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by

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In 1967, A. A. Tempelman announced generalizations of the Birkhoff and von Neumann ergodic theorems [6]. This paper supplies proofs of results similar to Tempelman’s. The main arguments are drawn from Calderon's paper [1]. The author has also had the benefit of reading Mrs. J. Chatard's work on the same problem [2].

PRELIMINARIES

Let \((M, \mathcal{M}, \mu)\) and \((G, \mathcal{J}, \gamma)\) be complete measure spaces, where \(\mu\) is \(\sigma\)-finite. Assume that \(G\) is a semigroup with product indicated by juxtaposition and that there is a map \((x, m) \mapsto x(m)\) from \(G \times M\) to \(M\), measurable with respect to \(G \times \mathcal{M}\) and such that \(x(y(m)) = xy(m)\) for all \(x, y \in G\) and \(m \in M\). Assume that \(\mu(x^{-1}F) \leq \mu(F)\) for all \(x \in G\) and \(F \in \mathcal{M}\), where \(x^{-1}F = \{ m : x(m) \in F \}\). Finally, assume that for all \(x \in G\) and \(E \in \mathcal{J}\), \(xE\) and \(Ex\) are measurable, \(\gamma(xE) = \gamma(E) = \gamma(Ex)\), and that \(x^{-1}E\) and \(Ex^{-1}\) are measurable, where \(x^{-1}E = \{ y \in G : xy \in E \}\) and \(Ex^{-1} = \{ y \in G : yx \in E \}\).

If \(x \in G\) and \(E\) and \(D\) are in \(\mathcal{J}\), then

\[
\gamma(E \cap x^{-1}D) = \gamma((x(E) \cap x^{-1}D)) = \gamma((x) \cap D),
\]

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so that
\[ \int_E \chi_D(xy) dy(y) = \int_{xE} \chi_D(y) dy(y), \]
where \( \chi_D \) is the characteristic function of \( D \). Therefore, if \( f \) is any integrable function on \( G \),
\[ \int_E f(xy) dy(y) = \int_{xE} f(y) dy(y), \]
and similarly,
\[ \int_E f(yx) dy(y) = \int_{xE} f(y) dy(y). \]

If \( f \) is any function on \( M \) and \( x \in G \), define \( f_x \) by \( f_x(m) = f(xm) \). If \( f \) is any nonnegative integrable function on \( M \),
\[ \int f_x d\mu \leq \int fd\mu. \]

Let \( A_n \) be a sequence of measurable subsets of \( G \) such that \( 0 < \gamma(A_n) < \infty \) for all \( n \). We shall use the following conditions on the \( A_n \).

I. \( n < m \) implies \( A_n \subseteq A_m \);

II. \( \lim_{n} \frac{\gamma(A_n \Delta xA_n)}{\gamma(A_n)} = \lim_{n} \frac{\gamma(A_n \Delta A_n x)}{\gamma(A_n)} = 0 \), for all \( x \in G \), where \( \Delta \) denotes symmetric difference;

III. for each \( k \) and \( n \), \( A_kA_n \) is measurable and \( \lim_{n} \frac{\gamma(A_kA_n \Delta A_n)}{\gamma(A_n)} = 0 \); and

IV. there exists \( K > 1 \) such that \( \gamma(A_n^{-1}A_n) \leq K(A_n) \) for all \( n \), where
\[ A_n^{-1}A_n = \{ x \in G : yx \in A_n \text{ for some } y \in A_n \}. \]

Let \( B \) be a real or complex Banach space with norm \( \| \cdot \| \) and dual \( B^* \). If \( \lambda \in B^* \) and \( b \in B \), \( \lambda(b) \) will be denoted by \( \lambda \cdot b \). If \( (N, \nu) \) is a measure space and if \( 1 \leq p < \infty \), \( L_p^B(N, \nu) \) will denote the set of equivalence classes of functions \( f : M \to B \) such that \( f \) is the limit in measure of simple functions and
\[ \| f \|_{L_p^B} = \left( \int \| f \|^p d\nu \right)^{1/p} < \infty. \]

