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Probability measures on H_p spaces

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1. INTRODUCTION

Let $(\Omega, \mathscr{A}, \mu)$ be a probability measure space, and let $(\mathscr{X}, \mathscr{B})$ denote a measurable space where $\mathscr{X}$ is a Banach space and $\mathscr{B}$ is the σ-algebra of Borel subsets of $\mathscr{X}$. Let $x(\omega)$ denote a $\mathscr{X}$-valued random variable; that is, x is a $\mathscr{X}$-valued measurable function defined on $(\Omega, \mathscr{A}, \mu)$. As a $\mathscr{X}$-valued function, $x(\omega)$ can be defined as (*i*) a Borel measurable function (i. e., $\{ \omega : x(\omega) \in B \} \in \mathscr{A}$ for all $B \in \mathscr{B}$,) (*ii*) a weakly measurable function, or (*iii*) a strongly measurable function. If $\mathscr{X}$ is separable, these three concepts of measurability are equivalent.

In the development of probability theory in Banach spaces and its various applications, a central role is played by the study of probability measures on Banach spaces. The probability measure μ and a $\mathscr{X}$-valued random variable x induce a probability measure (or distribution) on $\mathscr{X}$, namely $\nu_x = \mu \circ x^{-1}$; hence we obtain the induced probability measure space $(\mathscr{X}, \mathscr{B}, \nu_x)$. In applications, especially in the theory of random equations, it is necessary to construct and study probability measures on concrete Banach spaces. At the present time there is an extensive literature devoted to probability measures on concrete Banach spaces. We refer to the

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following books and papers, and references contained therein: Bharucha-Reid [*2*, chap. 1], Grenander [*5*, chap. 6], Jajte [*8*], Kampé de Fériet [*9*, *10*], Kuelbs [*11*], Kuelbs and Mandrekar [*12*] and Sato [*17*].

In this paper we consider probability measures on H_p spaces, that is the Hardy spaces of analytic functions. In 1960, Kampé de Fériet [*9*] introduced a method for the construction of probability measures on Banach spaces with Schauder bases. In Section 2 we extend Kampé de Fériet's method to complex Banach spaces; and in Section 3 we use these results to construct probability measures on H_p spaces, $1 < p < \infty$.

2. PROBABILITY MEASURES ON COMPLEX BANACH SPACES WITH SCHAUDER BASES

As is well-known, most of the observable properties of a physical system are given by linear functionals on the state or phase space associated with the system. Thus in many applications it is reasonable to assume that the linear functionals are measurable with respect to a probability measure on an appropriate function space. Such measures, called L-measures, were first introduced and studied by Fréchet and Mourier (cf. [*14*]). In this section we consider the construction of L-measures on complex Banach spaces with Schauder bases.

We first introduce some definitions which will be used in this section.

DEFINITION 2.1. — Let X be a linear topological space. A sequence $\{ e_n \}$ of elements of X is called a *topological basis* for X if for each $x \in X$ there corresponds a unique sequence $\{ \xi_n \}$ of scalars such that the series

$$\sum_{n=0}^{\infty} \xi_n e_n \tag{2.1}$$

converges to x in X.

If $\{ e_n \}$ is a topological basis for X, then there is a corresponding sequence $\{ e_n^* \}$ of linear functionals, called *coefficient functionals*, such that

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n \tag{2.2}$$

for all $x \in X$. The sequences $\{ e_n \}$ and $\{ e_n^* \}$ are *biorthogonal* in the sense that $e_n^*(e_m) = \delta_{n,m}$ (Kronecker delta). The coefficient functionals e_n^* belong to the algebraic dual of X; however they may not belong to the topological dual of X.

DEFINITION 2.2. — A topological basis for X is called a *Schauder basis* for X if all the associated coefficient functionals are continuous on X.

