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Hilbert spaces for massless particles with nonvanishing helicities

by

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ABSTRACT. – The paper contains a complete description of the phase spaces for massless particles with different helicities. Explicit formulae for reproducing kernels for investigated Hilbert spaces are given. The author demonstrates a full symmetry between scalar products and reproducing kernels which are strictly related to the twistor propagator. © Elsevier, Paris

Key words: Twistors, massless particles, Hilbert space, reproducing kernel, scalar product.

RÉSUMÉ. – L'article fournit la description complète des espaces de phase pour les particules sans masses, avec différentes hélicités. Nous donnons une formule explicite pour les noyaux reproduisants des espaces d'Hilbert en question. Nous exhibons une symétrie entre produits scalaires et noyaux reproduisants directement reliée au propagateur des twisteurs. © Elsevier, Paris

1. INTRODUCTION

Many works have been written about twistors (see for example [21]) since Penrose introduced them in 1967 [17]. That is why I don't want to say how important they were and are for mathematical physics. I will give only brief motivation that inspired me to write this paper.

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First of all anybody who is interested in massless particles must encounter twistors. Twistors are very natural objects for describing systems with conformal symmetry [13, 14]. Many authors have approached them from more mathematical rather than physical point of view. But a quantisation of classical objects as a procedure which is of special interest to physicists, requires many fundamental objects to be defined like Hilbert spaces, like scalar products, orthonormal bases and reproducing kernels. These objects are investigated in the present work. So, the next chapter recalls well known realizations of the representations of the group $SU(2,2)$ [12] and introduces notation and suitable definitions which will be necessary later. In chapter 3, by means of Lerner's version of the inverse Penrose transform [15], the orthonormal basis from the space of square integrable functions on $\mathbf{C}^2$ to the first cohomology group on $\mathbf{PT}^{\pm}$ is transformed. Hilbert space for massless particles with nonvanishing helicity is defined by means of a special choice of twistors which define a covering of $\mathbf{PT}^{\pm}$ by two open subsets. Chapter 4 generalizes this results. Hilbert spaces of massless particles are described together with explicit formulas for scalar products. The objects which define scalar products can be treated as reproducing kernels as well and that fact is investigated in detail in chapter 5. Theorem 2 which is formulated there gives a complete description of the Hilbert spaces for massless particles with different helicities and frequences, as well as a description of scalar products and reproducing kernels.

I have found that the twistor propagator described in [6, 8, 9] has many common features with reproducing kernels. As I shaw it turns out that they give the same cohomology classes. Some remarks about this are included in the last chapter. My formulas seem more suitable for calculations than those given in [6, 8, 9]. Ginsberg and Eastwood needed a special construction to find scalar products for positive helicities, but in my approach it is not necessary. There is a full symmetry between the cases of positive and negative helicities.

The present paper is concerned with the mathematical background of quantisation of massless particles which are classically described by $\mathbf{PT}^{\pm}$ spaces [13]. The next paper, which will appear soon, will show the applications of the obtained results to physics.

2. THE RECOLLECTION

Twistor space $\mathbf{T}$ is the pair $(\mathbf{C}^4, <,>)$ where $\mathbf{C}^4$ is linear space of 4-tuples of complex numbers and $<,>$ denotes an Hermitian form of signature $(+,+,-,-)$. The group of automorphisms of $\mathbf{T}$ is $SU(2,2)$

which preserves the twistor form $<,>$. Among its homogeneous spaces there are some which play crucial role in this work, namely:

$\mathbf{M}^{++}$, $\mathbf{M}^{--}$ - the space of 2 dimensional linear subspaces in $\mathbf{T}$ which are positive (negative) with respect to the twistor form.

$\mathbf{PT}^{+}$, $\mathbf{PT}^{-}$ - the spaces of 1 dimensional linear subspaces positive (negative) with respect to $<,>$.

$\mathbf{F}^{+,++}$, $\mathbf{F}^{-,--}$ - flag manifolds of lines inside 2-planes with positive lines and 2-planes or negative, respectively.

