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J. W. COHEN

I. GREENBERG

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## Distribution of crossings of level $K$ in a busy cycle of the $M/G/1$ queue

by

J. W. COHEN and I. GREENBERG (\*)

Technological University, Delft, Holland.

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SUMMARY. — For the  $M/G/1$  queuing system the distribution of the number of crossings from above of a level  $K$  by the virtual delay time during a busy cycle is derived; the busy cycle may be finite or infinite. Also the Laplace-Stieltjes transform of the distribution of the time of the first such crossing (if there is such a crossing) is obtained; similarly for the distribution of the time between two successive crossings from above during a busy cycle, and for the distribution of the time between the last crossing from above and the end of the busy cycle, if there is such a last crossing.

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### 1. INTRODUCTION

Consider a single server queue  $M/G/1$  with traffic intensity  $a$ . The average interarrival time is denoted by  $\alpha$ , so that the average service time is  $\alpha a$ ;  $B(t)$  will represent the distribution function of the service times, with  $B(0+) = 0$ ; further

$$\beta(\rho) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-\rho t} dB(t), \quad \text{Re } \rho \geq 0.$$

The virtual delay time at time  $t$  of the queueing process is denoted by  $v_t$ . In the figure below a realisation of  $v_t$  during a busy cycle  $c$  is shown. This realisation of  $v_t$  has two crossing from above of level  $K$  during the busy cycle  $c$ ; here  $K$  is a positive constant.

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(\*) The second author was on a leave of absence from the Dep. Ind. Eng. and Op. Res. of New York University.

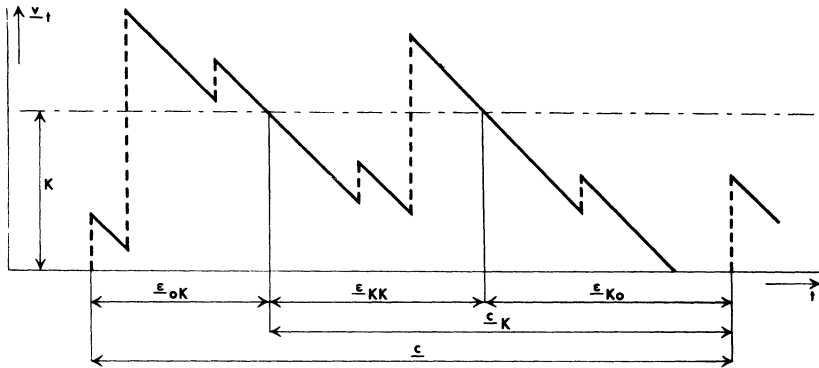


FIG. 1.

In the figure a number of other variables are indicated:  $c_K$  denotes the length of a busy cycle starting with an initial virtual delay time equal to  $K$ ;  $\epsilon_{0K}$  is the time from the start of the busy cycle until the moment of the first crossing from above of level  $K$ , if during the busy cycle  $v_t$  exceeds  $K$ ;  $\epsilon_{KK}$  is the time between the moments of two successive crossings from above of level  $K$  during a busy cycle; finally,  $\epsilon_{K0}$  is the time between the moment of the last crossing from above and the end of the busy cycle, if there is a last crossing from above.

The main purpose of this paper is to study the distribution of  $\Pi_K$ , the number of crossings of level  $K$  from above by  $v_t$  during a busy cycle.

## 2. THE DISTRIBUTIONS OF $\epsilon_{0K}$ , $\epsilon_{KK}$ AND $\epsilon_{K0}$

From the queueing theory of the  $M/G/1$  system it is well known (cf. [1]) that the function of  $\eta$

$$(2.1) \quad \beta(\eta) + (\eta - \rho)\alpha - 1, \quad \text{Re } \rho \geq 0,$$

has one zero  $\delta(\rho)$  with  $\text{Re } \delta(\rho) > 0$  if  $\text{Re } \rho > 0$ , or if  $a > 1$  and  $\text{Re } \rho \geq 0$ ; if  $\rho = 0$  and  $a < 1$  it has a single zero at  $\eta = 0$ , while for  $\rho = 0$ ,  $a = 1$  it has a zero of multiplicity two at  $\eta = 0$ . The zero  $\delta(\rho)$  is a continuous function of  $\rho$  for  $\text{Re } \rho \geq 0$ . Moreover, for  $\text{Re } \rho \geq 0$

$$(2.2) \quad E \{ e^{-\rho \epsilon} \} = \frac{1 + \alpha\rho - \alpha\delta(\rho)}{1 + \alpha\rho},$$

$$(2.3) \quad E \{ e^{-\rho \epsilon_K} \} = \frac{1}{1 + \alpha\rho} e^{-\delta(\rho)K}.$$

