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Stability estimates of an inverse problem for the stationary transport equation

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ABSTRACT. – For the stationary transport equation on a bounded open convex set Ω, one can study the following inverse problem: recovering the absorption coefficient and the collision kernel from the albedo operator on the boundary. In this work we will consider the question of stability for this inverse problem. © Elsevier, Paris

RÉSUMÉ. – Nous étudions le problème inverse suivant concernant les équations de transport stationnaires sur un domaine ouvert convexe Ω : comment retrouver le coefficient d'absorption et le noyau de collision à partir de l'opérateur d'albedo sur le bord du domaine ? Dans cet article, nous considérons la stabilité de ce problème inverse. © Elsevier, Paris

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In this article we are concerned with stability estimates of an inverse boundary value problem for the stationary linear transport equation. Let $\Omega \subset \mathbb{R}^3$ be a bounded open convex set with C^1 boundary $\partial\Omega$. We assume that V is an open set in $\mathbb{R}^3$ and $V \subseteq \{v \in \mathbb{R}^3\colon\ 0 < \lambda_1 \leqslant |v| \leqslant \lambda_2\}$.

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Additionally, let $\Gamma_\pm = \{(x, v) \in \partial\Omega \times V;\ n(x) \cdot v \gtrless 0\}$, where $n(x)$ is the outer normal to $\partial\Omega$ at $x \in \partial\Omega$. Now consider the following boundary value problem

$$\begin{cases} -v \cdot \nabla_x f(x, v) - \sigma_a(x) f(x, v) + \int_V k(v', v) f(x, v')\, dv' = 0, \\ f|_{\Gamma_-} = f_-. \end{cases} \tag{1.1}$$

Here $\sigma_a(x)$ is called the absorption coefficient and $k(v', v)$ is called the collision kernel. Further, we impose widely used admissible conditions on $\sigma_a(x)$ and $k(v', v)$.

(i) $0 \leqslant \sigma_a(x) \in L^\infty(\Omega)$.

(ii) $0 \leqslant k(v', \cdot) \in L^1(V)$ for a.e. $v' \in V$ and $\int_V k(v', v)\, dv := \sigma_p(v') \in L^\infty(V)$.

To make sure that the boundary value problem (1.1) is uniquely solvable, we will assume that

$$\|\tau\sigma_p\|_{L^\infty} < 1, \tag{1.2}$$

where $\tau = \tau_- + \tau_+$ and $\tau_\pm(x, v) = \min\{t \geqslant 0\colon\ x \pm tv \in \partial\Omega\}$. In fact, besides (1.2) we also need $\|\tau\sigma_a\|_{L^\infty} < \infty$. But this condition is trivially satisfied under admissibility of σ_a and our V. Now given an admissible pair $(\sigma_a(x), k(v', v))$ satisfying (1.2), one can show that (1.1) is uniquely solvable whenever $f_- \in L^1(\Gamma_-, d\xi)$ (see [1]). Here $d\xi = |n(x) \cdot v|\, d\mu(x)\, dv$ is a measure on $\Gamma_\pm$, where $d\mu(x)$ is the Lebesgue measure on $\partial\Omega$. We then define the so-called *albedo* operator $\mathcal{A}$ for (1.1)

$$\mathcal{A}: f_- \to f|_{\Gamma_+}. \tag{1.3}$$

In [1], we can see that $\mathcal{A}$ is a bounded operator from $L^1(\Gamma_-, d\xi)$ to $L^1(\Gamma_+, d\xi)$. Now the inverse problem for (1.1) is to recover $\sigma_a(x)$ and $k(v', v)$ from the knowledge of $\mathcal{A}$. Applying Choulli and Stefanov's results in [1] to our case here, one can show that $\sigma_a(x)$ and $k(v', v)$ are uniquely determined by the albedo operator $\mathcal{A}$. Note that Choulli and Stefanov considered more general σ_a and k in [1] and they also gave a reconstruction procedure. In this work, we are interested in the stability estimate of this inverse problem. One would like to know how $\sigma_a(x)$ and $k(v', v)$ depend on $\mathcal{A}$. This is a very important question from numerical point of view. To state the results, let us first introduce some notations. Hereafter, $\sigma_a(x)$ is replaced by $\sigma(x)$. Also, let $\|\cdot\|_*$ be the operator norm $L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$. Now we are ready to state the results of this article.

THEOREM 1.1. – *Let*

$$\mathcal{M} = \{u(x) \in H^{3/2+\tilde{r}}(\Omega):\ u(x) \geqslant 0,\ \operatorname{supp} u \subseteq \Omega,\ \text{and } \|u\|_{H^{3/2+\tilde{r}}} \leqslant M\}$$

for some $\tilde{r} > 0$ *and* $M > 0$. *Further, let*

$$\mathcal{N} = \{0 \leqslant k(v', v):\ k(v', \cdot) \in L^1(V) \text{ for a.e. } v' \in V,\ k(\cdot, v) \in C^0(V) \text{ for all } v \in V,\ \|\sigma_p\|_{L^\infty} < \infty \text{ and } \|\tau\sigma_p\|_{L^\infty} < 1\}.$$

Assume that $(\sigma_i(x), k_i(v', v)) \in \mathcal{M} \times \mathcal{N}$ *and* $\mathcal{A}_i$ *are the associated aldedo operators,* $i = 1, 2$. *Then for absorption coefficients we have*

$$\|\sigma_1 - \sigma_2\|_{H^{3/2+r}} \leqslant C\|\mathcal{A}_1 - \mathcal{A}_2\|_*^{\theta}, \tag{1.4}$$

where $0 \leqslant r < \tilde{r}$, $\theta = (\tilde{r} - r)/(\tilde{r} + 2)$, *and* $C = C(\Omega, \lambda_1, \lambda_2, r, \tilde{r})$, *while for collision kernels we get*

$$\|k_1 - k_2\|_{L^1(V \times V)} \leqslant \tilde{C}\|\mathcal{A}_1 - \mathcal{A}_2\|_*, \tag{1.5}$$

where $\tilde{C} = \tilde{C}(\Omega, M, \lambda_1, \lambda_2)$.

Note that for the case of pure absorption, i.e., $k = 0$, we can obtain the same estimates (1.4) by some elementary arguments (see Section 3). The interesting point of our result is that the estimate (1.4) is derived when k is nontrivial. And moreover, the constant C in (1.4) turns out to be the same as in the case $k = 0$. Additionally, we are able to obtain the stability estimate for the collision kernel here. This is the most difficult part of the whole business. The main step in the proof of Theorem 1.1 is to choose appropriate approximations of the identity and cut-off functions. The key to this approach is that the kernel of $\mathcal{A}$ has a special structure (see Section 2).

Previously, some stability estimates for inverse problems involving the stationary transport equation have been obtained by Romanov in [4]. He used different observable data from ours. Furthermore, his results required smallness assumptions on the absorption coefficient and the collision kernel. For stability estimates for different inverse problems, the reader is referred to a review article by Isakov [2].

This paper is organized as follows. In Section 2 we will review some results in [1] which are needed in our work. In Section 3 we prove the stability estimate for the pure absorption case, i.e., $k=0$, which is closely related to the stability estimate for the X-ray transform. We include this result here for completeness. Section 4 is devoted to the proof of Theorem 1.1.

2. PRELIMINARIES

Under the assumption that $\sigma(x)$ and $k(v',v)$ are admisible and satisfy (1.2), we obtained that the albedo operator $\mathcal{A}$ is a bounded operator $L^1(\Gamma_-,d\xi)\to L^1(\Gamma_+,d\xi)$. In fact, one can show that the kernel $a(x,v,x',v')$ of $\mathcal{A}$ is of the form $a=\alpha+\beta+\gamma$, where

$$\alpha = \mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma(x-pv)\,dp}\delta_{\{x-\tau_-(x,v)v\}}(x')\delta(v-v'),$$

$$\beta = \int_0^{\tau_-(x,v)} \mathrm{e}^{-\int_0^s\sigma(x-pv)\,dp}\,\mathrm{e}^{-\int_0^{\tau_-(x-sv,v')}\sigma(x-sv-pv')\,dp}k(v',v)$$

$$\times\,\delta_{\{x-sv-\tau_-(x-sv,v')v'\}}(x')\,ds,$$

$$|n(x')\cdot v'|^{-1}\gamma\in L^\infty(\Gamma_-;L^1(\Gamma_+,d\xi)),$$

where $\delta_{\{x'\}}(x)$ is a distribution on $\partial\Omega$ defined by

$$(\delta_{\{x'\}},\phi)=\int\delta_{\{x'\}}(x)\phi(x)d\mu(x)=\phi(x')$$

and δ is the standard delta function in $\mathbb{R}^3$ (see [1] for details).