\( L_p^B \) is a Banach space, and if \( (N, \nu) \) is \( \sigma \)-finite, its dual is \( L_q^{B^*} \), where \( q = \infty \) if \( p = 1 \), and \( \frac{1}{q} + \frac{1}{p} = 1 \) otherwise.
If \( g \in L^B_q \) and \( f \in L^B_p \),

\[
(1) \quad \int_{A_n} \int_{\gamma} |g(m) \cdot f_x(m)| d\mu(m) d\gamma(x) \leq \int_{A_n} \int_{\gamma} ||g(m)|| ||f_x(m)|| d\mu d\gamma \leq ||g||^B_q ||f||^B_p(\gamma(A_n)).
\]

If we choose \( g \) so that \( ||g(m)|| > 0 \) a.e., then (1) and Fubini's theorem imply that for almost every \( m \),

\[
\int_{A_n} ||f_x(m)|| d\gamma(x)
\]
exists. It follows that for almost every \( m \), \( f_{\gamma}(m) \in L^B_1(\gamma|_{A_n}) \) and hence that

\[
(2) \quad g(m) \cdot \int_{A_n} f_x(m) d\gamma(x) = \int_{A_n} g(m) \cdot f_x(m) d\gamma(x) \quad [3, \text{III}.6.10, \text{III}.2.22].
\]

Applying Fubini's theorem again and using (2), we obtain

\[
\int_{A_n} \int_{\gamma} g(m) \cdot f_x(m) d\mu d\gamma = \int_{\gamma} g(m) \left( \int_{A_n} f_x(m) d\gamma(x) \right) d\mu.
\]

Hence, the map \( x \mapsto f_x \) from \( A_n \) to \( L^B_p \) is integrable in the sense of Pettis [5], and the integral is equal to \( \int_{A_n} f_x(m) d\gamma(x) \) almost everywhere. Define

\[
\pi_n : L^B_p \rightarrow L^B_p \quad \text{by} \quad \pi_n(f) \equiv \frac{1}{\gamma(A_n)} \int_{A_n} f_x d\gamma(x).
\]

Clearly, \( \pi_n \) is a continuous linear operator of norm less than or equal to one.

**THE ERGODIC THEOREMS**

THEOREM 1 (von Neumann's Mean Ergodic Theorem). — If the \( A_n \) satisfy II and if \( 1 < p < \infty \) or if \( p = 1 \) and \( \mu(M) < \infty \), then there is \( \pi(f) \in L^B_p \) such that \( \lim_n ||\pi_n(f) - \pi(f)||_p = 0 \) and such that \( \pi(f_x) = \pi(f) = \pi(f)_x \) for all \( x \in G \). If \( p = 1 \), \( \int \pi(f) d\mu = \int fd\mu \). \( \pi \) is the projection of \( L^B_p \) onto \( I^B_p \) along \( M^B_p \), where \( I^B_p \) is the subspace of invariant functions and \( M^B_p \) is the closed subspace generated by \( \{ f_x - f : f \in L^B_p, x \in G \} \).
THEOREM 2 (Wiener-Calderon Dominated Convergence Theorem). Suppose that the $A_n$ satisfy I, III, and IV. If $f$ is a nonnegative integrable function and if for $\alpha > 0$,

$$E_\alpha = \{ m : \sup_{n=1,\ldots,\infty} \pi_n(f)(m) \geq \alpha \},$$

then

$$\mu(E_\alpha) \leq \frac{K}{\alpha} \int f d\mu.$$

THEOREM 3 (Birkhoff's Individual Ergodic Theorem). If the $A_n$ satisfy I-IV and if $f \in L_p^B$, where $1 \leq p < \infty$, then $\pi_n(f)$ converges almost everywhere. If $1 < p < \infty$ or if $p = 1$ and $\mu(M) < \infty$, then $\pi_n(f)$ converges almost everywhere to the $\pi(f)$ of Theorem 1.