The stochastic variables  $\underline{\varepsilon}_{0K}$ ,  $\underline{\varepsilon}_{KK}$  and  $\underline{\varepsilon}_{K0}$  are defined by

$$(2.4) \quad \underline{\varepsilon}_{0K} \stackrel{\text{def}}{=} \inf_{0 < t < \underline{c}} \{ t : \underline{v}_{t-} > K > \underline{v}_{t+} \mid \sup_{0 < \tau < \underline{c}} \underline{v}_{\tau} > K, \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

$\stackrel{\text{def}}{=} \infty$ , if no such finite  $t$  exists for the given conditions;

$$(2.5) \quad \underline{\varepsilon}_{KK} \stackrel{\text{def}}{=} \inf_{0 < t < \underline{c}_K} \{ t : \underline{v}_{t-} > K > \underline{v}_{t+} \mid \sup_{0 < \tau < \underline{c}_K} \underline{v}_{\tau} > K, \underline{v}_{0+} = K \},$$

$\stackrel{\text{def}}{=} \infty$ , if no such finite  $t$  exists for the given conditions;

$$(2.6) \quad \underline{\varepsilon}_{K0} \stackrel{\text{def}}{=} \inf_{0 < t < \underline{c}_K} \{ t : \underline{v}_{t-} = 0 < \underline{v}_{t+} \mid \sup_{0 < \tau < \underline{c}_K} \underline{v}_{\tau} < K, \underline{v}_{0+} = K \}.$$

Define for  $z \geq 0$ ,

$$(2.7) \quad F(z) \stackrel{\text{def}}{=} \Pr \{ \underline{c} < z, \sup_{0 < t < \underline{c}} \underline{v}_t < K \mid \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

$$(2.8) \quad H(z) \stackrel{\text{def}}{=} \Pr \{ \underline{c}_K < z, \sup_{0 < \tau < \underline{c}_K} \underline{v}_{\tau} < K \mid \underline{v}_{0+} = K \},$$

$$(2.9) \quad f(\rho) = \int_0^{\infty} e^{-\rho z} dF(z) \quad , \quad h(\rho) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-\rho z} dH(z), \quad \text{Re } \rho \geq 0.$$

Expressions for  $f(\rho)$  and  $h(\rho)$  have been derived in [2] (cf. (5.20) and (5.21) of [2]). These relations are : for  $\text{Re } \rho \geq 0, \text{Re } \eta > \text{Re } \delta(\rho)$

$$(2.10) \quad f(\rho) = \frac{1}{1 + \alpha\rho} \frac{\frac{1}{2\pi i} \int_{C_{\eta}} e^{\eta K} \frac{\beta(\eta)}{\beta(\eta) + (\eta - \rho)\alpha - 1} d\eta}{\frac{1}{2\pi i} \int_{C_{\eta}} e^{\eta K} \frac{1}{\beta(\eta) + (\eta - \rho)\alpha - 1} d\eta},$$

$$(2.11) \quad h(\rho) = \frac{1}{1 + \alpha\rho} \left[ \frac{1}{2\pi i} \int_{C_{\eta}} e^{\eta K} \frac{\alpha d\eta}{\beta(\eta) + (\eta - \rho)\alpha - 1} \right]^{-1};$$

here the integrals are to be read as

$$\frac{1}{2\pi i} \int_{C_{\eta}} \dots d\eta = \lim_{b \rightarrow \infty} \int_{R-ib}^{R+ib} \dots d\eta \quad , \quad R = \text{Re } \eta.$$

In [2] it has been shown that, if  $\underline{v}_t = u$  at some moment  $t$  with  $0 \leq u \leq K$ , then with probability one the system reaches the empty state in a finite time or passes level  $K$  (from below) in a finite time.

