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Extreme value distribution for the M/G/1 and the G/M/1 queueing systems

by

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SUMMARY.— For the supremum of the virtual delay time in a busy cycle and for the supremum of the actual waiting times of the customers served in a busy cycle the Laplace-Stieltjes transforms of the distribution functions have been found recently. Also for the supremum of the number of customers simultaneously present in the system during a busy cycle the generating function of the distribution is known. For every one of these variables the limit distribution of the maximum of these variables over a finite number of busy cycles is derived in the present paper. These limit distributions are obtained for the queueing systems M/G/1 and G/M/1 and for traffic intensities equal to one and less than one.

1. SOME RELATIONS FOR THE M/G/1 SYSTEM

For the M/G/1 queueing system denote by \underline{v}_t the virtual waiting time at time t , by \underline{x}_t the number of customers in the system at time t and by \underline{w}_n the actual waiting time of the n th arriving customer with $\underline{w}_1 = 0$. Further \underline{c} will denote the duration of a busy cycle and n the number of customers served in a busy cycle. Define

$$\begin{aligned}\underline{v}_{\max} &\stackrel{\text{def}}{=} \sup_{0 < t < \underline{c}} \underline{v}_t, \\ \underline{w}_{\max} &\stackrel{\text{def}}{=} \sup_{1 \leq n \leq n} \underline{w}_n, \\ \underline{x}_{\max} &\stackrel{\text{def}}{=} \sup_{0 < t < \underline{c}} \underline{x}_t,\end{aligned}$$

so that \underline{v}_{\max} is the supremum of the virtual waiting time in a busy cycle, \underline{w}_{\max} is the supremum of all actual waiting times of a busy cycle and \underline{x}_{\max} the maximum number of customers simultaneously present in a busy cycle.

Denoting by $B(t)$ the distribution function of the service times and by α the average interarrival time then with

$$\beta(\rho) = \int_0^{\infty} e^{-\rho t} dB(t), \quad \operatorname{Re} \rho \geq 0, \quad B(0+) = 0, \quad \beta = \int_0^{\infty} t dB(t) < \infty,$$

we have

$$(1.1) \quad \Pr \{ \underline{v}_{\max} < v \} = \frac{\frac{1}{2\pi i} \int_{C_n} e^{\eta v} \frac{\beta(\eta)}{\beta(\eta) + \alpha\eta - 1} d\eta}{\frac{1}{2\pi i} \int_{C_n} \frac{e^{\eta v}}{\beta(\eta) + \alpha\eta - 1} d\eta}, \quad \operatorname{Re} \eta > \delta; \quad v > 0,$$

$$= 0, \quad v < 0,$$

$$(1.2) \quad \Pr \{ \underline{w}_{\max} < w \} = \frac{\frac{1}{2\pi i} \int_{C_n} e^{\eta w} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1}}{\frac{1}{2\pi i} \int_{C_n} \frac{e^{\eta w}}{1 - \alpha\eta \beta(\eta) + \alpha\eta - 1} d\eta}, \quad \frac{1}{\alpha} > \operatorname{Re} \eta > \delta; \quad w > 0,$$

$$= 0, \quad w < 0,$$

and for $x = 1, 2, \dots$,

$$(1.3) \quad \Pr \{ \underline{x}_{\max} \leq x \} = \frac{\frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^x} \frac{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\}}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega}}{\frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^{x+1}} \frac{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\}}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega}}, \quad |\omega| < \mu.$$

Here we used the notation

$$\frac{1}{2\pi i} \int_{C_n} \dots d\eta = \lim_{b \rightarrow \infty} \int_{R-ib}^{R+ib} \dots d\eta, \quad R = \operatorname{Re} \eta,$$

and D_ω is a circle in the complex ω -plane with center at $\omega = 0$ and radius $|\omega|$, the positive direction of integration being counter clockwise. By δ is denoted the larger zero of $\beta(\eta) + \alpha\eta - 1$ with $\operatorname{Re} \eta \geq 0$, while μ is the smaller

zero inside or on the unit circle of $\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega$. It is well known (cf. Takacs [1]) that if $a \stackrel{\text{def}}{=} \beta/\alpha \leq 1$ then $\delta = 0$, $\mu = 1$; the zeros δ and μ have multiplicity one if $a \neq 1$, if $a = 1$ they have multiplicity two. The relations (1.1) and (1.3) have been derived by Takacs [2] and by Cohen [3], [4], [5], while the relation (1.2) has been obtained by Cohen [6].

Let \underline{w} and \underline{x} be stochastic variables with distribution functions given by

$$(1.4) \quad E \{ e^{-\rho \underline{w}} \} = \frac{(1 - a)\alpha\rho}{\beta(\rho) + \alpha\rho - 1}, \quad \text{Re } \rho \geq 0, \quad a < 1,$$

$$(1.5) \quad E \{ \omega^{\underline{x}} \} = (1 - a) \frac{(1 - \omega)\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\}}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega}, \quad |\omega| \leq 1, \quad a < 1,$$

so that the distribution of \underline{w} is the stationary distribution of the (virtual or actual) waiting time for the M/G/1 queue, and the distribution of \underline{x} is the stationary distribution of the number of customers present in the M/G/1 queueing system.

