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A simplified proof of the Sevastyanov theorem on branching processes

by

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SUMMARY. — The paper presents a new proof of a well known theorem (Sevastyanov) on necessary and sufficient conditions for the degeneration of a branching processes with n types of particles.

SOMMAIRE. — Cet article présente une démonstration nouvelle d'un théorème bien connu (Sevastyanov) sur les conditions nécessaires et suffisantes pour la dégénérescence des processus en cascade avec n types de particules.

One of the most important theorems on branching processes with n-types of particles (n-dimensional Galton-Watson processes) is the Sevastyanov's theorem on degeneration. The original proof of this theorem, as presented in [*1*], is complicated and is based on both probabilistic and non-probabilistic arguments. It is the author's belief that the proof presented in this paper is simpler. Moreover, it used analytic tools only.

We shall consider discrete-time-parameter processes only, i. e. we shall suppose that the time-parameter t assumes the values $t = 0, 1, 2, \ldots$ We shall denote by $\mathcal{P}$ a Markovian homogeneous branching process with n types of particles. We shall call $\mathcal{P}$ shortly a *n-dimensional branching process.* We shall distinguish the particles by indices $i = 1, 2, \ldots, n$. The basic set of indices will be denoted by $I = [1, 2, \ldots, n]$. If $A = [0, 1, 2, \ldots]$, then A^n is the state space of $\mathcal{P}$. The states of $\mathcal{P}$ will be denoted by $a = [a_1, a_2, \dots, a_n]$. To denote special vectors, we shall write $\bar{0} = [0, \dots, 0]$, $\bar{1} = [1, \dots, 1]$ and e_i will be the i-th unit vector. We shall denote by $P_i(t, a)$ the probability of transition from the state e_i to the state a after t time units. For $B \subset A^n$, $P_i(t, B)$ will be the corresponding probability of B, i. e.

$$P(t, B) = \sum_{a \in B} P_i(t, a).$$

We shall denote by $F_i(t, x)$ the generating function of $P_i(t, a)$, i. e.

$$F_i(t, x) = \sum_{a \in A^n} x_1^{a_1} \dots x_n^{a_n} P_i(t, a),$$

where $x = [x_1 \dots x_n] \in [0, 1]^n$. We shall write $F(t, x)$ instead of $[F_1(t, x), \dots, F_n(t, x)]$. It is well known that

$$F(s + t), x)) = F(s, F(t, x)). \tag{1}$$

For $i, j \in I$, we shall write

$$M_{ij}(t) = \sum_{a \in A^n} a_j P_i(t, a).$$

We shall suppose that all $M_{ij}(t)$ are finite and we shall denote by $M(t)$ the moment matrix $(M_{ij}(t))_{i,j \in I}$. In all symbols we have introduced the time-parameter t will be omitted, if $t = 1$. It is well known that $M(t) = M^t$. The maximal characteristic number of M will be denoted by R. Since the branching process $\mathscr{P}$ is uniquely determined by the basic vector $F(x) = [F_1(x) \dots F_n(x)]$, we may speak of a branching process defined by the generating functions $F_1(x), \dots, F_n(x)$.

The subsets of the basic index set $I = [1, 2, \dots, n]$ will be denoted by J, K or I_j. If $J \subset I$, $c(J)$ will denote the number of elements of J. If $x = [x_1, \dots, x_n]$, then $x^{(J)}$ will denote the $c(J)$-dimensional vector the coordinate of which are x_i, $i \in J$. Generally, we shall express the fact that x_i belongs to the i-th particle by the index i only, not by the position of the coordinate x_i in vector; f. i. (x_1, x_2) and (x_2, x_1) will be the same vector for our purposes. This will simplify the forming of new vectors by sub-vectors; f. i. if $I = [1, 2, 3, 4]$, $J = [1, 3]$, $K = [2, 4]$, $y^{(J)} = [y_1, y_3]$, $z^{(K)} = [z_2, z_4]$, then $x = [y^{(J)}, z^{(K)}] = (y_1, y_3, z_2, z_4) = (y_1, z_2, y_3, y_4)$.

We shall write for $J \subset I$,

$$M^{(J)} = (M_{ij})_{i,j \in J} \tag{2}$$

$M^{(J)}$ is a $c(J)$-dimensional matrix and we shall denote its maximal characteristic number by $R^{(J)}$.