PROOF OF THEOREM 1

**Lemma 1.** If $f \in L_p^B$ and $1 < p < \infty$, then $C(f) = \langle \{ f_x : x \in G \} \rangle$ is weakly compact.

**Proof.** If $p > 1$, let $\frac{1}{q} + \frac{1}{p} = 1$. If $p = 1$, let $q = \infty$. $C(f)$ is weakly compact if for each sequence $x_m \in G$ and each sequence $\lambda_n \in L_q^{B^*}$ such that $\| \lambda_n \|_{q}^{B^*} \leq 1$ for all $n$, $\lim_{m} \lambda_n \cdot f_{x_m} = \lim_{n} \lambda_n \cdot f_{x_m}$ whenever each limit exists [4, p. 159].

Let $\varepsilon > 0$ and choose a simple function $g \in L_p^B$ such that $\| f - g \|_p < \varepsilon.$ $| \lambda_n \cdot g_{x_m} | \leq \| g \|_p$ for all $n$ and $m$, so that we may, by a diagonal process, choose a subsequence $g_{x_{m_k}}$ such that for each $n$, $\lim_{k} \lambda_n \cdot g_{x_{m_k}} \rightarrow a_n$. We may assume that $a_n \rightarrow a$. Similarly, we may choose a subsequence $\lambda_{n_k} \cdot g_{x_{m_k}}$ such that $\lim_{k} \lambda_{n_k} \cdot g_{x_{m_k}} = c_k$ for each $k$. Again, we may assume that $c_k \rightarrow c$. Since $\| \lambda_n \cdot f_{x_m} - \lambda_{n_k} \cdot g_{x_m} \|_p^B \leq \| \lambda_n \|_q^{B^*} \| f_{x_m} - g_{x_m} \|_p^B < \varepsilon$ for all $m, n$, it suffices to show that $a = c$.

Since the $\lambda_n$ are uniformly bounded, the sequence $\lambda_{n_k}$ has a weak star limit point, $\lambda_0$. $g = \sum_{i=1}^{\infty} h_i b_i$, where $h_i \in L_p$ and $b_i \in B$. Since $\| (h_i)_x \|_p \leq \| h_i \|_p$ for all $x \in G$, $\{ (h_i)_x : x \in G \}$ is weakly relatively compact for each $i$. Hence, $x_{m_k}$ has a subnet $x_{m_{k(i)}}$ such that $(h_{i_{x_{m_{k(i)}}}})$ converges weakly to some $h_{i0}$ for each $i$. Then, $a = \lambda_0 \cdot \sum b_i h_{i0} = c$. Q. E. D.
PROOF OF THEOREM 1. — Suppose that $p > 1$. \( \pi_n(f) \in C(f) \), \( \forall n \). Since $C(f)$ is weakly compact, $\pi_n(f)$ has a weak cluster point $\pi(f)$. Given $\varepsilon > 0$, there are $v \in L^p$ with $\| v \|^p < \varepsilon$ and $\alpha_i, f_{x_i}, i = 1, \ldots, m$, with
\[
0 \leq \alpha_i \leq 1, \quad \sum_{i=1}^{m} \alpha_i = 1,
\]
such that
\[
\pi(f) = \sum_{n=1}^{\infty} \alpha_i f_{x_i} + v.
\]
Hence,
\[
\pi(f) - f = \sum_{i=1}^{m} \alpha_i (f_{x_i} - f) + v.
\]
For every $x \in G$,
\[
\| \pi_n(f)_x - \pi_n(f) \|^B_p \leq \frac{\gamma(A_n \triangle x A_n)}{\gamma(A_n)} \| f \|^p \rightarrow 0,
\]
so that $\pi(f)$ is invariant and $\pi_n(\pi(f)) = \pi(f)$ for all $n$. Hence,
\[
\pi(f) - \pi_n(f) = \sum_{i=1}^{m} \alpha_i \pi_n(f_{x_i} - f) + \pi_n(v).
\]
Since $\| \pi_n(v) \|^B_p < \varepsilon$ and since for all $x \in G$,
\[
\lim_{n} \| \pi_n(f_x) - \pi_n(f) \|^B_p \leq \lim_{n} \frac{\gamma(A_n \triangle A_n x)}{\gamma(A_n)} \| f \|^p = 0,
\]
it follows that $\lim_n \| \pi_n(f) - \pi(f) \|^B_p = 0$.