DEFINITION 2.3. — A basis $\{ e_i \}$ for a Banach space $\mathscr{X}$ is said to be *boundedly complete* if for each sequence $\{ \alpha_i \}$ of scalars such that the sequence

$$\left\{ \left\| \sum_{i \leq n} \alpha_i e_i \right\|, \quad n \geq 1 \right\}$$

is bounded, there is an $x \in \mathscr{X}$ such that $\alpha_i = e_i^*(x)$ for all i; and so

$$x = \sum_{i=1}^{\infty} \alpha_i e_i.$$

We refer to the books of Day [3], Marti [13], and Singer [18] for general discussions of Banach spaces with Schauder bases.

We now consider the construction of L-measures on complex Banach spaces with Schauder bases.

DEFINITION 2.4. — Let $\mathscr{X}$ be a Banach space. A probability measure ν on $\mathscr{X}$ is called an L-*measure* if all bounded linear functionals on $\mathscr{X}$ are ν-measurable.

In the construction of the measure we use Kolmogorov's consistency theorem for separable standard Borel spaces. A Borel space $(\mathscr{X}, \mathscr{B})$, where $\mathscr{B}$ is generated by a countable class in $\mathscr{B}$, is called *standard* if there is a complete separable metric space $\mathscr{Y}$ such that the σ-algebras $\mathscr{B}$ and $\mathscr{B}_{\mathscr{Y}}$ are σ-isomorphic; that is, there is a one-one map of $\mathscr{B}$ into $\mathscr{B}_{\mathscr{Y}}$ preserving countable set operations. We refer to Parthasarathy [15] for a discussion of separable standard Borel spaces and Kolmogorov's consistency theorem.

Let $\mathscr{X}$ be a complex Banach space with basis $\{ e_n \}$, $\| e_n \| = 1$. Let Z be the space of complex numbers; and let Z^∞ be the linear space of all complex sequences, that is, $Z^\infty = \{ \xi : \xi = (\xi_1, \xi_2, \ldots), \xi_i \in Z \}$. Since $\mathscr{X}$ has a basis, there is a linear map A which is bijective and bicontinuous between $\mathscr{X}$ and a subset $\tilde{\mathscr{X}} \subset Z^\infty$,

$$x = A\xi, \quad \xi = A^{-1}x, \quad x \in \mathscr{X}, \ \xi \in \tilde{\mathscr{X}}. \tag{2.3}$$

$\tilde{\mathscr{X}}$ becomes a Banach space with norm given by

$$\| \xi \|_{\sim} = \sup_{k \geq 1} \left\| \sum_{i=1}^{k} \xi_i e_i \right\|. \tag{2.4}$$

Let us put

$$[x_n] = \sum_{i=1}^{n} \xi_i e_i, \qquad [x]_{n,k} = \sum_{i=n+1}^{n+k} \xi_i e_i \tag{2.5}$$

$$p_n(\xi_1, \ldots, \xi_n) = \| [x]_n \| \tag{2.6}$$

$$p_{n,k}(\xi_{n+1}, \ldots, \xi_{n+k}) = \| [x]_{n,k} \| = p_n(0, \ldots, 0, \xi_{n+1}, \ldots, \xi_{n+k}) \tag{2.7}$$

Clearly the expansion $x = \sum_{n \geq 1} \xi_n e_n$ converges strongly if and only if

$$\inf_{n \geq 1} \sup_{k \geq 1} \| [x]_{n,k} \| = 0. \tag{2.8}$$

Thus the set $\tilde{\mathscr{X}}$, associated with $\mathscr{X}$, can be defined as follows

$$\tilde{\mathscr{X}} = \{ \xi \in Z^\infty : \inf_{n \geq 1} \sup_{k \geq 1} \| [x]_{n,k} \| = 0 \}. \tag{2.9}$$