If $J \subset I$, we shall denote by $\mathscr{P}^{(J)}$ the $c(J)$-dimensional branching process (with particle-indices $i \in J$), defined by the generating functions

$$F_i^{(J)}(x^{(J)}) = F_i(x^{(J)}, \bar{1}^{(I-J)}), \qquad i \in J.$$

Let $P_i^{(J)}(a)$ be the transition probabilities of $\mathscr{P}^{(J)}$. Then

$$(3) \qquad P_i^{(J)}(B) = P_i(B \times A^{c(I-J)})$$

for each $B \subset A^{c(J)}$.

It follows from (3) that

(4) $M^{(J)}$ *(defined by (2)) is the moment matrix of* $\mathscr{P}^{(J)}$.

Let $J \subset K \subset I$; J will be called *closed in* K *(with respect to* $\mathscr{P}$*)*, if

$$P_i(\{a : a_j = 0 \text{ for all } j \in K - J\}) = 1 \text{ for each } i \in J.$$

It follows from (3) that

(5) J *is closed in* K *with respect to* $\mathscr{P}$ *if and only if it is closed in* K *with respect to* $\mathscr{P}^{(K)}$.

An index set $J \subset I$ will be called *decomposable (with respect to* $\mathscr{P}$*)*, if if there exist two non-empty and disjoint set $J_1 \subset J$, $J_2 \subset J$ such that $J_1 \cup J_2 = J$ and J_1 is closed in J. J will be called *indecomposable* if it is not decomposable. Clearly

(6) J *is indecomposable if and only if* $M^{(J)}$ *is indecomposable.*

Also

(7) J *is indecomposable with respect to* $\mathscr{P}$ *if and only if it is indecomposable with respect to* $\mathscr{P}^{(J)}$.

An index set J will be called *final (with respect to* $\mathscr{P}$*)*, if it is indecomposable and if

$$P_i\left(\left\{ a : \sum_{j \in J} a_j = 1 \right\}\right) = 1 \quad \text{for each} \quad i \in J.$$

It follows again from (3) that

(8) J *is final with respect to* $\mathscr{P}$ *if and only if it is final with respect to* $\mathscr{P}^{(J)}$.

We shall call

$$P_i = \lim_{t\to\infty} P_i(t, 0) = \lim_{t\to\infty} F_i(t, 0) \qquad (i = 1, \ldots, n)$$

the *degeneration probabilities* of $\mathscr{P}$ and $p = (p_1, \ldots, p_n)$ the *degeneration-probability vector* of $\mathscr{P}$. We shall call $\mathscr{P}$ *degenerate* if $p = \bar{1}$. It is well known that

(9) $$F(p) = p$$

and that

(10) P *is degenerate if and only if* $\bar{1}$ *is the only solution in* $[0, 1]^n$ *of the system* $F(x) = x$.

In the proof of the main theorem we shall need two lemmas.

LEMMA A. — *If* $J \subset I$, $K \subset I$, $J \cap K = \emptyset$, $J \cup K = I$, J *closed in* I *and if both* $\mathscr{P}^{(J)}$ *and* $\mathscr{P}^{(K)}$ *are degenerate, then* $\mathscr{P}$ *is degenerate.*

Proof. — Let $p = (p_1, \ldots, p_n)$ be the degeneration-probability vector of $\mathscr{P}$. Since J is closed in I, $F_i(x)$ with $i \in J$ does not depend on x_j with $j \in K$ and, consequently

(11) $$F_i^{(J)}(p^{(J)}) = F_i(p^{(J)}, \bar{1}^{(K)}) = F_i(p) = p_i$$

for all $i \in J$ by (9). From (10), (11) and the assumption that $\mathscr{P}^{(J)}$ is degenerate it follows that

$$p_i = 1 \qquad \text{for} \qquad i \in J$$

Then

$$F_i^{(K)}(p^{(K)}) = F_i(\bar{1}^{(J)}, p^{(K)}) = F_i(p) = p_i \quad \text{for all} \quad i \in K$$

and again

$$p_i = 1 \qquad \text{for} \qquad i \in K.$$

Hence $p = \bar{1}$.