Next we give two lemmas, which will be needed in many occasions.

LEMMA 2.1. – *Let $f\in L^1(\Omega\times V)$, then*

$$\int_{\Omega\times V} f(x,v)\,dx\,dv=\int_{\Gamma_\mp}\int_0^{\tau_\pm(x',v)} f(x'\pm tv,v)\,dt\,d\xi(x',v).$$

Proof. – See [1]. □

LEMMA 2.2. – *Assume that $f_-(x',v)\in L^1(\Gamma_-,d\xi)$. Then*

$$\int_{\Gamma_+} f_-(x-\tau_-(x,v)v,v)\,d\xi(x,v)=\int_{\Gamma_-} f_-(x',v)\,d\xi(x',v). \tag{2.1}$$

Proof. – Note that

$$\int_{\Gamma_+} f_-\big(x - \tau_-(x,v), v\big)\, d\xi(x,v)$$
$$= \int_V \int_{\{x:\, n(x)\cdot v>0\}} f_-\big(x - \tau_-(x,v)v, v\big)\big(n(x)\cdot v\big)\, d\mu(x)\, dv.$$

It suffices to show that

$$\int_{\{x:\, n(x)\cdot v>0\}} f_-\big(x - \tau_-(x,v)v, v\big)\big(n(x)\cdot v\big)\, d\mu(x)$$
$$= \int_{\{x':\, n(x')\cdot v<0\}} f_-(x', v)\big|n(x')\cdot v\big|\, d\mu(x'),$$

for any fixed $v \in V$. Let us denote $f(x,v) = f_-(x - \tau_-(x,v)v, v)$ for all $x \in \overline{\Omega}$. Then one can see that $(v\cdot\nabla_x) f(x,v) = 0$ a.e. Therefore, we have

$$0 = \int_\Omega (v\cdot\nabla_x) f(x,v)\, dx = \int_{\partial\Omega} f(x,v)\big(n(x)\cdot v\big)\, d\mu(x)$$
$$= \int_{\{x:\, n(x)\cdot v>0\}} f_-\big(x - \tau_-(x,v)v, v\big)\big(n(x)\cdot v\big)\, d\mu(x)$$
$$+ \int_{\{x':\, n(x')\cdot v<0\}} f_-(x', v)\big(n(x')\cdot v\big)\, d\mu(x'). \tag{2.2}$$

Now the formula (2.1) follows from (2.2). □

3. THE PURE ABSORPTION CASE

In this section, we will prove the stability estimate for $\sigma(x)$ in terms of $\mathcal{A}$ when the collision kernel k is zero.

THEOREM 3.1. – *Assume that the collision kernel* $k = 0$ *in* (1.1). *Let* $\mathcal{M}$ *be defined as in Theorem* 1.1. *Then for* $\sigma_i(x) \in \mathcal{M}$ *and their associated albedo operators* $\mathcal{A}_i$, $i = 1, 2$, *we have the estimate* (1.4).

Proof. – Consider

$$\begin{cases} -v\cdot\nabla_x f(x,v) - \sigma(x) f(x,v) = 0, & (x,v) \in \Omega\times V, \\ f|_{\Gamma_-} = f_-(x', v) \in L^1(\Gamma_-, d\xi). \end{cases} \tag{3.1}$$

It is readily seen that

$$f(x,v)=\mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma(x-pv)\,dp}f_-(x-\tau_-(x,v)v,v)$$

is the unique solution to (3.1). Let $\sigma_1(x)$ and $\sigma_2(x)$ be any two functions in $\mathcal{M}$. Also, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the associated albedo operators. Therefore, for any $f_-\in L^1(\Gamma_-,d\xi)$ we have

$$\begin{aligned}\|(\mathcal{A}_1-\mathcal{A}_2)f_-\|_{L^1(\Gamma_+,d\xi)}&=\int_{\Gamma_+}\left|\mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma_1(x-pv)dp}-\mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma_2(x-pv)\,dp}\right|\\&\quad\times\left|f_-(x-\tau_-(x,v)v,v)\right|d\xi(x,v)\\&=\int_{\Gamma_+}\mathrm{e}^{\eta(x,v)}\left|\int_0^{\tau_-(x,v)}(\sigma_1-\sigma_2)(x-pv)\,dp\right|\\&\quad\times\left|f_-(x-\tau_-(x,v)v,v)\right|d\xi(x,v),\end{aligned}\tag{3.2}$$

where $\eta(x,v)$ is some point between

$$\int_0^{\tau_-(x,v)}\sigma_1(x-pv)\,dp\quad\text{and}\quad\int_0^{\tau_-(x,v)}\sigma_2(x-pv)\,dp.$$

Now since $\sigma_i\in\mathcal{M},\ i=1,2$, we get that

$$\eta(x,v)\geqslant-Md/\lambda_1,\tag{3.3}$$

where $d=\operatorname{diam}\Omega$. By substituting (3.3) into (3.2) we have that

$$\begin{aligned}\left\|(\mathcal{A}_1-\mathcal{A}_2)f_-\right\|_{L^1(\Gamma_+,d\xi)}&\geqslant C_1\int_{\Gamma_+}\left|\int_0^{\tau_-(x,v)}(\sigma_1-\sigma_2)(x-pv)\,dp\right|\\&\quad\times\left|f_-(x-\tau_-(x,v)v,v)\right|d\xi(x,v),\end{aligned}\tag{3.4}$$

where $C_1=\mathrm{e}^{-Md/\lambda_1}$. Now since $\operatorname{supp}\sigma_i(x)\in\Omega$, we can see that

$$\int_0^{\tau_-(x,v)}(\sigma_1-\sigma_2)(x-pv)\,dp=\frac{1}{|v|}X(\sigma_1-\sigma_2)\left(x,\frac{v}{|v|}\right),\tag{3.5}$$

where X means the X-ray transform, i.e.,

$$(Xg)(x,\omega)=\int_{-\infty}^{\infty} g(x+s\omega)\,ds, \quad \forall \omega\in \mathbf{S}^2.$$

Note that the value of $Xg(x,\omega)$ remains the same if we move x along the direction ω. Therefore, we have

$$X(\sigma_1-\sigma_2)\left(x,\frac{v}{|v|}\right)=X(\sigma_1-\sigma_2)\left(x-\tau_-(x,v)v,\frac{v}{|v|}\right).$$

Now we can obtain that

$$\begin{aligned}
&\int_{\Gamma_+}\frac{1}{|v|}\left|X(\sigma_1-\sigma_2)\left(x,\frac{v}{|v|}\right)\right|\left|f_-(x-\tau_-(x,v)v,v)\right|d\xi(x,v)\\
&\quad=\int_{\Gamma_+}\frac{1}{|v|}\left|X(\sigma_1-\sigma_2)\left(x-\tau_-(x,v)v,\frac{v}{|v|}\right)\right|\left|f_-(x-\tau_-(x,v)v,v)\right|d\xi\\
&\quad=\int_{\Gamma_-}\frac{1}{|v|}\left|X(\sigma_1-\sigma_2)\left(x',\frac{v}{|v|}\right)\right|\left|f_-(x',v)\right|d\xi(x',v),
\end{aligned}\tag{3.6}$$

where the second equality follows from Lemma 2.2. Combining (3.4) and (3.6), we have

$$\left\|(\mathcal{A}_1-\mathcal{A}_2)f_-\right\|_{L^1(\Gamma_+,d\xi)}\geqslant C_2\left\|X(\sigma_1-\sigma_2)f_-\right\|_{L^1(\Gamma_-,d\xi)},\tag{3.7}$$

where $C_2=C_1/\lambda_2$. The inequality (3.7) now implies that

$$\|\mathcal{A}_1-\mathcal{A}_2\|_*\geqslant C_2\sup_{\|f_-\|_{L^1(\Gamma_-,d\xi)}\leqslant 1}\left\|X(\sigma_1-\sigma_2)f_-\right\|_{L^1(\Gamma_-,d\xi)}.\tag{3.8}$$