Because of the isomorphism A between $\mathscr{X}$ and $\tilde{\mathscr{X}} \subset Z^\infty$, a measure $\tilde{\nu}$ on $\tilde{\mathscr{X}}$ will induce a measure ν on $\mathscr{X}$. So, it is enough if we construct a measure on $\tilde{\mathscr{X}}$, and we do this by using Kolmogorov's consistency theorem. Let $Z^1 = Z$, let Z^n be the space of all n-tuples $(\xi_1, \ldots, \xi_n)$ of complex numbers, and Z^∞ the space of all complex sequences. We define maps π_n and $\pi_{n,m}$ $(1 \leq m \leq n)$ as follows

$$\pi_n : Z^\infty \to Z^n; \quad \pi_n(\xi_1, \xi_2, \ldots) = (\xi_1, \ldots, \xi_n) \tag{2.10}$$

$$\pi_{n,m} : Z^n \to Z^m; \quad \pi_{n,m}(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_m) \tag{2.11}$$

If $\mathscr{B}_n$ denotes the Borel σ-algebra in Z^n, then the Borel space $(Z^n, \mathscr{B}_n)$ is a separable standard Borel space for $n \geq 1$. Let $\tilde{\nu}_1, \tilde{\nu}_2, \ldots$, be a sequence of measures defined on $\mathscr{B}_1, \mathscr{B}_2, \ldots$, respectively, satisfying the following consistency conditions:

$$\tilde{\nu}_n(Z^n) = 1, \quad \tilde{\nu}_m(B) = \tilde{\nu}_n(\pi_{n,m}^{-1}(B)), \quad \text{for all} \quad B \in \mathscr{B}_m. \tag{2.12}$$

From Kolmogorov's consistency theorem for standard separable Borel spaces (cf. Parthasarathy [*15*]), we obtain a unique measure $\tilde{\nu}$ on $(Z^\infty, \mathscr{B}_\infty)$ such that

$$\tilde{\nu}(Z^\infty) = 1, \quad \tilde{\nu}_n(B) = \tilde{\nu}(\pi_n^{-1}(B)), \quad \text{for all} \quad B \in \mathscr{B}_n. \tag{2.13}$$

First we note that the π_n's and $\pi_{n,m}$'s are measurable and the p_n's and $p_{n,k}$'s are continuous. Thus $p_{n,k}$ is $\tilde{\nu}_{n+k}$-measurable. $\tilde{\mathscr{X}}$ is defined by the functions $p_{n,k}$ under countable operations. Hence from the properties (2.13) of $\tilde{\nu}$, we obtain

LEMMA 2.1. — *The set $\tilde{\mathscr{X}} \subset \mathrm{Z}^\infty$ is ν-measurable.*

The isomorphism between $\mathscr{X}$ and $\tilde{\mathscr{X}}$ gives the following result:

THEOREM 2.1. — *Let $\{ \tilde{\nu}_n \}$ be a sequence of measures defined on $\{ \mathscr{B}_n \}$ satisfying conditions* (2.12), *and let $\tilde{\nu}$ be the measure on Z^∞ obtained through Kolmogorov's theorem from $\{ \tilde{\nu}_n \}$. Then by the isomorphism* (2.3), *$\tilde{\nu}$ induces a measure ν on $\mathscr{X}$.*

ν is obtained in a natural way. Let $\tilde{\mathscr{B}}$ be the restriction of $\mathscr{B}_\infty$ to $\tilde{\mathscr{X}}$, that is, $\tilde{\mathscr{B}} = \tilde{\mathscr{X}} \cap \mathscr{B}_\infty$. For all $\mathrm{B} \subset \mathscr{X}$ such that $\mathrm{B} = \mathrm{A}\tilde{\mathrm{B}}$, $\tilde{\mathrm{B}} \in \tilde{\mathscr{B}}$, put

$$\nu(\mathrm{B}) = \tilde{\nu}(\tilde{\mathrm{B}}). \tag{2.14}$$

Thus we have the measure space $(\mathscr{X}, \mathscr{B}, \nu)$:

$$(\mathscr{X}, \mathscr{B}, \nu) = \mathrm{A}(\tilde{\mathscr{X}}, \tilde{\mathscr{B}}, \tilde{\nu})\,; \quad (\tilde{\mathscr{X}}, \tilde{\mathscr{B}}, \tilde{\nu}) = \mathrm{A}^{-1}(\mathscr{X}, \mathscr{B}, \nu).$$