Consequently,

$$(2.12) \quad f(0) = \Pr \{ \underline{c} < \infty, \sup_{0 < \tau < \underline{c}} \underline{v}_{\tau} < K \mid \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

$$1 - f(0) = \Pr \{ \sup_{0 < \tau < \underline{c}} \underline{v}_{\tau} \geq K \mid \underline{v}_{0-} = 0, \underline{v}_{0+} > 0 \},$$

$$(2.13) \quad h(0) = \Pr \{ \underline{c}_K < \infty, \sup_{0 < \tau < \underline{\varepsilon}_K} v_\tau < K \mid v_{0+} = K \},$$

$$1 - h(0) = \Pr \{ \sup_{0 < \tau < \underline{\varepsilon}_K} v_\tau \geq K \mid v_{0+} = K \}.$$

A finite busy cycle which has at least one crossing of level  $K$  is the sum of one  $\underline{\varepsilon}_{0K}$ , of one  $\underline{\varepsilon}_{K0}$  and of a random number of variables  $\underline{\varepsilon}_{KK}$ . Since for the M/G/1 system the interarrival times are all independent, and negative exponentially distributed with the same parameter it follows that every crossing of level  $K$  from above by  $v_t$  is a regeneration point; consequently the variables  $\underline{\varepsilon}_{0K}$ ,  $\underline{\varepsilon}_{KK}$  and  $\underline{\varepsilon}_{K0}$  defined in (2.4), ..., (2.6) are independent variables. This conclusion leads to the following relations: for  $\text{Re } \rho \geq 0$ ,

$$(2.14) \quad E \{ e^{-\rho \underline{\varepsilon}} \} = f(\rho) + (1 - f(0))E \{ e^{-\rho \underline{\varepsilon}_{0K}} \} E \{ e^{-\rho \underline{\varepsilon}_K} \},$$

$$(2.15) \quad E \{ e^{-\rho \underline{\varepsilon}_K} \} = h(\rho) \sum_{m=0}^{\infty} [(1 - h(0))E \{ e^{-\rho \underline{\varepsilon}_{KK}} \}]^m$$

$$= \frac{h(\rho)}{1 - (1 - h(0))E \{ e^{-\rho \underline{\varepsilon}_{KK}} \}}.$$

From (2.14) and (2.15) and from (2.2) and (2.3) we have for  $\text{Re } \rho \geq 0$ ,

$$(2.16) \quad E \{ e^{-\rho \underline{\varepsilon}_{0K}} \} = \{ 1 + \alpha \rho - \alpha \delta(\rho) - (1 + \alpha \rho)f(\rho) \} \frac{e^{K\delta(\rho)}}{1 - h(0)},$$

$$(2.17) \quad E \{ e^{-\rho \underline{\varepsilon}_{KK}} \} = \frac{1}{1 - h(0)} \{ 1 - (1 + \alpha \rho)h(\rho)e^{K\delta(\rho)} \},$$

whereas from (2.6), (2.8) and (2.13)

$$(2.18) \quad E \{ e^{-\rho \underline{\varepsilon}_{K0}} \} = \frac{h(\rho)}{h(0)}.$$

These relations describe the distributions of the variables  $\underline{\varepsilon}_{0K}$ ,  $\underline{\varepsilon}_{KK}$  and  $\underline{\varepsilon}_{K0}$ . It follows

$$(2.19) \quad \Psi_{0K} \stackrel{\text{def}}{=} \Pr \{ \underline{\varepsilon}_{0K} < \infty \} = \{ 1 - \alpha \delta(0) - f(0) \} \frac{e^{K\delta(0)}}{1 - f(0)},$$

$$(2.20) \quad \Psi_{KK} \stackrel{\text{def}}{=} \Pr \{ \underline{\varepsilon}_{KK} < \infty \} = \frac{1}{1 - h(0)} \{ 1 - h(0)e^{K\delta(0)} \}.$$

It is of some interest to consider the relations obtained above for the case  $a \leq 1$ .