Next it is not hard to show that

$$\sup_{\|f_-\|_{L^1(\Gamma_-,d\xi)}\leqslant 1}\left\|X(\sigma_1-\sigma_2)f_-\right\|_{L^1(\Gamma_-,d\xi)}=\left\|X(\sigma_1-\sigma_2)\right\|_{L^\infty(\Gamma\mathbf{S}^2_-)},\tag{3.9}$$

where $\Gamma\mathbf{S}^2_-=\{(x',\omega)\in\partial\Omega\times\mathbf{S}^2\colon\ n(x')\cdot\omega<0\}$. Further, we can see that

$$\left\|X(\sigma_1-\sigma_2)\right\|_{L^2(T\mathbf{S}^2)}\leqslant C_3\left\|X(\sigma_1-\sigma_2)\right\|_{L^\infty(\Gamma\mathbf{S}^2_-)},\tag{3.10}$$

where

$$\|Xg\|^2_{L^2(TS^2)} = \int_{S^2} \|Xg(\cdot,\omega)\|^2_{L^2(\omega^\perp)}\, d\omega.$$

For the X-ray transform, we have the following estimate (see [3])

$$\|g\|_{H^{-1/2}} \leqslant C_4 \|Xg\|_{L^2(TS^2)}, \tag{3.11}$$

for $g \in H^{-1/2}(\Omega)$ and supp $g \subseteq \Omega$. Combining (3.8)–(3.11), we get that

$$\|\sigma_1 - \sigma_2\|_{H^{-1/2}} \leqslant C_5 \|\mathcal{A}_1 - \mathcal{A}_2\|_*. \tag{3.12}$$

Now by the interpolation formula, we have

$$\begin{aligned} \|\sigma_1 - \sigma_2\|_{H^{3/2+r}} &\leqslant C_6 \|\sigma_1 - \sigma_2\|^{1-\theta}_{H^{3/2+\tilde{r}}} \|\sigma_1 - \sigma_2\|^{\theta}_{H^{-1/2}} \\ &\leqslant C_7 \|\sigma_1 - \sigma_2\|^{\theta}_{H^{-1/2}}, \end{aligned} \tag{3.13}$$

where $0 \leqslant r < \tilde{r}$ and $\theta = (\tilde{r} - r)/(\tilde{r} + 2)$. By using (3.12) and (3.13), we immediately obtain the estimate (1.4) and Theorem 3.1. □

4. PROOF OF THEOREM 1.1

We will divide the proof into two parts. One is for σ's and the other is for k's.

Part 1. – To begin with, let us choose

$$0 \leqslant \psi \in C_0^\infty(\mathbb{R}^n), \quad \int \psi(x)\, dx = 1, \quad \text{and} \quad \psi(0) = 1.$$

Next let $(x_0', v_0') \in \Gamma_-$, i.e., $n(x_0') \cdot v_0' < 0$, be fixed. We now define

$$\psi^\varepsilon_{v_0'}(v') = \frac{1}{\varepsilon^n} \psi\left(\frac{v' - v_0'}{\varepsilon}\right).$$

Now we choose $\varepsilon > 0$ sufficiently small so that

$$Q^\varepsilon(x_0') \times \operatorname{supp} \psi^\varepsilon_{v_0'}(v') \subseteq \Gamma_-,$$

where $Q^\varepsilon(x_0') := B^\varepsilon(x_0') \cap \partial\Omega$, where $B^\varepsilon(x_0')$ is the ε-ball in $\mathbb{R}^n$ centered at x_0'. Next let $\phi^\varepsilon_{x_0'}(x')$ be such that:

- $\operatorname{supp} \phi^\varepsilon_{x_0'}(x') \subseteq Q^\varepsilon(x_0')$;

- $\int_{Q^\varepsilon(x_0')} \phi^\varepsilon_{x_0'}(x')\,d\mu(x') = 1$ and

$$\lim_{\varepsilon\to 0} \int_{\partial\Omega} f(x')\phi^\varepsilon_{x_0'}(x')\,d\mu(x') = f(x_0'), \quad \text{for all } f(x') \in C^0(\partial\Omega).$$

To see that $\phi^\varepsilon_{x_0'}(x')$ do exist, let us argue in the following way. Let O^ε be an open set in $\mathbb{R}^2$ such that $\rho: O^\varepsilon \to Q^\varepsilon(x_0')$ is a coordinate function of $\partial\Omega$ near x_0' and $D(y) := \rho_{y_1} \times \rho_{y_2} \neq 0$, $\forall y = (y_1, y_2) \in O^\varepsilon$. For simplicity, we also take $\rho(0) = x_0'$. Assume that $0 \leqslant w(y) \in C_0^\infty(\mathbb{R}^2)$, $\operatorname{supp} w(y/\varepsilon) \subseteq O^\varepsilon$ and satisfies $\int w(y)\,dy = 1$, $w(0) = 1$. Now let

$$\phi^\varepsilon_{x_0'}(x') = \frac{1}{\varepsilon^2 |D(\rho^{-1}(x'))|} w\left(\frac{\rho^{-1}(x')}{\varepsilon}\right).$$

Then we have $\phi^\varepsilon_{x_0'}(x') \geqslant 0$ and

$$\int_{Q^\varepsilon(x_0')} \phi^\varepsilon_{x_0'}(x')\,d\mu(x') = \int_{O^\varepsilon} \frac{1}{\varepsilon^2|D(y)|} w\left(\frac{y}{\varepsilon}\right)|D(y)|\,dy$$
$$= \frac{1}{\varepsilon^2}\int_{O^\varepsilon} w\left(\frac{y}{\varepsilon}\right)dy = 1,$$

and

$$\lim_{\varepsilon\to 0} \int_{Q^\varepsilon(x_0')} f(x')\phi^\varepsilon_{x_0'}(x')\,d\mu(x')$$
$$= \lim_{\varepsilon\to 0} \int_{O^\varepsilon} f(\rho(y)) \frac{1}{\varepsilon^2|D(y)|} w\left(\frac{y}{\varepsilon}\right)|D(y)|\,dy$$
$$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\int f(\rho(y))\, w\left(\frac{y}{\varepsilon}\right)dy. \tag{4.1}$$

Since ρ is at least a homeomorphism, we get that $f(\rho(y))$ is a continuous function. Therefore, we obtain that the limit in (4.1) is equal to $f(\rho(0)) = f(x_0')$. Finally, we define

$$f^\varepsilon_{x_0',v_0'}(x', v') = \frac{1}{|n(x')\cdot v'|}\phi^\varepsilon_{x_0'}(x')\psi^\varepsilon_{v_0'}(v').$$

Now we need to choose an appropriate cut-off function. Let

$$\tilde{\chi}^\varepsilon_{x_0',v_0'}(x', v') = \tilde{\phi}^\varepsilon_{x_0'}(x')\tilde{\psi}^\varepsilon_{v_0'}(v'),$$

where

$$\widetilde{\phi}^{\varepsilon}_{x_0'}(x') = \begin{cases} 1, & x' \in Q^{\varepsilon}(x_0'), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\widetilde{\psi}^{\varepsilon}_{v_0'}(v') = \begin{cases} 1, & v' \in \operatorname{supp} \psi^{\varepsilon}_{v_0'}, \\ 0, & \text{otherwise}. \end{cases}$$

Now let $f_- = f^{\varepsilon}_{x_0', v_0'}$ then we have

$$(\mathcal{A}_1 - \mathcal{A}_2) f^{\varepsilon}_{x_0', v_0'}(x, v) \tag{4.2}$$
$$= \int_{\Gamma_-} \left(a_1(x, v, x', v') - a_2(x, v, x', v')\right) f^{\varepsilon}_{x_0', v_0'}(x', v')\, d\mu(x')\, dv'.$$

Further, let us set $\chi^{\varepsilon}(x, v) = \widetilde{\chi}^{\varepsilon}_{x_0', v_0'}(x - \tau_-(x, v)v, v)$ for $(x, v) \in \Gamma_+$. Our plan here is to multiply $\chi^{\varepsilon}(x, v)$ on both sides of (4.2) and then integrate the new formula over Γ_+. We first deal with the right-hand side of (4.2) and prove the following two claims.