LEMMA B. — *If, for* $J \subset I$, $\mathscr{P}^{(J)}$ *is not degenerate, then* $\mathscr{P}$ *is not degenerate.*

Proof. — Let us write $K = I - J$. For each $x = (x_1, \ldots, x_n)$ and each $i \in J$ we have

$$F_i^{(J)}(x^{(J)}) = F_i(x^{(J)}, \bar{1}^{(K)}) \geqq F_i(x).$$

By (1)

$$F_i^{(J)}(2, x^{(J)}) = F_i^{(J)}(F^{(J)}(x^{(J)})) \geqq F_i^{(J)}((F(x))^{(J)}$$
$$\geqq F_i(F(x)) = F_i(2, x) \qquad \text{for each} \qquad i \in J.$$

Generally

(12) $$F_i^{(J)}(t, x^{(J)}) \geqq F_i(t, x) \qquad \text{for each} \qquad i \in J \text{ and } t,$$

and denoting by p_i the degeneration probabilities of $\mathscr{P}$ and by $q_i (i \in \mathrm{J})$ the degeneration probabilities of $\mathscr{P}^{(\mathrm{J})}$, we have by (12)

$$(13) \qquad q_i \geqq p_i \qquad \text{for all} \qquad i \in \mathrm{J}.$$

According to the assumption, $q_i < 1$ for at least one $i \in \mathrm{J}$, and then $p_i < 1$ by (13). Hence, $\mathscr{P}$ is not degenerate.

THEOREM (Sevastyanov). — *$\mathscr{P}$ is degenerate if and only if* (*a*)$\mathrm{R} \leqslant 1$ *and* (*b*) *there are no final index sets.*

Proof. — Let us suppose that the conditions (*a*) and (*b*) are satisfied.

(*i*) We shall first assume that the moment matrix M is indecomposable. Let $p = (p_1, \ldots, p_n)$ be the degeneration-probability vector and let J be the set of all indices i for which $p_i < 1$. Let us suppose that J is non-empty.

Then for each $i \in \mathrm{I}$

$$(14) \qquad \mathrm{F}_i(p) = 1 + \sum_{j \in \mathrm{J}} \mathrm{M}_{ij}(p_j - 1) + \frac{1}{2} \sum_{j,k \in \mathrm{J}} \frac{\partial^2}{\partial x_j \partial x_k} \mathrm{F}_i(q)(p_j - 1)(p_k - 1)$$

where $q = (q_1, \ldots, q_n)$ is a vector such that

$$(15) \qquad \begin{aligned} p_i < q_i &< 1 \qquad \text{for} \qquad i \in \mathrm{J} \\ q_i &= 1 \qquad \text{for} \qquad i \notin \mathrm{J}. \end{aligned}$$

By (9) and (14) we have for each $i \in \mathrm{I}$

$$(16) \qquad \begin{aligned} \sum_{j=1}^{n} \mathrm{M}_{ij}(1-p_j) = \sum_{j \in \mathrm{J}} \mathrm{M}_{ij}(1-p_j) = 1 - p_j + \frac{1}{2} \sum_{j,k \in \mathrm{J}} \frac{\partial^2}{\partial x_j \partial x_k} \mathrm{F}_i(q)(1-p_j)(1-p_k) \\ \geqq 1 - p_j \geqq \mathrm{R}(1 - p_j). \end{aligned}$$

Hence, $\mathrm{M}(\bar{1} - p) \geqq \mathrm{R}(\bar{1} - p)$ and since J supposed to be non-empty, $\bar{1} - p$ is an eigen-vector belonging to R, according to a well-known theorem on non-negative matrices. But then

$$(17) \qquad \mathrm{M}(\bar{1} - p) = \mathrm{R}(\bar{1} - p)$$

and since M is indecomposable, $1 - p_i > 0$ for all $i \in \mathrm{I}$, i. e. $\mathrm{J} = \mathrm{I}$. It follows now from (16) and (17) that $\mathrm{R} = 1$ and

$$(18) \qquad \frac{\partial^2}{\partial x_j \partial x_k} \mathrm{F}_i(q) = 0 \qquad \text{for all} \qquad i, j, k \in \mathrm{I}.$$

By (15), $q_i > 0$ for all $i \in \mathrm{J} = \mathrm{I}$, and since F_i is a power series with non-negative coefficients $\mathrm{P}_i(a)$, it follows from (18), that

$$\mathrm{P}_i\left(\left\{ a \colon \sum_{j=1}^{n} a_j \geqq 2 \right\}\right) = 0 \quad \text{for all} \quad i \in \mathrm{I}, \tag{19}$$

Hence, $\mathrm{M}_{ij} = \mathrm{P}_i(e_j)$ for all $i, j \in \mathrm{I}$, and since

$$\sum_{j=1}^{n} \mathrm{P}_i(e_j) \leqslant 1,$$

M is a sub-stochastic matrix. On the other hand, we have proved that $\mathrm{R} = 1$, which implies that M is a stochastic matrix, i. e.

$$\mathrm{P}_i\left(\left\{ a \colon \sum_{j=1}^{n} a_j = 1 \right\}\right) = \sum_{j=1}^{n} \mathrm{P}_i(e_j) = \sum_{j=1}^{n} \mathrm{M}_{ij} = 1.$$

It follows that I is a final set of indices and this is a contradiction to the condition (*b*). We came to this contradiction on the basis of the assumption that J is non-empty. Hence, J must be empty, i. e. $p = \bar{1}$.