The case $p = 1$ follows from the case $p = 2$, since the $\pi_n$ are uniformly bounded on $L^1_B$ and since, if $\mu(M) < \infty$, $L^B_2$ is $B^B_2$-dense in $L^B_2$ and the $\| \cdot \|^B_2$-topology is stronger than the $L^B_1$-topology. Q. E. D.

PROOF OF THEOREM 2

The key step of the proof is the Wiener-Calderon covering argument made in proving Lemma 1 below.

Let $\setminus$ denote set theoretic difference.
LEMMA 1. — Let $h$ be a real-valued $\gamma$-integrable function defined on $A_kA_n$. Suppose that

1. $h(x) \geq 0$ for all $x \in A_kA_n \setminus A_n$;

2. for all $x \in A_n$, either $h(x) \geq 0$ or $h(x) \geq -\frac{\alpha}{K}$ and

$$\frac{1}{\gamma(A_n)} \int_{A_i} h(yx) d\gamma(y) \geq \alpha - \frac{\alpha}{K}$$

for some $i = 1, \ldots, K$, where $a > 0$.

Then, $\int_{\bigcup_{i=1}^{K} A_iA_n} hd\gamma \geq 0$.

Proof. — Let $\mathcal{M}_k$ be a maximal collection of disjoint subsets of the form $A_kx$, such that

$$x \in A_n \quad \text{and} \quad \frac{1}{\gamma(A_k)} \int_{A_n} h(yx) d\gamma(y) \geq \alpha - \frac{\alpha}{K}.$$ 

Given $\mathcal{M}_{i+1}$, where $k > i \geq 1$, let $\mathcal{M}_i$ be a maximal collection of sets of the form $A_ix$ where $x \in A_n$ and such that

$$\frac{1}{\gamma(A_i)} \int_{A_i} h(yx) d\gamma(y) \geq \alpha - \frac{\alpha}{K}$$

and the $A_ix$ are mutually disjoint and are disjoint from every set in $\mathcal{M}_j$ for $i + 1 \leq j \leq k$. Let $\mathcal{M} = \bigcup_{i=1}^{k} \mathcal{M}_i$.

Let $\mathcal{N} = \{ A_i^{-1}A_jx : A_jx \in \mathcal{M} \}$. Let $N = A_kA_n \cup \mathcal{N}$. Suppose $x \in N$ and $k(x) < 0$. Then $x \in A_n$ and for some $i = 1, \ldots, k$,

$$\frac{1}{\gamma(A_i)} \int_{A_i} h d\gamma \geq \alpha - \frac{\alpha}{K}, \quad A_ix \notin \mathcal{M}.$$ 

Therefore, there exists $A_jx' \in \mathcal{M}$ with $j \geq i$ such that $A_ix \cap A_jx' \neq \emptyset$. But then $x \in A_i^{-1}A_jx' \subset A_i^{-1}A_jx' \subset \mathcal{N}$. This contradicts $x \in N$. Hence, $h \geq 0$ on $N$ and

$$\int_{\bigcup_{i=1}^{K} A_iA_n} h \geq \int_{\bigcup_{i=1}^{K} A_iA_n} h - \frac{\alpha}{\gamma(A_k)} \gamma(\mathcal{N} \cup \mathcal{M}) \geq \left( \alpha - \frac{\alpha}{K} \right) \sum_{\mathcal{M}} \gamma(A_i x)$$

$$- \frac{\alpha}{K} \gamma(\mathcal{N} \cup \mathcal{M}) = \alpha \sum \gamma(A_i x) - \frac{\alpha}{K} \gamma(\mathcal{N}) \geq \frac{\alpha}{K} \sum \gamma(A_i^{-1}A_n x) - \frac{\alpha}{K} \gamma(\mathcal{N}) = 0.$$ 