CLAIM 4.1. –

$$\lim_{\varepsilon \to 0} \int_{\Gamma_+} \chi^{\varepsilon}(x, v) \int_{\Gamma_-} (\alpha_1 - \alpha_2)(x, v, x', v') f^{\varepsilon}_{x_0', v_0'}(x', v')\, d\mu(x')\, dv'\, d\xi(x, v)$$
$$= e^{-\int_0^{\tau_+(x_0', v_0')} \sigma_1(x_0' + p v_0')\, dp} - e^{-\int_0^{\tau_+(x_0', v_0')} \sigma_2(x_0' + p v_0')\, dp}. \tag{4.3}$$

Proof. – Note that

$$\int_{\Gamma_-} (\alpha_1 - \alpha_2)(x, v, x', v') f^{\varepsilon}_{x_0', v_0'}(x', v')\, d\mu(x')\, dv'$$
$$= \left(e^{-\int_0^{\tau_-(x, v)} \sigma_1(x - pv)\, dp} - e^{-\int_0^{\tau_-(x, v)} \sigma_2(x - pv)\, dp} \right)$$
$$\times f^{\varepsilon}_{x_0', v_0'}\left(x - \tau_-(x, v)v, v\right). \tag{4.4}$$

Next we observe that

$$\int_0^{\tau_-(x, v)} \sigma(x - pv)\, dp = \int_0^{\tau_+(x - \tau_-(x, v)v, v)} \sigma\left(x - \tau_-(x, v)v + pv\right) dp. \tag{4.5}$$

Let us denote

$$R(\sigma_1 - \sigma_2)(x', v') = e^{-\int_0^{\tau_+(x', v')} \sigma_1(x' + pv')\, dp} - e^{-\int_0^{\tau_+(x', v')} \sigma_2(x' + pv')\, dp}.$$

Then combining (4.4) and (4.5), we have that

$$\int_{\Gamma_+} \chi^\varepsilon(x,v) \int_{\Gamma_-} (\alpha_1-\alpha_2)(x,v,x',v') f^\varepsilon_{x_0',v_0'}(x',v')\,d\mu(x')\,dv'\,d\xi(x,v)$$

$$= \int_{\Gamma_+} \widetilde{\chi}^\varepsilon_{x_0',v_0'}\big(x-\tau_-(x,v)v,v\big) R(\sigma_1-\sigma_2)\big(x-\tau_-(x,v)v,v\big)$$

$$\times f^\varepsilon_{x_0',v_0'}\big(x-\tau_-(x,v)v,v\big)\,d\xi(x,v)$$

$$= \int_{\Gamma_-} \widetilde{\chi}^\varepsilon_{x_0',v_0'}(x',v) R(\sigma_1-\sigma_2)(x',v) f^\varepsilon_{x_0',v_0'}(x',v)\,d\xi(x',v)$$

$$= \int_{\Gamma_-} R(\sigma_1-\sigma_2)(x',v)\phi^\varepsilon_{x_0'}(x')\psi^\varepsilon_{v_0'}(v)\,d\mu(x')\,dv. \tag{4.6}$$

The second equality in (4.6) follows from Lemma 2.2 and the third equality is a consequence of choices of $\widetilde{\chi}^\varepsilon_{x_0',v_0'}$ and $f^\varepsilon_{x_0',v_0'}$. Since $R(\sigma_1-\sigma_2)(x',v)$ is continuous in (x',v), we get (4.3) by taking $\varepsilon\to 0$. □

CLAIM 4.2. –

$$\lim_{\varepsilon\to 0}\int_{\Gamma_+} \chi^\varepsilon(x,v)\int_{\Gamma_-}(\beta_1-\beta_2)(x,v,x',v')$$

$$\times f^\varepsilon_{x_0',v_0'}(x',v')\,d\mu(x')\,dv'\,d\xi(x,v)=0. \tag{4.7}$$

$$\lim_{\varepsilon\to 0}\int_{\Gamma_+} \chi^\varepsilon(x,v)\int_{\Gamma_-}(\gamma_1-\gamma_2)(x,v,x',v')$$

$$\times f^\varepsilon_{x_0',v_0'}(x',v')\,d\mu(x')\,dv'\,d\xi(x,v)=0. \tag{4.8}$$

Proof. – We first prove (4.7). Let us denote

$$E_i(s,x,v',v) = \mathrm{e}^{-\int_0^s \sigma_i(x-pv)\,dp}\,\mathrm{e}^{-\int_0^{\tau_-(x-sv,v')}\sigma_i(x-sv-pv')\,dp},$$
$$i=1,2. \tag{4.9}$$

Using the fact that $\sigma_i\in\mathcal{M}$ and Fubini's theorem, we have that

$$\left|\int_{\Gamma_+}\chi^\varepsilon(x,v)\int_{\Gamma_-}(\beta_1-\beta_2) f^\varepsilon_{x_0',v_0'}\,d\mu(x')\,dv'\,d\xi(x,v)\right|$$

$$\leqslant \int_{\Gamma_+}\chi^\varepsilon(x,v)\int_{\Gamma_-}\int_0^{\tau_-(x,v)} |E_1k_1-E_2k_2|(s,x,v',v)$$

$$\times\, \delta_{\{x-sv-\tau_-(x-sv,v')v'\}}(x')\, f^{\varepsilon}_{x'_0,v'_0}\, ds\, d\mu\, dv'\, d\xi$$

$$\leqslant C_8 \int_{\Gamma_+} \chi^{\varepsilon}(x,v) \int_V \int_0^{\tau_-(x,v)} (k_1+k_2)(v',v)$$

$$\times\, f^{\varepsilon}_{x'_0,v'_0}(x-sv-\tau_-(x-sv,v')v',v')\, ds\, dv'\, d\xi$$

$$\leqslant C_8 \int_V \int_{\Gamma_+} \widetilde{\psi}^{\varepsilon}_{v'_0}(v) \int_0^{\tau_-(x,v)} (k_1+k_2)(v',v)$$

$$\times\, f^{\varepsilon}_{x'_0,v'_0}(x-sv-\tau_-(x-sv,v')v',v')\, ds\, d\xi\, dv'$$

$$:= C_8 T. \tag{4.10}$$

Now by Lemma 2.1, we can see that

$$T = \int_V \int_{\Omega\times V} \widetilde{\psi}^{\varepsilon}_{v'_0}(v)(k_1+k_2)(v',v) f^{\varepsilon}_{x'_0,v'_0}(x-\tau_-(x,v')v',v')\, dx\, dv\, dv'$$

$$= \int_{\Omega\times V} f^{\varepsilon}_{x'_0,v'_0}(x-\tau_-(x,v')v') \int_V \widetilde{\psi}^{\varepsilon}_{v'_0}(v)(k_1+k_2)(v',v)\, dv\, dx\, dv'. \tag{4.11}$$

Applying Lemma 2.1 to (4.11) and from $k_i \in \mathcal{N}$, $i=1,2$, we get that

$$T = \int_{\Gamma_-} \int_0^{\tau_+(x',v')} f^{\varepsilon}_{x'_0,v'_0}(x',v') \int_V \widetilde{\psi}^{\varepsilon}_{v'_0}(v)(k_1+k_2)(v',v)\, dv\, dt\, d\xi(x',v')$$

$$\leqslant \frac{d}{\lambda_1} \sup_{v'\in V} \int_V \widetilde{\psi}^{\varepsilon}_{v'_0}(v)(k_1+k_2)(v',v)\, dv \int_{\Gamma_-} f^{\varepsilon}_{x'_0,v'_0}(x',v')\, d\xi(x',v')$$

$$\leqslant \frac{d}{\lambda_1} \sup_{v'\in V} \int_{v\in \operatorname{supp}\psi^{\varepsilon}_{v'_0}} \widetilde{\psi}^{\varepsilon}_{v'_0}(v)(k_1+k_2)(v',v)\, dv. \tag{4.12}$$

Now combining (4.10), (4.12) and taking $\varepsilon \to 0$, we have (4.7).