Further let $\underline{\sigma}$ be a negative exponentially distributed variable with expectation α and $\underline{\tau}$ a variable with distribution function $B(t)$. Assume that \underline{w} and $\underline{\sigma}$ are independent, and also that \underline{w} and $\underline{\tau}$ are independent. It follows from (1.1), . . . , (1.5) that for $a < 1$,

$$\begin{aligned} \Pr \{ \underline{v}_{\max} < v \} &= \frac{\Pr \{ \underline{w} + \underline{\tau} < v \}}{\Pr \{ \underline{w} < v \}}, & v > 0, \\ &= 0, & v < 0; \\ \Pr \{ \underline{w}_{\max} < w \} &= \frac{\Pr \{ \underline{w} < w \}}{\Pr \{ \underline{w} < w + \underline{\sigma} \}}, & w > 0, \\ &= 0, & w < 0, \\ \Pr \{ \underline{x}_{\max} \leq x \} &= 1 - \frac{\Pr \{ \underline{x} = x \}}{\Pr \{ \underline{x} \leq x \}}, & x = 0, 1, \dots \end{aligned}$$

From (1.1) for $v > 0$, $\text{Re } \eta > 0$, $a \leq 1$,

$$(1.6) \quad 1 - \Pr \{ \underline{v}_{\max} < v \} = \frac{\frac{\alpha}{2\pi i} \int_{C_\eta} e^{\eta v} \frac{\eta d\eta}{\beta(\eta) + \alpha\eta - 1}}{\frac{1}{2\pi i} \int_{C_\eta} e^{\eta v} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1}};$$

from (1.2) for $w > 0$, $\frac{1}{\alpha} > \operatorname{Re} \eta > 0$, $a \leq 1$,

$$(1.7) \quad 1 - \Pr \{ \underline{w}_{\max} < w \} = \frac{\frac{\alpha}{2\pi i} \int_{C_\eta} \frac{e^{\eta w}}{1 - \alpha\eta} \frac{\eta d\eta}{\beta(\eta) + \alpha\eta - 1}}{\frac{1}{2\pi i} \int_{C_\eta} \frac{e^{\eta w}}{1 - \alpha\eta} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1}};$$

and from (1.3) for $x = 2, 3, \dots$, $|\omega| < 1$,

$$(1.8) \quad 1 - \Pr \{ \underline{x}_{\max} \leq x \} = \frac{\frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^x} \frac{1 - \omega}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega}}{\frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^x} \frac{1}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega}}$$

Define

$$\begin{aligned} H(t) &\stackrel{\text{def}}{=} \frac{1}{\beta} \int_0^t \{1 - B(\tau)\} d\tau, & h(t) &\stackrel{\text{def}}{=} \frac{1}{\beta} \{1 - B(t)\}, & t > 0, \\ &= 0, & &= 0, & t < 0, \end{aligned}$$

so that $H(t)$ is a distribution function having a bounded and monotone density function $h(t)$. Define for $a \leq 1$

$$(1.9) \quad K(t, a) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a^n H^{n*}(t),$$

so that

$$(1.10) \quad \int_{0-}^{\infty} e^{-\eta t} d_t K(t, a) = \frac{\alpha\eta}{\beta(\eta) + \alpha\eta - 1}, \quad \operatorname{Re} \eta > 0.$$

Obviously, $K(t, 1)$ is the renewal function of a renewal process with $H(t)$ as renewal distribution. Since $H(t)$ has a density which is monotone and bounded $K(t, a)$ has for $a \leq 1$ a bounded derivative $k(t, a)$ (cf. Feller [7], p. 358) and

$$(1.11) \quad k(t, a) = \frac{d}{dt} K(t, a), \quad t > 0,$$

$$(1.12) \quad \int_0^{\infty} e^{-\eta t} k(t, a) dt = \frac{\alpha\eta}{\beta(\eta) + \alpha\eta - 1} - 1, \quad \operatorname{Re} \eta > 0.$$

Since for $a \leq 1$

$$\begin{aligned} \int_0^\infty \mathbf{K}(w + \tau, a) e^{-\tau/\alpha} \frac{d\tau}{\alpha} &= e^{w/\alpha} \int_{t=w}^\infty e^{-t/\alpha} \mathbf{K}(t, a) \frac{dt}{\alpha}, \\ &= \mathbf{K}(w, a) + e^{w/\alpha} \int_{t=w}^\infty e^{-t/\alpha} k(t, a) dt, \end{aligned}$$

we have for $w > 0$, $a \leq 1$,

$$(1.13) \quad \frac{d}{dw} \int_0^\infty \mathbf{K}(w + \tau, a) e^{-\tau/\alpha} \frac{d\tau}{\alpha} = e^{w/\alpha} \int_{t=w}^\infty e^{-t/\alpha} k(t, a) \frac{dt}{\alpha}.$$

It is easily seen that for $a \leq 1$, $0 < \operatorname{Re} \eta < \frac{1}{\alpha}$,

$$(1.14) \quad \begin{aligned} \int_0^\infty e^{-\eta w} d_w \int_0^\infty \mathbf{K}(w + t, a) e^{-t/\alpha} \frac{dt}{\alpha} &= \frac{1}{1 - \alpha\eta} \frac{\alpha\eta}{\beta(\eta) + \alpha\eta - 1}, \\ \int_0^\infty e^{-\eta w} d_w \int_0^\infty k(w + t, a) e^{-t/\alpha} \frac{dt}{\alpha} &= \frac{1}{1 - \alpha\eta} \frac{\alpha\eta^2}{\beta(\eta) + \alpha\eta - 1}. \end{aligned}$$