If  $a \leq 1$  then  $\delta(0) = 0$  so that

$$(2.21) \quad \Psi_{0K} = 1, \quad \Psi_{KK} = 1.$$

If  $a < 1$  then it is well known that the actual waiting time of the queuing process M/G/1 has a unique stationary distribution  $W(t)$  of which the Laplace-Stieltjes transform is given by the Polaczek-Khinchin formula

$$(2.22) \quad \int_{0-}^{\infty} e^{-\rho t} dW(t) = \frac{(1-a)\alpha\rho}{\beta(\rho) + \alpha\rho - 1}, \quad \text{Re } \rho \geq 0, \quad a < 1.$$

Define two nonnegative stochastic variables  $w$  and  $\tau$  with joint distribution

$$(2.23) \quad \Pr \{ \underline{w} < t_1, \underline{\tau} < t_2 \} = W(t_1)B(t_2), \quad 0 \leq t_1, \quad 0 \leq t_2,$$

so that  $\underline{w}$  and  $\underline{\tau}$  are independent by definition.

Using the inversion formula for the Laplace-Stieltjes transform it follows from (2.7), (2.8), (2.12), (2.13), (2.22) and (2.24)

$$(2.24) \quad f(0) = \frac{\Pr \{ \underline{w} + \underline{\tau} < K \}}{\Pr \{ \underline{w} < K \}}, \quad 1 - f(0) = \frac{\Pr \{ K - \underline{\tau} \leq \underline{w} < K \}}{\Pr \{ \underline{w} < K \}},$$

$$(2.25) \quad h(0) = \frac{\Pr \{ \underline{w} = 0 \}}{\Pr \{ \underline{w} < K \}}, \quad 1 - h(0) = \frac{\Pr \{ 0 < \underline{w} < K \}}{\Pr \{ \underline{w} < K \}}.$$

The relations for  $f(0)$  and  $h(0)$  for all  $a > 0$  have been found also by Takacs [3], who uses combinatorial methods.

### 3. DISTRIBUTION OF CROSSING OF LEVEL K

The distribution of  $\underline{\Pi}_K$ , the number of crossings from above of level K by  $\underline{v}_t$ , can be obtained by making use of the renewal property of these crossing points. It is necessary, however, to distinguish between the finite and infinite busy cycles if  $a > 1$ .

It follows

$$(3.1) \quad \Pr \{ \underline{\Pi}_K = m, \underline{c} < \infty \} = f(0), \quad m=0, \\ = (1-f(0))\Psi_{0K} \{ (1-h(0))\Psi_{KK} \}^{m-1} h(0), \quad m=1, 2, \dots,$$

$$(3.2) \quad \Pr \{ \underline{\Pi}_K = m, \underline{c} = \infty \} = (1-f(0))(1-\Psi_{0K}), \quad m=0, \\ = (1-f(0))\Psi_{0K} \{ (1-h(0))\Psi_{KK} \}^{m-1} (1-h(0))(1-\Psi_{KK}), \quad m=1, 2, \dots$$

From (3.1) and (3.2) it is easily verified that

$$\sum_{m=0}^{\infty} \Pr \{ \underline{\Pi}_K = m \} = 1,$$

so that the number of crossings of level K by  $\underline{v}_t$  during a busy period is finite with probability one.

From (3.1) and (3.2) it is found

$$(3.3) \quad E \{ \underline{\Pi}_K | \underline{c} < \infty \} = \frac{1 - \alpha\delta(0) - f(0)}{h(0)} \frac{e^{-K\delta(0)}}{1 - \alpha\delta(0)},$$

$$\text{Var} \{ \underline{\Pi}_K | \underline{c} < \infty \} = \frac{1 - \alpha\delta(0) - f(0)}{h^2(0)} \frac{e^{-2K\delta(0)}}{1 - \alpha\delta(0)} \left\{ 1 - h(0)e^{K\delta(0)} + \frac{f(0)}{1 - \alpha\delta(0)} \right\},$$

and if  $a > 1$

$$(3.4) \quad E \{ \underline{\Pi}_K | \underline{c} = \infty \} = \frac{1 - \alpha\delta(0) - f(0)}{h(0)} \frac{1 - e^{-K\delta(0)}}{\alpha\delta(0)},$$

$$\text{Var} \{ \underline{\Pi}_K | \underline{c} = \infty \} = \frac{1 - \alpha\delta(0) - f(0)}{h^2(0)} \frac{\{1 - e^{-K\delta(0)}\} e^{-K\delta(0)}}{\alpha\delta(0)}$$

$$\times \left\{ 1 - h(0)e^{K\delta(0)} + \frac{1 - f(0)}{\alpha\delta(0)} - \frac{1 - \alpha\delta(0) - f(0)}{\alpha\delta(0)} e^{K\delta(0)} \right\}.$$