Next, we want to establish (4.8). Note that

$$\left| \int_{\Gamma_+} \chi^{\varepsilon}(x,v) \int_{\Gamma_-} (\gamma_1-\gamma_2)(x,v,x',v') f^{\varepsilon}_{x'_0,v'_0}(x',v')\, d\mu(x')\, dv'\, d\xi(x,v) \right|$$

$$\leqslant \int_{\Gamma_+} \chi^{\varepsilon}(x,v) \int_{\Gamma_-} |n(x')\cdot v'|^{-1} |\gamma_1-\gamma_2|(x,v,x',v')$$

$$\times \phi^{\varepsilon}_{x'_0}(x')\psi^{\varepsilon}_{v'_0}(v')\,d\mu(x')\,dv'\,d\xi(x,v) := \widetilde{T}. \tag{4.13}$$

Since $|n(x')\cdot v'|^{-1}\gamma_i(x,v,x',v') \in L^{\infty}(\Gamma_-; L^1(\Gamma_+, d\xi))$ for $i = 1, 2$, we obtain that

$$\begin{aligned}
\widetilde{T} &\leqslant \sup_{(x',v')\in\Gamma_-} \int_{\Gamma_+} \chi^{\varepsilon}(x,v)\left|n(x')\cdot v'\right|^{-1}|\gamma_1-\gamma_2|(x,v,x',v')\,d\xi(x,v) \\
&\quad \times \int_{\Gamma_-} \phi^{\varepsilon}_{x'_0}(x')\psi^{\varepsilon}_{v'_0}(v')\,d\mu(x')\,dv' \\
&= \sup_{(x',v')\in\Gamma_-} \int_{\Gamma_+} \chi^{\varepsilon}(x,v)\left|n(x')\cdot v'\right|^{-1} \\
&\quad \times |\gamma_1-\gamma_2|(x,v,x',v')\,d\xi(x,v).
\end{aligned} \tag{4.14}$$

Now in view of (4.13) and taking $\varepsilon \to 0$ in (4.14), we immediately get (4.8). □

After establishing Claims 4.1 and 4.2, we are in a position to derive the estimate (1.4) in Theorem 1.1. Let $\chi^{\varepsilon}(x,v)$ and $f^{\varepsilon}_{x'_0,v'_0}(x',v')$ be defined as above. Then

$$\begin{aligned}
&\left\|\chi^{\varepsilon}(\mathcal{A}_1-\mathcal{A}_2)f^{\varepsilon}_{x'_0,v'_0}\right\|_{L^1(\Gamma_+,d\xi)} \\
&= \left|\int_{\Gamma_+}\chi^{\varepsilon}\int_{\Gamma_-}(\alpha_1-\alpha_2)f^{\varepsilon}_{x'_0,v'_0}\,d\mu(x')\,dv'\,d\xi \right. \\
&\quad + \int_{\Gamma_+}\chi^{\varepsilon}\int_{\Gamma_-}(\beta_1-\beta_2)f^{\varepsilon}_{x'_0,v'_0}\,d\mu(x')\,dv'\,d\xi \\
&\quad \left. + \int_{\Gamma_+}\chi^{\varepsilon}\int_{\Gamma_-}(\gamma_1-\gamma_2)f^{\varepsilon}_{x'_0,v'_0}\,d\mu(x')\,dv'\,d\xi\right|.
\end{aligned} \tag{4.15}$$

For the left-hand side of (4.15), we have

$$\begin{aligned}
\left\|\chi^{\varepsilon}(\mathcal{A}_1-\mathcal{A}_2)f^{\varepsilon}_{x'_0,v'_0}\right\|_{L^1(\Gamma_+,d\xi)} &\leqslant \left\|(\mathcal{A}_1-\mathcal{A}_2)f^{\varepsilon}_{x'_0,v'_0}\right\|_{L^1(\Gamma_+,d\xi)} \\
&\leqslant \|\mathcal{A}_1-\mathcal{A}_2\|_*\left\|f^{\varepsilon}_{x'_0,v'_0}\right\|_{L^1(\Gamma_-,d\xi)} \\
&= \|\mathcal{A}_1-\mathcal{A}_2\|_*.
\end{aligned} \tag{4.16}$$

Now taking $\varepsilon \to 0$ on the right-hand side of (4.15) and using Claim 4.1, Claim 4.2, and (4.16), we obtain that

$$\left|\mathrm{e}^{-\int_0^{\tau_+(x'_0,v'_0)}\sigma_1(x'_0+pv'_0)\,dp} - \mathrm{e}^{-\int_0^{\tau_+(x'_0,v'_0)}\sigma_2(x'_0+pv'_0)\,dp}\right| \leqslant \|\mathcal{A}_1-\mathcal{A}_2\|_*, \tag{4.17}$$

for any $(x_0', v_0') \in \Gamma_-$. Note that

$$\left| e^{-\int_0^{\tau_+(x_0',v_0')} \sigma_1(x_0'+pv_0')\,dp} - e^{-\int_0^{\tau_+(x_0',v_0')} \sigma_2(x_0'+pv_0')\,dp} \right|$$
$$\geqslant C_9 \left| \int_0^{\tau_+(x_0',v_0')} (\sigma_1 - \sigma_2)(x_0' + pv_0')\,dp \right|. \tag{4.18}$$

Combining (4.17), (4.18), and the relation (cf. (3.5))

$$\int_0^{\tau_+(x_0',v_0')} g(x_0' + pv_0')\,dp = \frac{1}{|v_0'|} Xg\left(x_0', \frac{v_0'}{|v_0'|}\right),$$

we have

$$\|X(\sigma_1 - \sigma_2)\|_{L^\infty(\Omega \times \mathbb{S}^2)} \leqslant C_{10} \|\mathcal{A}_1 - \mathcal{A}_2\|_*,$$

i.e.,

$$\|X(\sigma_1 - \sigma_2)\|_{L^\infty(\Gamma \mathbb{S}^2_-)} \leqslant C_{10} \|\mathcal{A}_1 - \mathcal{A}_2\|_*. \tag{4.19}$$

Having established (4.19), the remaining arguments of the proof are the same as in the proof of Theorem 3.1. The proof of the estimate (1.4) is now complete.

Part 2. – Here we will prove the estimate (1.5) in Theorem 1.1. As before, we want to choose appropriate approximations of the identity and cut-off functions. Let $(x_0', v_0') \in \Gamma_-$, i.e., $n(x_0') \cdot v_0' < 0$. Assume that $\psi_{v_0'}^{\varepsilon}(v')$ is replaced by $\psi_{v_0'}^{\varepsilon_1}(v')$ and $\phi_{x_0'}^{\varepsilon}(x')$ is replaced by $\phi_{x_0'}^{\varepsilon_2}(x')$, where ε_1 and ε_2 are chosen sufficiently small so that $\psi_{v_0'}^{\varepsilon_1}$ and $\phi_{x_0'}^{\varepsilon_2}$ satisfy all conditions for $\psi_{v_0'}^{\varepsilon}$ and $\phi_{x_0'}^{\varepsilon}$. Note that here we will take $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ in a different order. Also, the following condition is imposed

$$Q^{\varepsilon_2}(x_0') \times \operatorname{supp} \psi_{v_0'}^{\varepsilon_1} \subset \Gamma_-.$$