(*ii*) We shall suppose again that conditions (*a*) and (*b*) hold, but M will now be an arbitrary moment matrix. It is well-known that there exists index sets $\mathrm{I}_l \subset \mathrm{I}$ $(l = 1, 2, \ldots, k)$ such that I_l are disjoint,

$$\bigcup_{l=1}^{k} \mathrm{I}_l = \mathrm{I},$$

$\mathrm{M}^{(\mathrm{I}_l)}$ are indecomposable and

$$\mathrm{M}_{ij} = 0 \quad \text{for} \quad i \in \mathrm{I}_l, \quad j \in \mathrm{I}_{l+1} \cup \ldots \cup \mathrm{I}_k \quad (l = 1, 2, \ldots, k-1). \tag{20}$$

We shall write $\mathrm{J}_l = \mathrm{I}_1 \cup \ldots \cup \mathrm{I}_l$. By (6) and (7), I_l is indecomposable with respect to $\mathscr{P}^{(\mathrm{I}_l)}$, and it is well known that $\mathrm{R}^{(\mathrm{I}_l)} \geqq \mathrm{R}$. Hence, by (4), (8) and part (*i*) of this proof, each $\mathscr{P}^{(\mathrm{I}_l)}$ is degenerate. In particular, $\mathscr{P}^{(\mathrm{J}_1)}$ is degenerate. Let us suppose that we have already proved that $\mathscr{P}^{(\mathrm{J}_l)}$ is degenerate. It follows from (20) and (5) that J_l is closed in J_{l+1} with respect to $\mathscr{P}^{(\mathrm{J}_{l+1})}$ and hence $\mathscr{P}^{(\mathrm{J}_{l+1})}$ is degenerate according to Lemma A. By induction, $\mathscr{P}^{(\mathrm{J}_k)} = \mathscr{P}$ is degenerate.

We shall prove that conditions (*a*) and (*b*) are necessary

(*iii*) Let us suppose that there exists a final index set $\mathrm{J} \subset \mathrm{I}$. According to the definition of a final set,

$$\mathrm{F}_i^{(\mathrm{J})}(\bar{0}) = \mathrm{P}_i^{(\mathrm{J})}(\bar{0}) = 0 \quad \text{for all} \quad i \in \mathrm{J}.$$

Then $F_i^{(J)}(2, \bar{0}) = F_i^{(J)}(F^{(J)}(\bar{0})) = F_i^{(J)}(0) = 0$ and, generally, $F_i^{(J)}(t, \bar{0}) = 0$ for all $i \in J$ and all t. Hence, the process $\mathscr{P}^{(J)}$ is not degenerate and according to Lemma B, $\mathscr{P}$ is also not degenerate.

(*iv*) Let us suppose that $R > 1$. It is well-known from the spectral theory of non-negative matrices that there exists $j \in I$ and s such that $M_{jj}(s) > 1$. Let $\bar{\mathscr{P}}$ be a new branching process with the index set I, generated by basic generating functions $\bar{F}_i(x) = F_i(s, x)$. According to (1), the general generating functions of $\bar{\mathscr{P}}$ are $\bar{F}_i(t, x) = F_i(st, x)$ and, consequently,

$$\lim_{t \to \infty} \bar{F}_i(t, \bar{0}) = \lim_{t \to \infty} F_i(st, \bar{0}) = \lim_{t \to \infty} F_i(t, \bar{0}), \qquad i \in I. \tag{21}$$

Let us write $J = \{J\}$. Then $\bar{\mathscr{P}}^{(J)}$ is a one-dimensional subprocess of $\bar{\mathscr{P}}$ with the first moment $M_{jj}(s) > 1$ and according to a well-known theorem on one-dimensional branching processes, $\bar{\mathscr{P}}^{(J)}$ is not degenerate. By Lemma B, $\bar{\mathscr{P}}$ is also not degenerate. But, according to (21), $\mathscr{P}$ and $\bar{\mathscr{P}}$ have the same degeneration-probability vector and, consequently, $\mathscr{P}$ is not degenerate.

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[*1*] B. A. SEVASTYANOV, Theory of branching processes, (Russian) *Uspekhi mat. nauk.*, **6** (6), 1951, 47-99.

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