Further for $|\omega| < 1$, $x = 0, 1, \dots$,

$$(1.15) \quad \begin{aligned} \frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^{x+1}} \frac{1 - \omega}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega} &= \int_{0-}^\infty \frac{(t/\alpha)^x}{x!} e^{-t/\alpha} d_t \mathbf{K}(t, a), \\ \frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^{x+1}} \frac{1}{\beta \left\{ \frac{1}{\alpha}(1 - \omega) \right\} - \omega} &= \int_{0-}^\infty \frac{(t/\alpha)^x}{x!} e^{-t/\alpha} \mathbf{K}(t, a) \frac{dt}{\alpha}. \end{aligned}$$

From (1.6), (1.7) and (1.8) it follows easily by using the inversion formula for the Laplace-Stieltjes transform that for $a \leq 1$,

$$(1.16) \quad \begin{aligned} 1 - \Pr \{ v_{\max} < v \} &= \alpha \frac{k(v, a)}{\mathbf{K}(v, a)} = \alpha \frac{d}{dv} \log \mathbf{K}(v, a), \quad v > 0, \\ &= \alpha \frac{d}{dv} \log \Pr \{ \underline{w} < v \} \quad \text{if } a < 1; \end{aligned}$$

$$(1.17) \quad \begin{aligned} 1 - \Pr \{ \underline{w}_{\max} < w \} &= \frac{\alpha \int_0^\infty k(w + t, a) e^{-t/\alpha} dt}{\int_0^\infty \mathbf{K}(w + t, a) e^{-t/\alpha} dt} \\ &= \alpha \frac{d}{dw} \log \int_0^\infty \mathbf{K}(w + t, a) e^{-t/\alpha} dt, \\ &= \alpha \frac{d}{dw} \log \Pr \{ \underline{w} < w + \sigma \} \quad \text{if } a < 1, \quad w > 0, \end{aligned}$$

$$(1.18) \quad 1 - \Pr \{x_{\max} < x\} = \frac{\int_{0-}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} d_t \mathbf{K}(t, a)}{\int_{0-}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \mathbf{K}(t, a) \frac{dt}{\alpha}}, \quad x = 2, 3, \dots,$$

From the relations

$$\begin{aligned} E \{v_{\max}\} &= \int_0^{\infty} \{1 - \Pr \{v_{\max} < v\}\} dv, \\ E \{v_{\max}^2\} &= 2 \int_0^{\infty} v \{1 - \Pr \{v_{\max} < v\}\} dv, \end{aligned}$$

and

$$\Pr \{\underline{w} < 0+\} = 1 - a \quad \text{if } a < 1,$$

it is found that for $a < 1$ (cf. (1.16) and (1.17))

$$\begin{aligned} E \{v_{\max}\} &= \frac{\beta}{a} \log \frac{1}{1-a}, \\ E \{v_{\max}^2\} &= -2 \frac{\beta}{a} \int_0^{\infty} \log \{1 - \Pr \{\underline{w} \geq v\}\} dv, \\ E \{w_{\max}\} &= \frac{\beta}{a} \log \frac{\beta \left(\frac{1}{\alpha}\right)}{1-a}, \\ E \{w_{\max}^2\} &= -2 \frac{\beta}{a} \int_0^{\infty} \log \Pr \{\underline{w} < w + \underline{\sigma}\} dw. \end{aligned}$$

Since

$$E \{v_{\max}^2\} = 2 \frac{\beta}{a} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{n} [\Pr \{\underline{w} \geq w\}]^n dw,$$

and

$$\Pr \{\underline{w} \geq w\} < a,$$

we have

$$2 \frac{\beta}{a} \int_0^{\infty} \Pr \{\underline{w} \geq w\} dw < E \{v_{\max}^2\} < 2 \frac{\beta}{a} \sum_{n=1}^{\infty} \frac{a^{n-1}}{n} \int_0^{\infty} \Pr \{\underline{w} \geq w\} dw$$

so that since

$$\int_0^{\infty} \Pr \{\underline{w} \geq w\} dw = \frac{1}{2} \frac{a\beta}{1-a} \frac{\beta_2}{\beta^2},$$

with β_2 the second moment of $B(t)$, we obtain

$$\frac{\beta_2}{1-a} < E\{v_{\max}^2\} < \frac{\beta_2}{1-a} \frac{1}{a} \log \frac{1}{1-a}.$$

It is seen that the second moment of v_{\max} is finite if $\beta_2 < \infty$, a similar conclusion holds for w_{\max} and x_{\max} . It is noted that $E\{w_{\max}\}$ is finite if $a < 1$, while $E\{w\}$ is finite if $a < 1$ and $\beta_2 < \infty$.

