(1)$  has an arithmetic distribution,  $\phi_1(u_0) = 1$  for some  $u_0 \neq 0$ , whence  $\phi_t(u_0) = 1$  for all  $t$ . It follows [6, XV.1] that every  $\tau(t)$  is distributed over some lattice  $\{0, a, 2a, \dots\}$ . Since the set  $Z(\omega) = \{t : x_t(\omega) = 0\}$  is a. s. uncountable when 0 is regular [7], and  $Z(\omega)$  differs from the range of  $\tau(t, \omega)$ ,  $t \in \mathbb{R}_+$ , by a countable set, we have a contradiction.

Condition (b) may also be eliminated: if  $f \in L^1(\pi_0)$ , and  $E^0(\tau(1)) = \infty$ , the renewal theorems in [6] tell us that  $P_t f(0) \rightarrow 0$  ( $t \rightarrow \infty$ ). According to Theorem 6, the state space  $E$  breaks into several pieces on each of which a (possibly) different limiting distribution obtains (or possibly no limiting distribution). Incidentally, the presence of recurrent points is not necessary for the existence of limiting distributions.

## 3. AN EXAMPLE

We compute the measure  $\pi$  of § 2 explicitly for an arbitrary semilinear strong Markov process  $X$  having « characteristic »  $\{\beta, h(x)\}$ . These processes are studied at length in [7] to which we refer the reader for terminology. Suffice it to say that each such  $X$  is completely characterized (up to equivalence of transition functions) by a constant  $\beta \geq 0$  and a non-negative, right-continuous, non-increasing function  $h(x)$  such that

$$\int_0^x h(y)dy < \infty, \quad x > 0.$$

The trajectories  $X_t(\omega)$  are « saw-tooth functions », with possibly infinitely many teeth in finite time intervals, and whose zeros occur precisely at points of  $Q(\omega)$ , where  $Q(\omega)$  is the range of a subordinator  $\tau(s, \omega)$  having exponent  $g(\lambda) = \beta\lambda + \int_{(0, \infty]} (1 - e^{-\lambda y})\mu(dy)$ , with Lévy measure  $\mu$  on  $(0, \infty]$  determined by  $\mu(x, \infty] = h(x)$ ,  $x > 0$ . In fact,  $\tau(s)$  is the inverse local time at  $x = 0$  for the process  $X$ .

It was shown in [7] (by an entirely different method) that, if

$$\int_0^\infty h(x)dx < \infty,$$

$X$  has a unique invariant probability measure given by

$$\pi(dx) = c\alpha\delta_0(dx) + ch(x)dx,$$

where

$$c = \left( \alpha + \int_0^\infty h(x)dx \right)^{-1}.$$

This result, which generalizes an old result of Doob in renewal theory, will now be extended to the case where  $\int_0^\infty h(x)dx$  may be infinite. We assume, however, that  $h(\infty) = 0$ , so  $\tau(s)$  is not killed at a finite time. Also, let  $h(0+) = \infty$ , so  $x = 0$  is a regular point. (The irregular case is Doob's result.)

**THEOREM 7.** — The measure  $\pi(dx) = \beta\delta_0(dx) + h(x)dx$  is the unique (except for constants)  $\sigma$ -finite invariant measure for the strong Markov semilinear process  $X$  having characteristic  $\{\beta, h(x)\}$ .

Indeed, every state  $x$  is strictly recurrent for such processes, so all we need show is that

$$\pi_0(\Gamma) = E^0 \int_0^{\tau(1)} I_\Gamma(X_s) ds$$

has the desired form. Under  $P^0$ ,  $\tau(s)$  is a subordinator as described above, hence there is a Lévy decomposition [9, p. 33]

$$(3.1) \quad \tau(s) = \beta s + \int_{0+}^{\infty} v p(s, dv), \quad s \geq 0,$$

where  $p(s, dv)$  is the number of jumps suffered by  $\tau(s)$  of size  $j \in dv$  during time  $[0, s]$ . The number of jumps in disjoint size intervals is independent, as well as the number in disjoint time intervals. Moreover,  $p(s, dv)$  has a Poisson distribution with parameter  $s\mu(dv)$  ( $\mu$  the Lévy measure) and  $X_t$  has the representation

$$(3.2) \quad X_t = t - \sup \{ u \leq t : u \in Q \}, \quad t \geq 0,$$

where  $Q = Q(\omega)$  is the range of  $\tau(s, \omega)$ ,  $s \geq 0$ . Since  $\tau(0) = 0$   $P^0$ -a. s., (3.2) is well-defined and finite.