Q. E. D.
LEMMA 2. — Suppose that $f \in L_1(M)$ and that for each $m \in M$, either $f(m) \geq 0$ or $f(m) \geq -\frac{\alpha}{K}$ and for some $i = 1, \ldots, k$,

$$
\frac{1}{\gamma(A_k)} \int_{A_k} f(y) dy \gamma(y) \geq \alpha - \frac{\alpha}{K}.
$$

Then, $\int f d\mu \geq 0$.

Proof. — Let $n$ be a positive integer and for each $m \in M$, let $h_m : A_k \to A_n$ be defined by

$$
h_m(y) = \begin{cases} \mid f(ym) \mid, & \text{if } y \in A_k \setminus A_n, \\ f(ym), & \text{if } y \in A_n.
\end{cases}
$$

Let $M' = \{ m \in M : \mid f(ym) \mid \text{ is } \gamma\text{-integrable on } A_k \setminus A_n \}$. By Fubini's Theorem, $\mu(M \setminus M') = 0$. For each $m \in M'$, $h_m$ satisfies the assumptions of Lemma 1. Hence,

$$
0 \leq \int_{A_k \setminus A_n} h_m \gamma = \int_{A_k} f(y) \gamma(y) + \int_{A_k \setminus A_n} \mid f(ym) \mid d\gamma(y).
$$

Applying Fubini's Theorem, we have

$$
0 \leq \gamma(A_n) \int_M f d\mu + \gamma(A_k \setminus A_n) \int_M \mid f \mid d\mu,
$$

or

$$
0 \leq \int_M f d\mu + \frac{\gamma(A_k \setminus A_n)}{\gamma(A_n)} \mid f \mid.
$$

Let $n \to \infty$ and apply III. Q. E. D.

PROOF OF THEOREM 2. — It is sufficient to prove that

$$
\mu(F \cap E_k^x) \leq \frac{K}{\alpha} \int f d\mu,
$$

where $\mu(F) < \infty$ and

$$
E_k^x = \{ m \in M : \max_{i=1,\ldots,k} \pi_i(f)(m) \geq \alpha \}.
$$

Let

$$
h = f - \frac{\alpha}{K} \chi_{F \cap E_k^x}.
$$

Since $h$ satisfies the assumptions of Lemma 3, $\int h d\mu \geq 0$. Q. E. D.
PROOF OF THEOREM 3

Suppose at first that if $p = 1$, $\mu(M) < \infty$.

Let $\pi(f)$ be the limit defined by the Mean Convergence Theorem. Since $\pi_n(f - \pi(f)) = \pi_n(f) - \pi(f)$, one may suppose that $\pi(f) = 0$. We show that $\pi_n(f) \to 0$ a.e.

Let $\varepsilon > 0$. Choose $f^b$ bounded and such that $|| f^b - f ||_p^B < \frac{\varepsilon}{3}$. Choose $k$ such that $|| \pi_k(f^b) - \pi(f^b) ||_p^B < \frac{\varepsilon}{3}$. Then, $f = H + G$, where

$$H = (f - f^b) + (\pi_k(f^b) - \pi(f^b)) + \pi(f^b) \quad \text{and} \quad G = f^b - \pi_k(f^b).$$

Clearly, $|| H ||_p^B < \varepsilon$.