Now let P be a plane in $\mathbb{R}^3$ containing the vector v_0'. This is a plane in the velocity space. Let us denote $P_{v_0'} = P \cap V$. Assume that m is the normal vector to P. Let $S_{x'}$ be the intersection of $\partial\Omega$ and the plane having the normal m and passing $x' \in \operatorname{supp} \phi_{x_0'}^{\varepsilon_2}(x')$. Next we denote

$$\Pi_{\varepsilon_2} = \bigcup_{x' \in \operatorname{supp} \phi_{x_0'}^{\varepsilon_2}} S_{x'}.$$

Then Π_{ε_2} forms a band on $\partial\Omega$ since Ω is convex. We now define a cut-off function on $\partial\Omega$

$$\chi^{\varepsilon_2}(x)=\begin{cases}1, & \text{on } \Pi_{\varepsilon_2},\\ 0, & \text{otherwise.}\end{cases}$$

Further, let $\widetilde{\mathcal{V}}^{\varepsilon_3}$ (respectively $-\widetilde{\mathcal{V}}^{\varepsilon_3}$) be a closed ε_3-conical neighborhood of v_0' (respectively $-v_0'$) on $P_{v_0'}$, where ε_3 is the degree of the angle for $\widetilde{\mathcal{V}}^{\varepsilon_3}$ (also $-\widetilde{\mathcal{V}}^{\varepsilon_3}$). Next we set

$$\mathcal{V}^{\varepsilon_3}=\widetilde{\mathcal{V}}^{\varepsilon_3}\cup\left(-\widetilde{\mathcal{V}}^{\varepsilon_3}\right),$$

and

$$\widetilde{\Gamma}_+=\left\{(x,v)\in\Gamma_+ \colon\ v\in\left(P_{v_0'}\setminus\mathcal{V}^{\varepsilon_3}\right)\right\}.$$

Here we will take the measure on $\widetilde{\Gamma}_+$ to be

$$d\xi_{v_0'}(x,v)=|n(x)\cdot v|\,d\mu(x)\,dH_{v_0'},$$

where $dH_{v_0'}$ is the Lebesgue measure on $P_{v_0'}$.

Similar to Part 1, we proceed our proof with several claims.

CLAIM 4.3. –

$$\lim_{\varepsilon_1\to 0}\int_{\widetilde{\Gamma}_+}\chi^{\varepsilon_2}(x)\int_{\Gamma_-}|\alpha_1-\alpha_2| \times(x,v,x',v')\phi_{x_0'}^{\varepsilon_2}(x')\psi_{v_0'}^{\varepsilon_1}(v')\,d\mu(x')\,dv'\,d\xi_{v_0'}(x,v)=0, \tag{4.20}$$

for all ε_2, ε_3 sufficiently small.

Proof. – Set

$$G(x,v):=\left|\mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma_1(x-pv)\,dp}-\mathrm{e}^{-\int_0^{\tau_-(x,v)}\sigma_2(x-pv)\,dp}\right|.$$

Note that

$$\int_{\widetilde{\Gamma}_+}\chi^{\varepsilon_2}\int_{\Gamma_-}|\alpha_1-\alpha_2|\phi_{x_0'}^{\varepsilon_2}\psi_{v_0'}^{\varepsilon_1}\,d\mu(x')\,dv'\,d\xi_{v_0'} =\int_{\widetilde{\Gamma}_+}\chi^{\varepsilon_2}G(x,v)\phi_{x_0'}^{\varepsilon_2}\big(x-\tau_-(x,v)v\big)\psi_{v_0'}^{\varepsilon_1}(v)\,d\xi_{v_0'}. \tag{4.21}$$

Now we can take ε_1 to be sufficiently small so that

$$\operatorname{supp}\psi_{v_0'}^{\varepsilon_1} \cap \left(P_{v_0'} \setminus \mathcal{V}^{\varepsilon_3}\right) = \emptyset.$$

Therefore, the right-hand side of (4.21) vanishes as we first take $\varepsilon_1 \to 0$. Thus we get (4.20). □

CLAIM 4.4. –

$$\liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}(x) \int_{\Gamma_-} |\gamma_1 - \gamma_2|(x, v, x', v')\phi_{x_0'}^{\varepsilon_2}(x') \times \psi_{v_0'}^{\varepsilon_1}(v')\, d\mu(x')\, dv'\, d\xi_{v_0'}(x, v) = J_{x_0', v_0'}^{\varepsilon_3, \varepsilon_2} < \infty, \tag{4.22}$$

and

$$\lim_{\varepsilon_2\to 0} J_{x_0', v_0'}^{\varepsilon_3, \varepsilon_2} = 0, \tag{4.23}$$

$\forall\, \varepsilon_3$.

Proof. – Remember that

$$\left|n(x')\cdot v'\right|^{-1}\gamma_i \in L^\infty\big(\Gamma_-; L^1(\Gamma_+, d\xi)\big), \quad i = 1, 2.$$

So we have

$$\begin{aligned}
&\int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \int_{\Gamma_-} |\gamma_1 - \gamma_2|\phi_{x_0'}^{\varepsilon_2}\psi_{v_0'}^{\varepsilon_1}\, d\mu\, dv'\, d\xi_{v_0'} \\
&\quad\leqslant \int_{\Gamma_+} \chi^{\varepsilon_2} \int_{\Gamma_-} \left|n(x')\cdot v'\right|^{-1}|\gamma_1 - \gamma_2|\left|n(x')\cdot v'\right|\phi_{x_0'}^{\varepsilon_2}\psi_{v_0'}^{\varepsilon_1}\, d\mu\, dv'\, d\xi \\
&\quad\leqslant \lambda_2 \sup_{(x',v')\in\Gamma_-} \int_{\Gamma_+} \chi^{\varepsilon_2}\left|n(x')\cdot v'\right|^{-1}|\gamma_1 - \gamma_2|\, d\xi \int_{\Gamma_-} \phi_{x_0'}^{\varepsilon_2}\psi_{v_0'}^{\varepsilon_1}\, d\mu\, dv' \\
&\quad= \lambda_2 \sup_{(x',v')\in\Gamma_-} \int_{\Gamma_+} \chi^{\varepsilon_2}\left|n(x')\cdot v'\right|^{-1}|\gamma_1 - \gamma_2|\, d\xi.
\end{aligned} \tag{4.24}$$

It is easily seen that (4.22) follows from (4.24). In view of the definition of χ^{ε_2}, we obtain (4.23) from (4.24). □

Now we are ready to prove (1.5). To begin with, we know that

$$(\mathcal{A}_1 - \mathcal{A}_2)\phi_{x_0'}^{\varepsilon_2}\psi_{v_0'}^{\varepsilon_1}(x, v) = I_1 + I_2 + I_3, \tag{4.25}$$

where

$$I_1 = \int_{\Gamma_-} (\alpha_1 - \alpha_2)(x, v, x', v')\phi^{\varepsilon_2}_{x'_0}(x')\psi^{\varepsilon_1}_{v'_0}(v')\, d\mu(x')\, dv',$$

$$I_2 = \int_{\Gamma_-} (\beta_1 - \beta_2)(x, v, x', v')\phi^{\varepsilon_2}_{x'_0}(x')\psi^{\varepsilon_1}_{v'_0}(v')\, d\mu(x')\, dv',$$

$$I_3 = \int_{\Gamma_-} (\gamma_1 - \gamma_2)(x, v, x', v')\phi^{\varepsilon_2}_{x'_0}(x')\psi^{\varepsilon_1}_{v'_0}(v')\, d\mu(x')\, dv'.$$

Therefore we have

$$\int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}(x)\big|(\mathcal{A}_1 - \mathcal{A}_2)\phi^{\varepsilon_2}_{x'_0}\psi^{\varepsilon_1}_{v'_0}(x, v)\big|\, d\xi_{v'_0}(x, v)$$
$$= \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}(x)|I_1 + I_2 + I_3|\, d\xi_{v'_0}(x, v). \tag{4.26}$$