2. EXTREME VALUE DISTRIBUTIONS FOR M/G/1

Suppose the server is idle at time $t = 0$. Denote by $v_{\max}^{(j)}$, $w_{\max}^{(j)}$ and $x_{\max}^{(j)}$ the supremum of v_t , of w_t and of x_t in the j th busy cycle of the queueing system M/G/1, $j = 1, 2, \dots$. Obviously, $v_{\max}^{(j)}$, $j = 1, 2, \dots$, are independent, identically distributed variables with finite first moment if $a < 1$ and with finite second moment if $\beta_2 < \infty$. If $a < 1$ then the strong law of large numbers applies for the sequence $v_{\max}^{(j)}$, $j = 1, 2, \dots$; whereas if $\beta_2 < \infty$ the central limit theorem applies also for this sequence. Similar statements hold for the other sequences $w_{\max}^{(j)}$, $j = 1, 2, \dots$, and $x_{\max}^{(j)}$, $j = 1, 2, \dots$.

Define for $n = 1, 2, \dots$,

$$\begin{aligned} \underline{V}_n &\stackrel{\text{def}}{=} \max_{1 \leq j \leq n} v_{\max}^{(j)}, & \underline{W}_n &\stackrel{\text{def}}{=} \max_{1 \leq j \leq n} w_{\max}^{(j)}, \\ \underline{X}_n &\stackrel{\text{def}}{=} \max_{1 \leq j \leq n} x_{\max}^{(j)}, \end{aligned}$$

i. e. \underline{V}_n is the supremum of the virtual waiting time in n busy cycles, \underline{W}_n that of the actual waiting times in n busy cycles and \underline{X}_n the supremum of the number of customers present simultaneously in the system during n busy cycles. For these variables we shall derive some limit theorems.

THEOREM 1. — If $a = 1$ and β_2 , the second moment of $B(t)$, is finite then the distributions of $\frac{1}{n\beta} \underline{V}_n$, of $\frac{1}{n\beta} \underline{W}_n$ and of $\frac{1}{n} \underline{X}_n$ all converge for $n \rightarrow \infty$ to the distribution $G(x)$ with

$$G(x) = e^{-x^{-1}} \quad \text{for } x > 0, \quad = 0 \quad \text{for } x < 0.$$

Proof. Since $\beta_2/2\beta$ is the first moment of $H(t)$, and since $h(t)$ is monotone we have from renewal theory (cf. Feller [7], p. 358)

$$(2.1) \quad \lim_{t \rightarrow \infty} \frac{K(t, 1)}{t} = \frac{2\beta}{\beta_2}, \quad \lim_{t \rightarrow \infty} k(t, 1) = \frac{2\beta}{\beta_2}.$$

Hence from (1.16) since $a = 1$

$$(2.2) \quad \lim_{v \rightarrow \infty} v \{ 1 - \Pr \{ v_{\max}^{(j)} < v \} \} = \alpha = \beta.$$

From this relation and from

$$\Pr \left\{ \frac{1}{n\beta} V_n < x \right\} = [\Pr \{ v_{\max}^{(j)} < n\beta x \}]^n = \left\{ 1 - \frac{\beta}{n\beta x} + o\left(\frac{1}{n}\right) \right\}^n, \quad x > 0,$$

for $n \rightarrow \infty$ it follows immediately that

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{n\beta} V_n < x \right\} &= e^{-x^{-1}}, & x > 0, \\ &= 0, & x < 0, \end{aligned}$$

and the statement for \underline{V}_n has been proved.

From (1.17) for $a = 1$

$$(2.4) \quad 1 - \Pr \{ \underline{w}_{\max} < w \} = \alpha \frac{\int_0^\infty k(w + \tau, 1) e^{-\tau/\alpha} \frac{d\tau}{\alpha}}{\int_0^\infty K(w + \tau, 1) e^{-\tau/\alpha} \frac{d\tau}{\alpha}}, \quad w > 0.$$

For given $\varepsilon > 0$ a finite number $W(\varepsilon) > 0$ exists such that

$$\left| k(w, 1) - \frac{2\beta}{\beta_2} \right| < \varepsilon \quad \text{for all} \quad w > W(\varepsilon),$$

so that

$$\left| k(w + t, 1) - \frac{2\beta}{\beta_2} \right| < \varepsilon \quad \text{for all} \quad w > W(\varepsilon), \quad t \geq 0.$$

Consequently, since $k(t, 1)$ is bounded

$$\lim_{w \rightarrow \infty} \int_0^\infty k(t + w, 1) e^{-t/\alpha} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2} \int_0^\infty e^{-t/\alpha} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

Using (2.1) the same argumentation yields

$$\lim_{w \rightarrow \infty} \frac{1}{w} \int_0^\infty K(w + t, 1) e^{-t/\alpha} \frac{d\tau}{\alpha} = \lim_{w \rightarrow \infty} \int_0^\infty \frac{K(w + t, 1)}{w + t} e^{-t/\alpha} \left\{ 1 + \frac{t}{w} \right\} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

Hence from (2.4)

$$\lim_{w \rightarrow \infty} w \{ 1 - \Pr \{ \underline{w}_{\max}^{(j)} < w \} \} = \alpha = \beta,$$

so that, as above the statement for \underline{W}_n follows.