Consider an interval  $\Gamma = [0, a]$ . Putting (3.1) and (3.2) together it's easy to see that

$$E^0 \int_0^{\tau(1)} I_\Gamma(X_s) ds = E^0 \left[ \int_0^a v p(1, dv) + a p(1, (a, \infty)) + m(Q \cap [0, \tau(1)]) \right],$$

where  $m =$  Lebesgue measure. Now

$$E^0 \int_0^a v p(1, dv) + a E^0 p(1, (a, \infty)) = \int_0^a v \mu(dv) + a \mu(a, \infty) = \int_0^a h(x) dx.$$

LEMMA. —  $m(Q \cap [0, t]) = \beta A_t^0$ ,  $A_t^0 =$  local time at zero.

Assuming the lemma for a moment, set  $t = \tau(1)$  to obtain

$$E^0 m(Q \cap [0, \tau(1)]) = \beta,$$

thereby proving Theorem 7. As for the lemma, it follows easily from the information in [7] that the CAFs  $\beta A_t^0$  and  $\int_0^t I_0(X_s) ds$  have the same  $\lambda$ -potential for all  $\lambda > 0$ , hence are equivalent. ( $I_0$  is the indicator of the set  $\{0\}$ .)

Finally, we remark on the connection between the above example and

the stationary regenerative phenomena of Kingman [11], [12]. Recall that Kingman constructs a stationary version of a regenerative phenomenon with an arbitrary  $p$ -function  $p(t)$ ,  $t \in \mathbb{R}$ . More specifically, if  $\bar{\Omega} = \{0, 1\}^{\mathbb{R}}$  and  $\Omega = \bar{\Omega} - \{0\}$ , where  $0$  denotes the function  $0(t) \equiv 0$ , Kingman obtains regular measure  $\lambda$  on the  $\sigma$ -ring in  $\Omega$  generated by the compacts, and  $\lambda$  is finite on compacts. This measure has the property

$$(3.3) \quad \lambda \{ \omega : Z_{t_1} = 1, \dots, Z_{t_n} = 1 \} = \prod_{i=1}^{n-1} p(t_{i+1} - t_i)$$

(or  $\lambda \{ \omega : Z_t = 1 \} = 1$  for  $n = 1$ ), where  $Z_t$  are the coordinate functions on  $\Omega$ .

It is further shown that, if  $p(t)$  is *standard* (i. e.  $p(t) \rightarrow 1$  as  $t \rightarrow 0$ ), then  $\lambda$  has a unique,  $\sigma$ -finite, minimal extension on the Borel sets in  $\Omega$  (called « weak Borel sets » in [12]) with total mass  $\lambda(\Omega) = (\lim_{t \rightarrow \infty} p(t))^{-1}$ . If  $p(t)$  is not standard, either  $p(t) = a \bar{p}(t)$  for some  $a \in (0, 1)$  and standard  $\bar{p}(t)$ , in which case a similar result holds for  $\lambda$ , or  $p(t) = 0$  a. e., in which case no  $\sigma$ -finite extension to the Borel sets is possible.