We first deal with the left-hand side of (4.26). We have

$$\begin{aligned}\int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}\big|(\mathcal{A}_1 - \mathcal{A}_2)\phi^{\varepsilon_2}_{x'_0}\psi^{\varepsilon_1}_{v'_0}\big|\, d\xi_{v'_0} &\leqslant \big\|(\mathcal{A}_1 - \mathcal{A}_2)\phi^{\varepsilon_2}_{x'_0}\psi^{\varepsilon_1}_{v'_0}\big\|_{L^1(\Gamma_+, d\xi)} \\ &\leqslant \|\mathcal{A}_1 - \mathcal{A}_2\|_*\big\|\phi^{\varepsilon_2}_{x'_0}\psi^{\varepsilon_1}_{v'_0}\big\|_{L^1(\Gamma_-, d\xi)} \\ &\leqslant \lambda_2\|\mathcal{A}_1 - \mathcal{A}_2\|_*. \end{aligned}\tag{4.27}$$

From (4.26) and using (4.27), we get that

$$\int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}|I_2|\, d\xi_{v'_0} \leqslant \lambda_2\|\mathcal{A}_1 - \mathcal{A}_2\|_* + \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}\big(|I_1| + |I_3|\big)\, d\xi_{v'_0}. \tag{4.28}$$

Now let $E_i(s, x, v', v)$, $i = 1, 2$, be defined as in (4.9). Then

$$\begin{aligned} I_2 &= \int_{\Gamma_-} \int_0^{\tau_-(x,v)} \big(E_1(s, x, v', v)k_1(v', v) - E_2(s, x, v', v)k_2(v', v)\big) \\ &\quad \times \delta_{\{x - sv - \tau_-(x - sv, v')v'\}}(x')\phi^{\varepsilon_2}_{x'_0}(x')\psi^{\varepsilon_1}_{v'_0}(v')\, ds\, d\mu(x')\, dv' \\ &= \int_V \int_0^{\tau_-(x,v)} \big(E_1(s, x, v', v)k_1(v', v) - E_2(s, x, v', v)k_2(v', v)\big) \\ &\quad \times \phi^{\varepsilon_2}_{x'_0}\big(x - sv - \tau_-(x - sv, v')v'\big)\psi^{\varepsilon_1}_{v'_0}(v')\, ds\, dv'. \end{aligned}\tag{4.29}$$

According to (4.28) and (4.29), we have

$$\int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \tilde{I}_2 \, d\xi_{v_0'} \leqslant \lambda_2 \|\mathcal{A}_1 - \mathcal{A}_2\|_* + \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} (|I_1| + |I_3|) \, d\xi_{v_0'} + \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \widetilde{J} \, d\xi_{v_0'}, \quad (4.30)$$

where

$$\tilde{I}_2 = \left| \int_V \int_0^{\tau_-(x,v)} E_1(s, x, v', v)(k_1 - k_2)(v', v) \times \phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-(x - sv, v')v'\big) \psi_{v_0'}^{\varepsilon_1}(v') \, ds \, dv' \right|, \quad (4.31)$$

and

$$\widetilde{J} = \left| \int_V \int_0^{\tau_-(x,v)} (E_1 - E_2)(s, x, v', v) k_2(v', v) \times \phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-(x - sv, v')v'\big) \psi_{v_0'}^{\varepsilon_1}(v') \, ds \, dv' \right|. \quad (4.32)$$

We now focus on the left-hand side of (4.30). It is clear that the integrand of (4.31) is continuous in v'. So by Fatou's lemma we have

$$\liminf_{\varepsilon_1 \to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \tilde{I}_2 \, d\xi_{v_0'} \geqslant \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \liminf_{\varepsilon_1 \to 0} \tilde{I}_2 \, d\xi_{v_0'}$$

$$= \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2}(x) \big|(k_1 - k_2)(v_0', v)\big| \left| \int_0^{\tau_-(x,v)} E_1(s, x, v_0', v) \times \phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-(x - sv, v_0')v_0'\big) \, ds \right| d\xi_{v_0'}(x, v) := R_1. \quad (4.33)$$

With the choices of $\phi_{x_0'}^{\varepsilon_2}$ and $\widetilde{\Gamma}_+$, we can see that

$$\phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-(x - sv, v_0')v_0'\big) = 0$$

for all $(x, v) \in \widetilde{\Gamma}_+$, $x \notin \Pi_{\varepsilon_2}$, and $0 \leqslant s \leqslant \tau_-(x, v)$. Also, note that the integrand inside $\int_0^{\tau_-(x,v)}$ is non-negative. Therefore, from (4.33) and using Lemma 2.1 (with V being replaced by $P_{v_0'} \setminus \mathcal{V}^{\varepsilon_3}$) we obtain that

$$\begin{aligned} R_1 = & \int_{\widetilde{\Gamma}_+} \left|(k_1 - k_2)(v_0', v)\right| \int_0^{\tau_-(x,v)} E_1(s, x, v_0', v) \\ & \times \phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-(x - sv, v_0')v_0'\big)\, ds\, d\xi_{v_0'}(x, v) \\ = & \int_{(P_{v_0'} \setminus \mathcal{V}^{\varepsilon_3}) \times \Omega} \left|(k_1 - k_2)(v_0', v)\right| E_1(x, v_0', v) \\ & \times \phi_{x_0'}^{\varepsilon_2}\big(x - \tau_-(x, v_0')v_0'\big)\, dx\, dH_{v_0'} := R_2, \end{aligned} \tag{4.34}$$

where

$$E_1(x, v_0', v) = \mathrm{e}^{-\int_0^{\tau_-(x,v_0')} \sigma_1(x - pv_0')\, dp}\, \mathrm{e}^{-\int_0^{\tau_+(x,v)} \sigma_1(x+pv)\, dp}.$$

Since $\sigma_1(x) \in \mathcal{M}$, we get

$$\begin{aligned} R_2 \geqslant \mathrm{e}^{-\frac{2dM}{\lambda_1}} & \int_{P_{v_0'} \setminus \mathcal{V}^{\varepsilon_3}} \left|(k_1 - k_2)(v_0', v)\right| dH_{v_0'} \int_\Omega \phi_{x_0'}^{\varepsilon_2}\big(x - \tau_-(x, v_0')v_0'\big)\, dx \\ := R_3. & \end{aligned} \tag{4.35}$$

Now observe that

$$\begin{aligned} & \int_\Omega \phi_{x_0'}^{\varepsilon_2}\big(x - \tau_-(x, v_0')v_0'\big)\, dx \\ & = \int_{\{x' \in \partial\Omega:\ n(x')\cdot v_0' < 0\}} \int_0^{\tau_+(x', v_0')} \phi_{x_0'}^{\varepsilon_2}(x')\, dt \left|n(x') \cdot v_0'\right| d\mu(x') \\ & = \int_{\{x' \in \partial\Omega:\ n(x')\cdot v_0' < 0\}} \tau_+(x', v_0')\big(-n(x') \cdot v_0'\big)\phi_{x_0'}^{\varepsilon_2}(x')\, d\mu(x'). \end{aligned} \tag{4.36}$$

Since $\tau_+(x', v_0')(-n(x') \cdot v_0')$ in (4.36) is continuous in x', we have

$$\lim_{\varepsilon_2 \to 0} R_3 = \mathrm{e}^{-\frac{2dM}{\lambda_1}} \tau_+(x_0', v_0')\left|n(x_0') \cdot v_0'\right| \int_{P_{v_0'} \setminus \mathcal{V}^{\varepsilon_3}} \left|(k_1 - k_2)(v_0', v)\right| dH_{v_0'}. \tag{4.37}$$

Now it is helpful to summarize what we have done. By taking $\liminf_{\varepsilon_1\to 0}$ on both sides of (4.30) and using Claim 4.3, we get that

$$R_3 \leqslant \liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \tilde{I}_2 \, d\xi_{v_0'}$$
$$\leqslant \lambda_2 \|\mathcal{A}_1 - \mathcal{A}_2\|_* + J^{\varepsilon_3,\varepsilon_2}_{x_0',v_0'} + \liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \widetilde{J} \, d\xi_{v_0'}. \qquad (4.38)$$