For $x = 1, 2, \dots$,

$$\int_{W(\varepsilon)}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \left| k(t, 1) - \frac{2\beta}{\beta_2} \right| \frac{dt}{\alpha} \leq \varepsilon \int_{W(\varepsilon)}^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \frac{dt}{\alpha} \leq \varepsilon,$$

and

$$\begin{aligned} \lim_{x \rightarrow \infty} \int_0^{W(\varepsilon)} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} k(t, 1) \frac{dt}{\alpha} \\ \leq \max_{0 \leq t \leq W(\varepsilon)} k(t, 1) \cdot \lim_{x \rightarrow \infty} \int_0^{W(\varepsilon)} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \frac{dt}{\alpha} = 0. \end{aligned}$$

It follows

$$\lim_{x \rightarrow \infty} \int_0^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} \left(k(t, 1) - \frac{2\beta}{\beta_2} \right) \frac{dt}{\alpha} = 0,$$

or

$$\lim_{x \rightarrow \infty} \int_0^{\infty} \frac{(t/\alpha)^{x-1}}{(x-1)!} e^{-t/\alpha} k(t, 1) \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

In the same way it is shown that

$$\lim_{x \rightarrow \infty} \int_0^{\infty} \frac{(t/\alpha)^x}{x!} e^{-t/\alpha} \frac{K(t, 1)}{t} \frac{dt}{\alpha} = \frac{2\beta}{\beta_2}.$$

Hence from (1.12)

$$\lim_{x \rightarrow \infty} x \{ 1 - \Pr \{ \underline{x}_{\max}^{(j)} \leq x \} \} = 1,$$

the last relation leads as above to the statement for X_n . The theorem is proved.

THEOREM 2. — If $a < 1$, $\rho_0 > 0$ and $-\rho_0$ is the abscissa of convergence of $\beta(\rho)$ and if $\beta(-\rho_0 + 0) = \infty$ then for $-\infty < x < \infty$,

$$\lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\beta} \underline{V}_n < \frac{x + \log(nb_1)}{-\varepsilon\beta} \right\} = e^{-e^{-x}},$$

$$\lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\beta} \underline{W}_n < \frac{x + \log(nb_2)}{-\varepsilon\beta} \right\} = e^{-e^{-x}},$$

$$\lim_{n \rightarrow \infty} \Pr \left\{ \underline{X}_n < \frac{x + \log(nb_3)}{\log(1 - \alpha\varepsilon)} \right\} = e^{-e^{-x}},$$

with

$$b_1 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \alpha\varepsilon, \quad b_2 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \frac{\alpha\varepsilon}{1 - \alpha\varepsilon}, \quad b_3 = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \alpha\varepsilon(1 - \alpha\varepsilon),$$

$$\beta'(\rho) = - \int_0^{\infty} t e^{-\rho t} dB(t), \quad \text{Re } \rho > -\rho_0,$$

and ε is the zero of $\beta(\eta) + \alpha\eta - 1$, $\text{Re } \eta < 0$ which is nearest to the imaginary axis $\text{Re } \eta = 0$.

Proof. Since $\rho_0 > 0$ and $a < 1$, the function $\beta(\eta) + \alpha\eta - 1$ has for $\text{Re } \eta < 0$ a real zero. Denote by ε its real zero nearest to the axis $\text{Re } \eta = 0$. Clearly $\varepsilon > -\rho_0$. From

$$|\beta(\eta)| \leq \beta(\text{Re } \eta) = 1 - \alpha\varepsilon < |1 - \alpha\eta| \quad \text{for } \text{Re } \eta = \varepsilon, \eta \neq \varepsilon,$$

it follows that ε is the only zero with $\text{Re } \eta = \varepsilon$. From

$$|\beta(\eta)| \leq \beta(\text{Re } \eta) < |1 - \alpha\eta| \quad \text{for } \text{Re } \eta > \varepsilon$$

and from Rouché's theorem it is seen that $\beta(\eta) + \alpha\eta - 1$ has only one zero with $\text{Re } \eta > \varepsilon$; this zero is $\eta = 0$. Hence ε is the zero with $\text{Re } \eta < 0$ nearest to the axis $\text{Re } \eta = 0$. Moreover, ε is a single zero, since

$$\beta'(\varepsilon) + \alpha = \beta'(\varepsilon) + \frac{1 - \beta(\varepsilon)}{\varepsilon} = - \sum_{n=1}^{\infty} \int_0^{\infty} (n-1) \frac{(-\varepsilon)^{n-1} t^n}{n!} d\mathbf{B}(t) < 0,$$

the series being convergent. If $\beta(\eta) + \alpha\eta - 1$ has a second zero ε_1 with $\text{Re } \varepsilon_1 < 0$ then $-\rho_0 < \text{Re } \varepsilon_1 < \varepsilon$. Let C_ξ be a line parallel to the imaginary axis with $\text{Re } \varepsilon_1 < \text{Re } \xi < \varepsilon$ if ε_1 exists, otherwise $-\rho_0 < \text{Re } \xi < \varepsilon$. The function $\beta(\eta) + \alpha\eta - 1$ is analytic for $\text{Re } \eta > \text{Re } \xi$ and has single zeros at $\eta = \varepsilon$ and $\eta = 0$. From Cauchy's theorem it follows for

$$\begin{aligned} \text{Re } \eta > 0 > \varepsilon > \text{Re } \xi > \text{Re } \varepsilon_1 > -\rho_0 \\ \frac{\alpha}{2\pi i} \int_{C_n} e^{\eta v} \frac{\eta d\eta}{\beta(\eta) + \alpha\eta - 1} &= \frac{\alpha\varepsilon}{\alpha + \beta'(\varepsilon)} e^{\varepsilon v} \\ &+ \frac{\alpha}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{\xi d\xi}{\beta(\xi) + \alpha\xi - 1}, \quad v > 0, \\ \frac{1}{2\pi i} \int_{C_n} e^{\eta v} \frac{d\eta}{\beta(\eta) + \alpha\eta - 1} &= \frac{1}{\alpha - \beta} + \frac{1}{\alpha + \beta'(\varepsilon)} e^{\varepsilon v} \\ &+ \frac{1}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{d\xi}{\beta(\xi) + \alpha\xi - 1}, \quad v > 0. \end{aligned}$$