LEMMA 2.2. — *The measure ν on the Banach space $\mathscr{X}$ with Schauder basis $\{ e_n \}$ is an* L*-measure if and only if the coordinate functionals e_n^*'s are ν-measurable.*

Proof. — If ν is an L-measure, the measurability of $e_n^* \in \mathscr{X}^*$ follows from Definition 2.4 of an L-measure. Conversely, if e_n^*'s are measurable, then, the measurability of $x^* \in \mathscr{X}^*$ follows upon observing that

$$x^*(x) = \sum_{n \geq 1} e_n^*(x) x^*(e_n)$$

THEOREM 2.2. — *The measure ν defined by Theorem* 2.1 *is an* L*-measure.*

Proof. — If $\xi = (\xi_1, \xi_2, \dots) \in \mathrm{Z}^\infty$, let $\mathrm{P}_n(\xi) = \xi_n$ be the projection map to the n-th coordinate. Clearly P_n is $\tilde{\nu}$-measurable. Now the result follows by Lemma 2.2 when we note

$$e_n^*(x) = \xi_n = \mathrm{P}_n(\xi) = (\mathrm{P}_n \mathrm{A}^{-1})(x), \qquad x \in \mathscr{X}.$$

It is clear that $0 \leq \nu(\mathscr{X}) = \tilde{\nu}(\tilde{\mathscr{X}}) \leq 1$.

The next lemma gives a condition so that $\nu(\mathscr{X}) = \tilde{\nu}(\tilde{\mathscr{X}}) = 1$.

LEMMA 2.3. — *The measure space* $(\mathscr{X}, \mathscr{B}, \nu)$ *is a probability measure space if and only if, for any* $\varepsilon > 0$, *the sequence* $\{ \tilde{\nu}_n \}$ *of measures satisfies the following conditions:*

(i) *the consistency conditions* (2.12); *and*
(ii) $\inf_{\varepsilon>0} \sup_{n\geq 1} \inf_{k\geq 1} \nu_{n+k}(\mathrm{A}_{n,k}(\varepsilon)) = 1$ *where*

$$\mathrm{A}_{n,k}(\varepsilon) = \bigcap_{i=1}^{k} \{ (\xi_1, \ldots, \xi_{n+k}) : p_{n,i}(\xi_{n+1}, \ldots, \xi_{n+i}) < \varepsilon \}.$$

Next we consider the situation in terms of random elements with values in a complex Banach space $\mathscr{X}$ with basis $\{ e_n \}$. Let $x : (\Omega, \mathscr{A}, \mu) \to (\mathscr{X}, \mathscr{B})$ be a random element in $\mathscr{X}$. Then there exists a sequence $\{ \xi_n(\omega) \}$ of complex random variables such that

$$x(\omega) = \sum_{n\geq 1} \xi_n(\omega) e_n,$$

where the series converges almost surely. In a Banach space a weak basis is also a strong basis; hence the Banach space $\tilde{\mathscr{X}} \subset \mathrm{Z}^\infty$ obtained by considering $\{ e_n \}$ as a weak basis is the same as the space $\tilde{\mathscr{X}}$ obtained by considering $\{ e_n \}$ as a strong basis. Thus the almost sure convergence of

$$x(\omega) = \sum_{n\geq 1} \xi_n(\omega) e_n$$

is equivalent to the almost sure convergence of

$$x^*(x(\omega)) = \sum_{n\geq 1} \xi_n(\omega) x^*(e_n), \quad \text{for all} \quad x^* \in \mathscr{X}^*. \tag{2.15}$$

We continue to assume $\| e_n \| = 1$. Clearly the almost sure convergence of $\sum_{n\geq 1} \xi_n(\omega)$ gives the almost sure convergence of (2.15). Thus, using some almost sure convergence criteria, we obtain the next few results which we state without proof.