Next let $\varepsilon_2 \to 0$ in (4.38) and use Claim 4.4 and (4.37), we have

$$\tau_+(x_0', v_0')|n(x_0')\cdot v_0'| \int_{P_{v_0'}\setminus \mathcal{V}^{\varepsilon_3}} |(k_1 - k_2)(v_0', v)| \, dH_{v_0'}$$
$$\leqslant C_{11}\left(\|\mathcal{A}_1 - \mathcal{A}_2\|_* + \lim_{\varepsilon_2\to 0} \liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \widetilde{J} \, d\xi_{v_0'} \right). \qquad (4.39)$$

The inequality (4.39) is what we have obtained up to this point. Now we continue our proof by taking $\varepsilon_3 \to 0$ in (4.39). First of all, we note that

$$\lim_{\varepsilon_3\to 0} \int_{P_{v_0'}\setminus \mathcal{V}^{\varepsilon_3}} |(k_1 - k_2)(v_0', v)| \, dH_{v_0'} = \int_{P_{v_0'}} |(k_1 - k_2)(v_0', v)| \, dH_{v_0'} \qquad (4.40)$$

by the Lebesgue's convergence theorem. Hence, we have

$$\tau_+(x_0', v_0')|n(x_0')\cdot v_0'| \int_{P_{v_0'}} |(k_1 - k_2)(v_0', v)| \, dH_{v_0'}$$
$$\leqslant C_{11}\left(\|\mathcal{A}_1 - \mathcal{A}_2\|_* + \lim_{\varepsilon_3\to 0} \lim_{\varepsilon_2\to 0} \liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \widetilde{J} d\xi_{v_0'} \right). \qquad (4.41)$$

Now we need to estimate

$$\lim_{\varepsilon_3\to 0} \lim_{\varepsilon_2\to 0} \liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \widetilde{J} \, d\xi_{v_0'} := R_4.$$

One can see that

$$R_4 = \lim_{\varepsilon_3\to 0} \lim_{\varepsilon_2\to 0} \liminf_{\varepsilon_1\to 0} \int_{\widetilde{\Gamma}_+} \chi^{\varepsilon_2} \widetilde{J} \, d\xi_{v_0'}$$

$$\leqslant \lim_{\varepsilon_2\to 0} \limsup_{\varepsilon_1\to 0} \int_{\Gamma_+} \widetilde{J}\, d\xi \leqslant \lim_{\varepsilon_2\to 0} \int_{\Gamma_+} \limsup_{\varepsilon_1\to 0} \widetilde{J}\, d\xi$$

$$\leqslant \lim_{\varepsilon_2\to 0} \int_{\Gamma_+} \int_0^{\tau_-(x,v)} |E_1 - E_2|\big(x, v_0', v\big) k_2\big(v_0', v\big)$$

$$\times \phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-\big(x - sv, v_0'\big)v_0'\big)\, ds\, d\xi. \tag{4.42}$$

Next we observe that

$$|E_1 - E_2|\big(x, v_0', v\big)$$

$$= \left| \mathrm{e}^{-\int_0^{\tau_-(x,v_0')} \sigma_1(x - pv_0')\,dp}\, \mathrm{e}^{-\int_0^{\tau_+(x,v)} \sigma_1(x+pv)\,dp} - \mathrm{e}^{-\int_0^{\tau_-(x,v_0')} \sigma_2(x - pv_0')\,dp}\, \mathrm{e}^{-\int_0^{\tau_+(x,v)} \sigma_2(x+pv)\,dp} \right|$$

$$= \mathrm{e}^{-\int_0^{\tau_-(x,v_0')} \sigma_1(x - pv_0')\,dp}\, \mathrm{e}^{-\int_0^{\tau_+(x,v)} \sigma_2(x+pv)\,dp} \times \left| \mathrm{e}^{\int_0^{\tau_+(x,v)} (\sigma_2-\sigma_1)(x+pv)\,dp} - \mathrm{e}^{\int_0^{\tau_-(x,v_0')} (\sigma_1-\sigma_2)(x-pv_0')\,dp} \right|$$

$$\leqslant C_{12} \left| \int_0^{\tau_+(x,v)} (\sigma_2 - \sigma_1)(x + pv)\, dp - \int_0^{\tau_-(x,v_0')} (\sigma_1 - \sigma_2)\big(x - pv_0'\big)\, dp \right|$$

$$\leqslant C_{13} \| X(\sigma_1 - \sigma_2) \|_{L^\infty(\Omega\times\mathrm{S}^2)}.$$

We now recall the estimate (4.19) and obtain

$$\|E_1 - E_2\|_{L^\infty(\Omega\times V\times V)} \leqslant C_{14} \|\mathcal{A}_1 - \mathcal{A}_2\|_*. \tag{4.43}$$

By substituting (4.43) into (4.42) and using Lemma 2.1, we have

$$R_4 \leqslant C_{14} \|\mathcal{A}_1 - \mathcal{A}_2\|_* \lim_{\varepsilon_2\to 0} \int_{\Gamma_+} \int_0^{\tau_-(x,v)} k_2\big(v_0', v\big)$$

$$\times \phi_{x_0'}^{\varepsilon_2}\big(x - sv - \tau_-\big(x - sv, v_0'\big)v_0'\big)\, ds\, d\xi$$

$$= C_{14} \|\mathcal{A}_1 - \mathcal{A}_2\|_* \lim_{\varepsilon_2\to 0} \int_{\Omega\times V} k_2\big(v_0', v\big) \phi_{x_0'}^{\varepsilon_2}\big(x - \tau_-\big(x, v_0'\big)v_0'\big)\, dx\, dv$$

$$= C_{14} \|\mathcal{A}_1 - \mathcal{A}_2\|_* \int_V k_2\big(v_0', v\big)\, dv$$

$$\times \lim_{\varepsilon_2 \to 0} \int_{\Omega} \phi_{x_0'}^{\varepsilon_2}\left(x - \tau_-(x, v_0')v_0'\right) dx. \tag{4.44}$$

According to (4.36) and $k_2 \in \mathcal{N}$, it follows from (4.44) that

$$R_4 \leqslant C_{15} \|\mathcal{A}_1 - \mathcal{A}_2\|_*. \tag{4.45}$$

Therefore, combining (4.41) and (4.45) we have

$$\begin{aligned} &\tau_-(x_0', v_0')|n(x_0') \cdot v_0'| \int_{P_{v_0'}} |(k_1 - k_2)(v_0', v)| \, dH_{v_0'} \\ &\quad \leqslant C_{16} \|\mathcal{A}_1 - \mathcal{A}_2\|_*, \end{aligned} \tag{4.46}$$

where the right-hand side of (4.46) is independent of $(x_0', v_0') \in \Gamma_-$. Most importantly, the inequality (4.46) is uniformly in the class of $P_{v_0'}$. Note that the region formed by the union of $P_{v_0'}$ for any fixed v_0' is in fact V. In view of the spherical coordinates, we can see that

$$\begin{aligned} &\tau_-(x_0', v_0')|n(x_0') \cdot v_0'| \int_{V} |(k_1 - k_2)(v_0', v)| \, dv \\ &\quad \leqslant (\pi \lambda_2) C_{16} \|\mathcal{A}_1 - \mathcal{A}_2\|_*. \end{aligned} \tag{4.47}$$

Now since (4.47) holds for all $(x_0', v_0') \in \Gamma_-$, we have

$$\begin{aligned} &\mathrm{Vol}(\Omega) \|k_1 - k_2\|_{L^1(V \times V)} \\ &= \int_{\Omega \times V} \|(k_1 - k_2)(v', \cdot)\|_{L^1(V)} \, dx \, dv' \\ &= \int_{\Gamma_-} \int_0^{\tau_-(x', v')} \|(k_1 - k_2)(v', \cdot)\|_{L^1(V)} \, dt \, d\xi(x', v') \\ &= \int_{\Gamma_-} \tau_-(x', v')|n(x') \cdot v'| \|(k_1 - k_2)(v', \cdot)\|_{L^1(V)} \, d\mu(x') \, dv' \\ &\leqslant (\pi \lambda_2) C_{16} \|\mathcal{A}_1 - \mathcal{A}_2\|_* \int_{\Gamma_-} d\mu(x') \, dv'. \end{aligned} \tag{4.48}$$

The estimate (1.5) follows immediately from (4.48). The proof of Theorem 1.1 is now complete.

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