It is easily verified that

$$\lim_{v \rightarrow \infty} \frac{e^{\varepsilon v}}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{\xi d\xi}{\beta(\xi) + \alpha\xi - 1} = 0, \quad \lim_{v \rightarrow \infty} \frac{e^{\varepsilon v}}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{d\xi}{\beta(\xi) + \alpha\xi - 1} = 0.$$

Hence from (1.6) we obtain

$$\lim_{v \rightarrow \infty} e^{-\varepsilon v} \{ 1 - \text{Pr} \{ \underline{v}_{\max}^{(j)} < v \} \} = \frac{\alpha - \beta}{\alpha + \beta'(\varepsilon)} \alpha\varepsilon = b_1 > 0.$$

Therefore

$$\Pr \left\{ \frac{1}{\beta} \underline{V}_n < \frac{x + \log (nb_1)}{-\epsilon\beta} \right\} = \left[\Pr \left\{ \underline{L}_{\max}^{(j)} < \frac{x + \log (nb_1)}{-\epsilon\beta} \right\} \right]^n,$$

so that for $n \rightarrow \infty$

$$\Pr \left\{ \frac{1}{\beta} \underline{V}_n < \frac{x + \log (nb_1)}{-\epsilon\beta} \right\} = \left[1 - b_1 e^{-x - \log (nb_1)} + o\left(\frac{1}{n}\right) \right]^n,$$

i. e.

$$\lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\beta} \underline{V}_n < \frac{x + \log (nb_1)}{-\epsilon\beta} \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

This proves the statement for \underline{V}_n , that for \underline{W}_n is proved in the same way. The statement for \underline{X}_n is also analogous. Start from (1.3) and move the path of integration D_ω to a circle with radius $|\omega| > 1$ and such that the first zero of

$$\beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\} - \omega$$

outside the circle $|\omega| = 1$ is an interior point of this circle.

COROLLARY to theorem 2. For $a < 1$ the variables $\frac{1}{\beta} \frac{\underline{V}_n}{\log n}$, $\frac{1}{\beta} \frac{\underline{W}_n}{\log n}$ and $\frac{\underline{X}_n}{\log n}$ converge for $n \rightarrow \infty$ in probability to $-\frac{1}{\epsilon\beta}$, $-\frac{1}{\epsilon\beta}$ and $\frac{1}{\log(1 - \alpha\epsilon)}$, respectively.

Proof. For every fixed $x > 0$ it follows from theorem 2 that for $n \rightarrow \infty$

$$\Pr \left\{ \left| \frac{1}{\beta} \frac{\underline{V}_n}{\log n} + \frac{1}{\epsilon\beta} + \frac{\log b_1}{\epsilon\beta \log n} \right| > \frac{x}{-\epsilon\beta \log n} \right\} \rightarrow \{ e^{-e^x} + 1 - e^{-e^{-x}} \},$$

so that for every $z > 0$,

$$\Pr \left\{ \left| \frac{1}{\beta} \frac{\underline{V}_n}{\log n} + \frac{1}{\epsilon\beta} + \frac{\log b_1}{\epsilon\beta \log n} \right| > \frac{z}{-\epsilon\beta} \right\} \rightarrow e^{-n^z} + 1 - e^{-n^{-z}} \rightarrow 0$$

for $n \rightarrow \infty$,

and hence the statement for \underline{V}_n follows; the other statements are proved similarly.

During a busy cycle a realisation of \underline{v}_t may have a number of intersections with level K . There are no intersections at all if during the busy cycle the virtual delay time is always less than K . Denote by $\Pi_K^{(j)}$ the number of intersections from above with level K of \underline{v}_t in the j th busy cycle, $j = 1, 2, \dots$

Obviously, the variables $\Pi_K^{(j)}$, $j = 1, 2, \dots$, are independent and identically distributed variables. It has been shown in [8] that if $a \leq 1$ then

$$\Pr \{ \Pi_K^{(j)} = m \} = f(0), \quad m = 0, \\ = \{ 1 - f(0) \} \{ 1 - h(0) \}^{m-1} h(0), \quad m = 1, 2, \dots,$$

where

$$f(0) = \Pr \{ v_{\max} < K \}, \\ h(0) = \left[\frac{1}{2\pi i} \int_{c_n} e^{\eta K} \frac{\alpha d\eta}{\beta(\eta) + \alpha\eta - 1} \right]^{-1}, \quad \text{Re } \eta > 0.$$