THEOREM 2.3. — *A sequence* $\{ \xi_n(\omega) \}$ *of complex random variables defines an* L*-measure on* $\mathscr{X}$ *if* $\sum_{n\geq 1} \mathscr{E} \mid \xi_n(\omega) \mid < \infty$.

The above result follows from the fact that the almost sure convergence criterion that $\sum_{n \geq 1} \mathscr{E} \mid \xi_n \mid < \infty$ implies

$$P\left[\sum_{n \geq 1} \mid \xi_n \mid < \infty\right] = 1.$$

THEOREM 2.4. — *A sequence of complex random variables* $\{ \xi_n \}$ *defines an* L-*measure on* $\mathscr{X}$ *if a sequence* $\{ \varepsilon_n \}$ *of positive numbers, with either*

$$\sum_{n \geq 1} \varepsilon_n < \infty \quad \textit{or} \quad \varepsilon_n \downarrow 0,$$

exists such that $\sum_{n \geq 1} P[\mid \xi_n \mid > \varepsilon_n] < \infty$.

THEOREM 2.5. — *A sequence* $\{ \xi_n \}$ *of independent random variables defines an* L-*measure on* $\mathscr{X}$ *if and only if, for a fixed* $c > 0$, *all the following series converge*

(*i*) $$\sum_{n \geq 1} P\{ \mid \xi_n \mid > c \};$$

(*ii*) $$\sum_{n \geq 1} \int_{\{|\xi_n| < c\}} \xi_n(\omega) dP(\omega);$$

and

(*iii*) $$\sum_{n \geq 1} \int_{\{|\xi_n| < c\}} \xi_n^2(\omega) dP(\omega).$$

The above result follows from Kolmogorov's three series criterion.

THEOREM 2.6. — *A sequence* $\{ \xi_n \}$ *of independent random variables determines an* L-*measure on* $\mathscr{X}$ *if and only if*

$$\prod_{k=1}^{n} \varphi_k \to \varphi$$

and φ *is continuous at the origin, where the* φ_k*'s are the Fourier transforms of the* ξ_k*'s.*

This follows from a criterion, in terms of characteristic functions, for almost sure convergence.

We now consider Gaussian (or normal) random elements in a Banach space with basis.

DEFINITION 2.5. — A random element x in a Banach space $\mathscr{X}$ is said to be *Gaussian* (or *normal*) if, for $x^* \in \mathscr{X}^*$, the scalar random variable $x^*(x(\omega))$ is Gaussian.

We now state and prove the following result.

THEOREM 2.7. — *Let* $x(\omega) = \sum_{n=1}^{\infty} \xi_n(\omega)e_n$ *be a random element in* $\mathscr{X}$. $x(\omega)$ *is a Gaussian random element in* $\mathscr{X}$ *if and only if* $\xi_n(\omega)$ *is a Gaussian random variable for each* $n \geq 1$.

Proof. — Let $x(\omega)$ be a Gaussian random element in $\mathscr{X}$. Applying Definition 2.5 to the coefficient functional $e_n^* \in \mathscr{X}^*$, $\xi_n(\omega) = e_n^*(x(\omega))$ is a Gaussian random variable for each n. Conversely, let $\xi_n(\omega)$ be a Gaussian random variable for each $n \geq 1$. For each $x^* \in \mathscr{X}^*$, the series

$$x^*(x(\omega)) = \sum_{n \geq 1} \xi_n(\omega)x^*(e_n)$$

converges almost surely. Being the limit of Gaussian random variables, $x^*(x(\omega))$ is Gaussian for each $x^* \in \mathscr{X}^*$. Hence the sufficiency.