Let  $p(t)$  be a standard  $p$ -function with canonical measure  $\mu$  [11]. It is known [7] that a regenerative phenomenon  $D_t$  having  $p(t)$  as its  $p$ -function is obtained by taking  $D_t = \{X_t = 0\}$  where  $X$  is the semilinear strong Markov process with characteristic  $\{1, h(x)\}$ , where  $h(x) = \mu(x, \infty)$ ,  $x > 0$ . In this situation, we can take our probability space to be the set  $W$  of all « saw-tooth » functions, or some other suitable function space, and we again get a stationary regenerative phenomenon by taking  $Z_t = I_{\{X_t = 0\}}$ , and using the measure  $P^\pi$  constructed above. This is always  $\sigma$ -finite (finite if  $\int_0^\infty h(x) dx$  is finite) and gives an explicit representation for standard stationary regenerative phenomena.

Consider now a semilinear strong Markov process  $X$  having characteristic  $\{0, h(x)\}$ . Then  $D_t = \{X_t = 0\}$  is a (non-standard) regenerative phenomenon having  $p(t) \equiv 0$  (if  $h(0+) = \infty$ ) or  $p(t) = 0$  except for countably many  $t$  (if  $h(0+) < \infty$ ) (see [7]). From the point of view of regenerative phenomena, this is a degenerate situation, and there is no finite stationary version on the space  $\Omega$ . Nevertheless, it is a typical situation for Markov processes: the measure  $P^\pi$  still provides a stationary version of  $X_t$  on  $W$ . Of course the random variables  $Z_t = I_{\{X_t = 0\}}$  are trivial now:  $P^\pi(Z_t = 1) = P^\pi(X_t = 0) = \pi(\{0\}) = 0$ .

## REFERENCES

- [1] J. AZEMA, M. DUFLO and D. REVUZ, Mesure invariante des processus de Markov. *Lecture Notes in Math.*, no. 88, Springer-Verlag, Berlin, New York, 1969.
- [2] S. BERMAN, Local times and sample function properties of stationary Gaussian processes. *Trans. Amer. Math. Soc.*, t. **137**, 1969, p. 277-299.
- [3] R. M. BLUMENTHAL and R. K. GETTOOR, *Markov Processes and Potential Theory*. Academic Press, New York, 1968.
- [4] L. BREIMAN, *Probability*. Addison-Wesley, Reading, Massachusetts, 1968.
- [5] H. CRAMÉR and M. R. LEADBETTER, *Stationary and Related Stochastic Processes*. John Wiley, New York, 1967.
- [6] W. FELLER, *An Introduction to Probability Theory and its Applications*, Vol. II. John Wiley, New York, 1966.
- [7] J. HOROWITZ, Semilinear Markov processes, subordinators, and renewal theory, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **24**, 1972, p. 167-193.
- [8] J. HOROWITZ, Short proof of a « Radon-Nikodym theorem » for additive functionals, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **25**, 1973, p. 159-162.
- [9] K. ITÔ and H. P. MCKEAN, *Diffusion Processes and their Sample Paths*. Springer, Berlin, 1965.
- [10] M. KAC and D. SLEPIAN, Large excursions of Gaussian processes. *Ann. Math. Stat.*, t. **30**, 1959, p. 1215-1228.
- [11] J. F. C. KINGMAN, The stochastic theory of regenerative events. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **2**, 1964, p. 180-224.
- [12] J. F. C. KINGMAN, Stationary regenerative phenomena. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **15**, 1970, p. 1-18.
- [13] J. LAZARO and P. A. MEYER, Méthodes des martingales et théorie des flots. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **18**, 1971, p. 116-140.
- [14] J. MECKE, Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie verw. Geb.*, t. **9**, 1967, p. 36-58.
- [15] D. REVUZ, Mesures associées aux fonctionnelles additives de Markov I. *Trans. Amer. Math. Soc.*, t. **148**, 1970, p. 501-531.
- [16] C. RYLL-NARDZEWSKI, Remarks on processes of calls. *Proc. Fourth Berkeley Symp. Math. Stat. and Prob.*, t. **2**, 1961, p. 455-466.
- [17] H. TOTOKI, Time changes of flows. *Mem. Fac. Sci. Kyūshū Univ. Ser. A.*, t. **20**, 1966, p. 27-55.

(Manuscrit reçu le 4 septembre 1972).