Denote by E_K the state with K customers left behind in the system at a departure. Let $\Delta_K^{(j)}$ represent the number of times that state E_K occurs during the j th busy cycle. Obviously, $\Delta_K^{(j)}$, $j = 1, 2, \dots$, are independent and identically distributed variables. It has been shown in [9] that if $a \leq 1$ then

$$\Pr \{ \Delta_K^{(j)} = m \} = f(1), \quad m = 0, \\ = \{ 1 - f(1) \} \{ 1 - h(1) \}^{m-1} h(1), \quad m = 1, 2, \dots,$$

where

$$f(1) = \Pr \{ x_{\max} \leq K \}, \\ h(1) = \left[\frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^{K+1}} \frac{\beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\}}{\beta \left\{ \frac{1}{\alpha} (1 - \omega) \right\} - \omega} \right]^{-1}, \quad |\omega| < 1.$$

Define

$$\underline{P}_{K,n} \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} \Pi_K^{(j)}, \quad \underline{L}_{K,n} \stackrel{\text{def}}{=} \max_{1 \leq j \leq n} \Delta_K^{(j)},$$

then we have :

THEOREM 3. — If $a \leq 1$ then

$$\lim_{n \rightarrow \infty} \Pr \left\{ \underline{P}_{K,n} < \frac{x + \log \left\{ n \frac{1 - f(0)}{1 - h(0)} \right\}}{-\log \{ 1 - h(0) \}} \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty, \\ \lim_{n \rightarrow \infty} \Pr \left\{ \underline{L}_{K,n} < \frac{x + \log \left\{ n \frac{1 - f(1)}{1 - h(1)} \right\}}{-\log \{ 1 - h(1) \}} \right\} = e^{-e^{-x}}, \quad -\infty < x < \infty.$$

Proof. It is easily verified that

$$\Pr \{ \Pi_K^{(j)} \geq m \} = \frac{1 - f(0)}{1 - h(0)} \exp \{ m \log (1 - h(0)) \}$$

from which the statement of the theorem follows as in the preceding theorem.

Similarly for $\underline{L}_{K,n}$.

As before we obtain.

COROLLARY to theorem 3. For $a \leq 1$ the variables $\frac{P_{K,n}}{\log n}$ and $\frac{L_{K,n}}{\log n}$ converge for $n \rightarrow \infty$ in probability to $\frac{1}{-\log \{1 - h(0)\}}$ and $\frac{1}{-\log \{1 - h(1)\}}$, respectively.

It is noted that if $B(t) = 1 - e^{-t/\beta}$ for $t > 0$ then $\varepsilon\beta = -(1 - a)$, $b_1 = (1 - a)$, $b_2 = a(1 - a)$, $b_3 = a^{-1}(1 - a)$, $1 - \alpha\varepsilon = a^{-1}$,

$$\begin{aligned} f(0) &= \frac{1 - e^{-(1-a)K/\beta}}{1 - ae^{-(1-a)K/\beta}}, & h(0) &= \frac{1 - a}{1 - ae^{-(1-a)K/\beta}}, & a < 1, \\ &= \frac{K/\beta}{1 + K/\beta}, & &= \frac{1}{1 + K/\beta}, & a = 1, \\ f(1) &= \frac{1 - a^K}{1 - a^{K+1}}, & h(1) &= \frac{1 - a}{1 - a^{K+1}}, & a < 1, \\ &= \frac{K}{1 + K}, & &= \frac{1}{1 + K}, & a = 1. \end{aligned}$$

3. EXTREME VALUE DISTRIBUTIONS FOR G/M/1

Denote by $A(t)$ the distribution function of the interarrival times for the queuing system G/M/1 ;

$$\alpha(\rho) = \int_0^\infty e^{-\rho t} dA(t), \quad \text{Re } \rho \geq 0, \quad A(0+) = 0, \quad \alpha = \int_0^\infty t dA(t) < \infty.$$

For the system G/M/1 the variables v_{\max} , w_{\max} and x_{\max} will have the same meaning as those for the system M/G/1, and similarly for \underline{V}_n , \underline{W}_n and \underline{X}_n .

For $a \leq 1$ we have (cf. Cohen [3], [5], [6]),

$$(3.1) \quad 1 - \Pr \{ v_{\max} < v \} = 0, \quad v < 0, \\ = \left\{ \frac{1}{2\pi i} \int_{C_\xi} e^{\xi v} \frac{\beta d\xi}{\alpha(\xi) + \beta\xi - 1} \right\}^{-1}, \quad \text{Re } \xi > \psi, \quad v > 0,$$

$$(3.2) \quad 1 - \Pr \{ w_{\max} < w \} = 0, \quad w < 0, \\ = \left\{ \frac{1}{2\pi i} \int_{C_\xi} \frac{e^{\xi w}}{1 - \beta\xi} \frac{\beta d\xi}{\alpha(\xi) + \beta\xi - 1} \right\}^{-1}, \quad \frac{1}{\beta} > \text{Re } \xi > \psi, \quad w > 0,$$

$$(3.3) \quad 1 - \Pr \{ x_{\max} \leq x \} \\ = \left\{ \frac{1}{2\pi i} \int_{D_\omega} \frac{d\omega}{\omega^x} \frac{1}{\alpha \left\{ \frac{1}{\beta}(1 - \omega) \right\} - \omega} \right\}^{-1}, \quad |\omega| < \varphi, \quad x = 1, 2, \dots,$$