If  $a < 1$  then  $\underline{c}$ ,  $\underline{\varepsilon}_{0K}$  and  $\underline{\varepsilon}_{KK}$  are finite with probability one; in this case it follows from (3.1), (3.3), (2.24) and (2.25)

$$(3.5) \quad \Pr \{ \underline{\Pi}_K = m \} = \frac{\Pr \{ \underline{w} + \underline{\tau} < K \}}{\Pr \{ \underline{w} < K \}}, \quad m = 0,$$

$$= \frac{\Pr \{ K - \underline{\tau} \leq \underline{w} < K \}}{\Pr \{ \underline{w} < K \}} \left[ \frac{\Pr \{ 0 < \underline{w} < K \}}{\Pr \{ \underline{w} < K \}} \right]^{m-1} \frac{\Pr \{ \underline{w} = 0 \}}{\Pr \{ \underline{w} < K \}},$$

$$E \{ \underline{\Pi}_K \} = \frac{1}{1-a} \Pr \{ K - \underline{\tau} \leq \underline{w} < K \}, \quad m = 1, 2, \dots,$$

$$\text{Var} \{ \underline{\Pi}_K \} = \frac{1}{(1-a)^2} \Pr \{ K - \underline{\tau} \leq \underline{w} < K \} [\Pr \{ 0 < \underline{w} < K \}$$

$$+ \Pr \{ \underline{w} + \underline{\tau} < K \}].$$

In [2] the distribution of  $\Phi_K$ , the number of overflows during a wet period (or busy cycle) of an M/G/1 dam with finite capacity K has been derived. It appears that for  $a < 1$ ,  $\Phi_K$  and  $\underline{\Pi}_K$  have the same distribution.

Putting

$$\beta = a\alpha,$$

so that  $\beta$  is the mean service time then for the M/M/1 queueing system the results of this section specialize as follows

$$f(0) = \frac{1 - e^{-(1-a)K/\beta}}{1 - ae^{-(1-a)K/\beta}}, \quad h(0) = \frac{1 - a}{1 - ae^{-(1-a)K/\beta}}, \quad a \neq 1,$$

$$= \frac{K/\beta}{1 + K/\beta}, \quad = \frac{1}{1 + K/\beta}, \quad a = 1,$$

$$\Psi_{0K} = \Psi_{KK} = 1 - \alpha\delta(0) = a^{-1}, \quad a > 1,$$

$$= 1, \quad a \leq 1;$$

for  $a < 1$ ,

$$E \{ \underline{\Pi}_K \} = e^{-(1-a)K/\beta}, \quad \text{Var} \{ \underline{\Pi}_K \} = \frac{1+a}{1-a} e^{-(1-a)K/\beta} \{ 1 - e^{-(1-a)K/\beta} \};$$

for  $a = 1$ ,

$$E \{ \underline{\Pi}_K \} = 1, \quad \text{Var} \{ \underline{\Pi}_K \} = 2K/\beta;$$

for  $a > 1$ ,

$$E \{ \underline{\Pi}_K | \underline{c} < \infty \} = e^{(1-a)K/\beta},$$

$$\text{Var} \{ \underline{\Pi}_K | \underline{c} < \infty \} = \frac{1+a}{1-a} e^{(1-a)K/\beta} \{ e^{(1-a)K/\beta} - 1 \},$$

$$E \{ \underline{\Pi}_K | \underline{c} = \infty \} = \frac{1 - e^{(1-a)K/\beta}}{a - 1},$$

$$\text{Var} \{ \underline{\Pi}_K | \underline{c} = \infty \} = \frac{\{ e^{(1-a)K/\beta} - a \} \{ e^{(1-a)K/\beta} - 1 \}}{(1-a)^2}.$$

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