COROLLARY 2.1. — *An* L*-measure on* $\mathscr{X}$ *is Gaussian if and only if the coefficient functionals are Gaussian.*

3. L-MEASURES ON THE HARDY SPACES H_p

In this section we use the method given in the last subsection to construct L-mesures on the Hardy spaces H_p.

DEFINITION 3.1. — The *Hardy space* H_p $(1 \leq p < \infty)$ is the space of all functions $f(z)$ holomorphic in the unit disc $|z| < 1$, such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p d\theta \tag{3.1}$$

is bounded for $r \in [0, 1)$. The norm of an element in H_p is given by

$$\| f \|_p = \lim_{r \to 1} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(re^{i\theta}) |^p \right]^{1/p}. \tag{3.2}$$

We refer to the books of Duren [*4*], Hoffman [7] and Porcelli [*16*] for detailed discussions of H_p spaces, their properties, and their relationship to other function spaces.

In order to apply the construction given in Section 2 we need a Schauder basis in H_p. It is known that the trigonometric exponentials $1, e^{i\theta}, e^{2i\theta}, \ldots$ form a Schauder basis in H_p, $1 < p < \infty$. Recently, Akutowicz [*1*] has given a method of constructing a Schauder basis for H_p $(1 \leq p < \infty)$. However, Singer [*18*, p. 201] has pointed out that Akutowicz' result is not correct; hence in this paper we will use the usual Schauder basis, and restrict our attention to the spaces H_p, $1 < p < \infty$.

All the necessary and sufficient conditions that a sequence $\{ \xi_n \}$ be associated with a function $f(z) \in H_p$ is equivalent to the definition of the set $\tilde{H}_p \subset Z^\infty$ which is isomorphic to H_p. As seen in the last section, we have a family of pairs of isomorphic measure spaces $(H_p, \mathscr{A}_p, \nu_p)$ and $(\tilde{H}_p, \tilde{\mathscr{A}}_p, \tilde{\nu}_p)$, $1 < p < \infty$. Since $H_q \subset H_p$, for $p < q$, we have $\tilde{H}_q \subset \tilde{H}_p \subset Z^\infty$. We also note that H_p for $1 < p < \infty$ is reflexive. In this case we have the following result:

THEOREM 3.1. — *If* $\{ \xi_i \}$ *is a sequence of complex numbers such that the sequence* $\left\{ \sum_{i \leq n} \xi_i e_i(z),\ n \geq 1 \right\}$ *is norm bounded in* H_p, *then* $\{ \xi_i \} \in \tilde{H}_p \subset Z^\infty$, *for* $1 < p < \infty$.

The proof of this result follows from the fact that if a Banach space with Schauder basis $\{ e_i \}$ is reflexive, then the basis is boundedly complete (cf. [*3*, p. 70], [*13*, p. 36]).

We also remark that the series $\sum_{n \geq 0} \xi_n e_n(z)$ converges to $f(z) \in H_p$ if and only if

$$\inf_{n \geq 0} \sup_{k \geq 1} \left\| \sum_{j=n+1}^{n+k} \xi_j e_j(z) \right\|_p = 0. \tag{3.3}$$

Since $\| e_n \|_p = 1$, and

$$\left\| \sum_{j=n+1}^{n+k} \xi_j e_j \right\|_p \leq \sum_{j=n+1}^{n+k} | \xi_j | \quad \| e_j \|_p = \sum_{j=n+1}^{n+k} | \xi_j |,$$

we obtain the following sufficient condition.