here Ψ is the larger zero of $\alpha(\xi) + \beta\xi - 1$ with $\text{Re } \xi \geq 0$, and φ is the smaller zero of $\alpha \left\{ \frac{1}{\beta}(1 - \omega) \right\} - \omega$ with $|\omega| \leq 1$. If $a = 1$ then $\psi = 0$, $\varphi = 1$, whereas for $a < 1$ both φ and ψ are positive with multiplicity one. Put

$$\begin{aligned} N(t) &\stackrel{\text{def}}{=} 0, & t < 0, \\ &= \int_0^t \{1 - A(u)\} \frac{du}{\alpha}, & t > 0, \end{aligned}$$

and

$$M(t) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \{N(t)\}^{n*}, \quad t > 0,$$

so that $M(t)$ is the renewal function of the renewal process with $N(t)$ as renewal distribution. As in section 1 (cf. the derivation of (1.10), ..., (1.12)) we have from (3.1), ..., (3.3) for $a = 1$

$$\begin{aligned} 1 - \Pr \{ \underline{v}_{\max} < v \} &= \{ M(v) \}^{-1}, & v > 0, \\ 1 - \Pr \{ \underline{w}_{\max} < w \} &= \left\{ \int_0^{\infty} M(w+t) e^{-t/\beta} \frac{dt}{\beta} \right\}^{-1}, & w > 0, \\ 1 - \Pr \{ \underline{x}_{\max} \leq x \} &= \left\{ \int_0^{\infty} \frac{(t/\beta)^{x-1}}{(x-1)!} e^{-t/\beta} M(t) \frac{dt}{\beta} \right\}^{-1}, & x = 1, 2, \dots \end{aligned}$$

If the second moment α_2 of $A(t)$ is finite then from renewal theory

$$\lim_{t \rightarrow \infty} \frac{M(t)}{t} = \frac{2\alpha}{\alpha_2}.$$

The same argumentation as used in the proof of theorem 1 leads immediately to

THEOREM 4. — If $a = 1$ and $\alpha_2 < \infty$ then the distribution functions of $\frac{2\alpha}{n\alpha_2} \underline{V}_n$, of $\frac{2\alpha}{n\alpha_2} \underline{W}_n$ and of $\frac{2\alpha^2}{n\alpha_2} \underline{X}_n$ all converge to $G(x)$ for $n \rightarrow \infty$.

Further

THEOREM 5. — If $a < 1$ then for $-\infty < x < \infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\beta} \underline{V}_n < \frac{x + \log(nc_1)}{\psi\beta} \right\} &= e^{-e^{-x}}, \\ \lim_{n \rightarrow \infty} \Pr \left\{ \frac{1}{\beta} \underline{W}_n < \frac{x + \log(nc_2)}{\psi\beta} \right\} &= e^{-e^{-x}}, \\ \lim_{n \rightarrow \infty} \Pr \left\{ \underline{X}_n < \frac{x + \log(nc_1)}{-\log(1 - \alpha\psi)} \right\} &= e^{-e^{-x}}, \end{aligned}$$

with

$$c_1 = \frac{\alpha'(\psi) + \beta}{\beta}, \quad c_2 = \frac{\alpha'(\psi) + \beta}{\beta}(1 - \beta\psi),$$

$$\alpha'(\rho) = - \int_0^\infty te^{-\rho t}dA(t), \quad \text{Re } \rho \geq 0.$$

Proof. From (3.1) we have for $\text{Re } \xi > \psi > \text{Re } \eta > 0, v > 0$

$$\frac{1}{2\pi i} \int_{C_\xi} e^{z v} \frac{\beta d\xi}{\alpha(\xi) + \beta\xi - 1} = \frac{\beta e^{\psi v}}{\alpha'(\psi) + \beta} + \frac{1}{2\pi i} \int_{C_\eta} e^{v\eta} \frac{\beta d\eta}{\alpha(\eta) + \beta\eta - 1},$$

so that, since $\alpha(\eta) + \eta\beta - 1$ has no zeros for $0 < \text{Re } \eta < \psi$, it immediately follows from (3.1) that

$$\lim_{v \rightarrow \infty} e^{\psi v} \{ 1 - \Pr \{ \underline{v}_{\max}^{(j)} < v \} \} = \frac{\alpha'(\psi) + \beta}{\beta} = \frac{1}{\beta} \left\{ \alpha'(\psi) + \frac{1 - \alpha(\psi)}{\psi} \right\} > 0.$$

From this relation the statement for \underline{V}_n follows as in the proof of theorem 2. The proof of the statement for \underline{W}_n is similar. To prove the statement for \underline{X}_n move the path of integration D_ω to a circle with radius $|\zeta|$ and such that $\varphi < |\zeta| < 1$, and observe that $\varphi = 1 - \alpha\psi$. The statement for \underline{X}_n is now easily derived.

COROLLARY to theorem 5. For $a < 1$ the variables $\frac{1}{\beta \log n} \underline{V}_n, \frac{1}{\beta \log n} \underline{W}_n$ and $\frac{\underline{X}_n}{\log n}$ converge for $n \rightarrow \infty$ in probability to $\frac{1}{\psi\beta}, \frac{1}{\psi\beta}$ and $\frac{1}{-\log(1 - \alpha\psi)}$, respectively.

The proof is analogous to that of the corollary of theorem 2 in the preceding section.

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