$\pi_n(G)$ converges to zero almost everywhere since, for almost every $m$,

$$|| \pi_n(G)m || = \frac{1}{\gamma(A_n)} \left| \int_{A_n} \frac{1}{\gamma(A_k)} \left( \int_{A_k} f^b(m) d\gamma(x) - \int_{A_k} f^b(m) d\gamma(y) \right) \right|$$

$$\leq \frac{1}{\gamma(A_k)} \int_{A_k} \frac{1}{\gamma(A_n)} \left( \int_{A_n} \int_{A_n} (f^b(m) - f^b(m)) d\gamma(y) \right) d\gamma(x)$$

$$\leq \frac{1}{\gamma(A_k)} \int_{A_k} \frac{1}{\gamma(A_n)} \left( \int_{A_n} \int_{A_n} \int_{A_n} f^b(m) d\gamma(y) d\gamma(x) \right)$$

$$\leq \frac{\gamma(A_n \Delta A_k \Delta A_n)}{\gamma(A_n)} \sup || f^b || \to 0 \text{ as } n \to \infty.$$

Therefore, if $\delta > 0$,

$$\mu \{ m : \lim_n || \pi_n(f)|m|| > 3\delta \} \leq \mu \{ m : \lim_n \pi_n(|| H(m) ||) > 2\delta \}.$$

If $p = 1$, by Theorem 2 we obtain

$$\mu \{ m : \lim_n \pi_n(|| H(m) ||) > 2\delta \} \leq \frac{K}{2\delta} || H ||_1^B < \frac{K\varepsilon}{2\delta}.$$

Since $\varepsilon$ is arbitrarily small, we obtain $\lim_n \pi_n(f) = 0$ a.e.

If $p > 1$, let

$$H^\delta(m) = \begin{cases} || H(m) || & \text{if } || H(m) || \geq \delta, \\ 0, & \text{otherwise.} \end{cases}$$

$$|| H || \leq H^\delta + \delta,$$ so that $\mu \{ m : \lim_n \pi_n(|| H(m) ||) \geq 2\delta \} \leq \mu \{ m : \lim_n \pi_n(H^\delta(m)) \geq \delta \}.$
Furthermore,
\[ H^a \in L_1 \quad \text{and} \quad \| H^a \|_1 \leq \left( \frac{\| H \|_1^B}{\delta^{p-1}} \right)^p < \frac{\varepsilon^p}{\delta^{p-1}} \]
so that by Theorem 2,
\[ \mu \{ m : \limsup_n \pi_n(H^a(m)) \geq \delta \} \leq K \left( \frac{\varepsilon}{\delta} \right)^p \quad \text{and} \quad \lim_n \pi_n(f) = 0 \]
amost everywhere.

We now remove the assumption that \( \mu(M) < \infty \) as in the case \( p = 1 \) and prove that \( \pi_n(f) \) converges almost everywhere. We call a set \( E \in \mathcal{M} \) invariant if \( xE \subset E, \forall x \in G \). It is possible to find a sequence of invariant sets, \( I_k \), of finite measure and such that if \( I \) is invariant and measurable and if \( \mu(I \cap \bigcup_k I_k) = 0 \), then either \( \mu(I) = 0 \) or \( \mu(I) \equiv \infty \). By what we have already proved, \( \pi_n(f) \) converges on each \( I_k \) and hence on \( I = \bigcup_k I_k \).

Let \( \varepsilon > 0 \) and let \( f^b \) be a bounded function such that \( \| f - f^b \|_1^B < \varepsilon \).
\[ \{ m \in M \setminus \bar{I} : \limsup_n \pi_n(f)(m) \| > 2\delta \} \subset \{ m \in M : \limsup_n \pi_n(\| f - f^b \|(m)) \| > \delta \} \]
\[ \cup \{ m \in M \setminus \bar{I} : \limsup_n \pi_n(\| f^b(m) \|) \| > \delta \} \]
The measure of the first set on the right is bounded by \( \frac{K\varepsilon}{\delta} \). It is easy to show that the second set is invariant. Since it is bounded by \( \frac{K}{\delta} \| f^b \|_1^B \) it must have measure zero. Q. E. D.

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