LEMMA 3.1. — *The series* $\sum_{n\geq 0} \xi_n e_n(z)$ *converges in* H_p *to a function* $f(z) \in H_p$, *if*

$$\inf_{n\geq 0} \sup_{k\geq 1} \sum_{j=n+1}^{n+k} |\xi_j| = 0. \tag{3.4}$$

It is well-known that the H_p norm, denoted by $\|\,.\,\|_p$, is equivalent to the L_p norm $[.]_p$, of the boundary function

$$g(\theta) = \lim_{r\to 1} f(re^{i\theta}), \qquad f \in H_p, \tag{3.5}$$

and

$$\| f \|_p = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(\theta)|^p d\theta \right]^{1/p}. \tag{3.6}$$

If $\{ \varepsilon_n(\theta) \}$ denotes the sequence of boundary functions of the sequence $\{ e_n(z) \}$, then the series $\sum_{n\geq 0} \xi_n e_n(z)$ converges strongly in H_p to a function $f(z) \in H_p$ if and only if

$$\inf_{n\geq 0} \sup_{k\geq 1} \left[\sum_{j=n+1}^{n+k} \xi_j \varepsilon_j \right]_p = 0. \tag{3.7}$$

Let A_p denote the isomorphism between H_p and $\tilde{H}_p \subset Z^\infty$, then

$$\left.\begin{aligned} x &= A_p \tilde{x} \\ \tilde{x} &= A_p^{-1} x \end{aligned}\right\}, \tag{3.8}$$

where

$$x(z) = \sum_{n\geq 0} \xi_n e_n(z).$$

We now give a necessary and sufficient condition that a probability measure ν_p on H_p be an L-measure.

THEOREM 3.2. — *A measure* ν_p *on* $(H_p, \mathscr{A}_p)$ *is an* L-*measure if and only if there corresponds a sequence* $\{ \xi_n(\omega),\ n \geq 0 \}$ *of complex random variables given by the transformation* (3.8) *such that either*

$$P\left\{ \inf_{n\geq 0} \sup_{k\geq 1} \left\| \sum_{j=n+1}^{n+k} \xi_j(\omega) e_j(z) \right\|_p = 0 \right\} = 1, \tag{3.9}$$

or

$$P\left\{ \inf_{n\geq 0} \sup_{k\geq 1} \left[\sum_{j=n+1}^{n+k} \xi_j(\omega)\varepsilon_j(\theta) \right]_p = 0 \right\} = 1. \tag{3.10}$$

This is an immediate consequence of relations (3.3) and (3.7).

From (3.4) we obtain a sufficient condition.

COROLLARY 3.1. — *A measure ν_p on $(H_p, \mathscr{A}_p)$ is an L-measure if there exists a sequence $\{ \xi_n, n \geq 0 \}$ of complex random variables such that*

$$P\left\{ \inf_{n\geq 0} \sup_{k\geq 1} \sum_{j=1}^{k} |\xi_{n+j}| = 0 \right\} = 1. \tag{3.11}$$

From Theorem 3.1 we get the following sufficient condition for the spaces H_p, $1 < p < \infty$.

THEOREM 3.2. — *A sequence of complex random variables $\{ \xi_n(\omega), n \geq 0 \}$ determines an L-measure ν_p on $(H_p, \mathscr{A}_p)$, $1 < p < \infty$, if there is a bounded non-negative real random variable $\eta(\omega)$ such that*

$$\sup_n \left\| \sum_{i=0}^{n} \xi_i(\omega)e_i(z) \right\| \leq \eta(\omega) \text{ a. s.} \tag{3.12}$$

We now state and prove a number of results on L-measures and L*-measures on H_p spaces. Let $(H_p, \mathscr{A}_p, \nu_p)$, $1 < p < \infty$, be a class of probability spaces, where the probability measure ν_p is an L-measure.

THEOREM 3.3. — *The L-measure ν_p on H_p is also an L-measure on H_q for $1 < p \leq q < \infty$.*

Proof. — Since $H_q \subset H_p$ for $1 \leq p \leq q$, we have $\tilde{H}_q \subset \tilde{H}_p \subset Z^\infty$. By construction, $\tilde{H}_p$ is $\tilde{\nu}$-measurable for each $p > 1$, where $\tilde{\nu}$ is the measure on Z^∞ obtained by the Kolmogorov extension theorem. $\tilde{H}_q \subset \tilde{H}_p$ implies that $\tilde{H}_q \subset \mathscr{A}_p$ and thus $\mathscr{A}_q \subset \mathscr{A}_p$ for $q \geq p$. The measure $\tilde{\nu}$ on Z^∞ induces the measures ν_p, $1 < p < \infty$. Let $B \in \mathscr{A}_q \subset \mathscr{A}_p$; then there exists a $B \in \mathscr{A}_q$ such that $\tilde{B} = \tilde{A}_q B$ and $\nu_q(B) = \tilde{\nu}(\tilde{B})$. Similarly $\tilde{\nu}(\tilde{B}) = \nu_p(B)$. Therefore $\nu_q(B) = \nu_p(B)$, for each $B \in \mathscr{A}_q$. Thus ν_p is an L-measure on H_q.

We have seen that ν_q is the restriction of ν_p on $(H_q, \mathscr{A}_q)$. Hence we have

COROLLARY 3.2. — $\nu_p(H_p - H_q) = 0$, *for* $1 < p \leq q < \infty$.

THEOREM 3.4. — *Let ν_q be an L-measure on the Borel measurable space $(H_q, \mathscr{A}_q)$. Then there exists an L-measure ν_p on $(H_p, \mathscr{A}_p)$, for $1<p\leq q<\infty$, such that ν_q is the restriction of ν_p on $\mathscr{A}_q \subset \mathscr{A}_p$.*

Proof. — By hypothesis ν_q is an L-measure on H_q. Because of the isomorphism between H_q and $\tilde{H}_q$, there is a probability measure $\tilde{\nu}_q$ on $\tilde{H}_q \subset Z^\infty$. We define a measure λ on $(Z^\infty, \mathscr{B}_\infty)$ as follows:

$$\lambda(B) = \tilde{\nu}_q(B \cap \tilde{H}_q) \quad \text{for} \quad B \in \mathscr{B}_\infty.$$

Clearly λ is a measure. Let $\tilde{\lambda}_p$ be the restriction of λ on $\tilde{H}_p$. By the isomorphism between $\tilde{H}_p$ and H_p we obtain a measure ν_p on H_p. If $B \in \mathscr{A}_q \subset \mathscr{A}_p$, then

$$\begin{aligned} \nu_p(B) &= \tilde{\lambda}_p(A_p^{-1}B) = \tilde{\lambda}_p(E) \\ &= \tilde{\lambda}_q(E \cap \tilde{H}_q) = \tilde{\lambda}_q(E) \\ &= \nu_q(B). \end{aligned}$$

Since ν_q is a probability measure on H_q, so is ν_p on H_p. The coefficient functionals are continuous and hence measurable.

Let $\mathscr{X}^*$ be the dual of a separable Banach space $\mathscr{X}$. We now define an L*-measure.

DEFINITION 3.2. — A measure ν^* on $\mathscr{X}^*$ is called and *L*-measure* if all the elements in the weak-dual of $\mathscr{X}^*$ are ν^*-measurable.

Clearly L*-measures are less restrictive than L-measures on $\mathscr{X}^*$. A measure ν on $\mathscr{X}^*$ is an L-measure if each $x^{**} \in \mathscr{X}^{**}$ is ν-measurable. If the space is reflexive, then $\mathscr{X}^{**} = \mathscr{X}$, and the notions of L*-measure and L-measure coincide. Because of the reflexivity of H_p for $1 < p < \infty$ and the equivalence of the definitions of L* and L-measures on reflexive spaces, we have the following result:

THEOREM 3.5. — *An L*-measure on $H_p\left(\cong H_{p'}^*, \frac{1}{p} + \frac{1}{p'} = 1\right)$, $1<p<\infty$, is also an L-measure (Thus, constructing an L*-measure on H_p, $1 < p < \infty$, is equivalent to constructing an L-measure on H_p).*

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