For the above objects, there are natural double fibrations [16]:

$$\alpha_{\pm} : \mathbf{F}^{\pm,\pm,\pm} \longrightarrow \mathbf{PT}^{\pm} \tag{1}$$

$$\beta_{\pm} : \mathbf{F}^{\pm,\pm,\pm} \longrightarrow \mathbf{M}^{\pm\pm} \tag{2}$$

where $\alpha_{\pm}$, $\beta_{\pm}$ are projections on the first and the second component of the pairs belonging to the flag manifolds.

$\mathbf{M}^{++}$ and $\mathbf{M}^{--}$ have common Śilov boundary which is the real Minkowski space $\mathbf{M}^{00}$. In the approach presented here $\mathbf{M}^{00}$ is the space of 2 dimensional subspaces in $\mathbf{T}$ such that the restriction of the twistor form to them equals zero.

Let us consider solutions of the zero rest mass equations on $\mathbf{M}^{00}$ [16, 18, 21]:

$$\begin{aligned} \nabla^{AA'_1}\varphi_{A'_1,\dots,A'_n} &= 0 \\ \Box\varphi = \nabla^{AA'}\nabla_{AA'}\varphi &= 0 \\ \nabla^{A_1A'}\varphi_{A_1,\dots,A_n} &= 0 \end{aligned} \tag{3}$$

where $A, A_1, A'_1, \dots,$ are spinor indices and $\nabla_{AA'}$ means differentiation with respect to $x^{AA'}$ - spinor coordinates on M^{00}.

The solutions of the above equations are zero rest mass fields with helicities respectively: $s = \frac{n}{2}$, $s = 0$, $s = -\frac{n}{2}$. Fourier analysis splits every solution into two parts one positive and one negative: $\varphi = \varphi^{+} + \varphi^{-}$ (analogously for $n \neq 0$) where $\varphi^{\pm}$ extend as holomorphic functions to $\mathbf{M}^{++}$ and $\mathbf{M}^{--}$ respectively, by means of the Laplace transform *Lap*. The corresponding linear spaces of solutions, which split into positive and negative parts, are denoted as follows:

$\mathcal{Z}'_n(\mathbf{M}^{++})$ positive frequency solutions with helicities $s = \frac{n}{2}$.

$\mathcal{Z}'_n(\mathbf{M}^{--})$ negative frequency solutions with helicities $s = \frac{n}{2}$.

and analogously: $\mathcal{Z}_n(\mathbf{M}^{++})$, $\mathcal{Z}_n(\mathbf{M}^{--})$, $\mathcal{Z}_0(\mathbf{M}^{++})$, $\mathcal{Z}_0(\mathbf{M}^{--})$. When holomorphic objects are considered the operator $\nabla_{AA'}$ is $\nabla_{AA'} = \frac{\partial}{\partial \hat{Z}^{AA'}}$

where the complex matrices $\hat{Z}^{AA'}$ are coordinates of points from $\mathbf{M}^{++}$, $\mathbf{M}^{--}$.

The spaces $\mathcal{Z}_n$ and $\mathcal{Z}'_n$ can be equipped with the structure of Hilbert space. Let us consider the spaces [12] $L'_n(\partial\Gamma^{\pm})$ and $L_n(\partial\Gamma^{\pm})$ defined in the following way: $\Gamma^{\pm}$ are upper and lower parts of a null-cone respectively. $L'_n(\partial\Gamma^{\pm})$ and $L_n(\partial\Gamma^{\pm})$ are Hilbert spaces whose elements are the sections of the bundles $\bigodot^n(\mathbf{C}^2)' \longrightarrow \partial\Gamma^{\pm}$ or $\bigodot^n \mathbf{C}^2 \longrightarrow \partial\Gamma^{\pm}$ satisfying the conditions:

$$\xi^{A'_1} f_{A'_1,\ldots,A'_n}(\hat{K}) = 0 \quad \text{for} \quad L'_n(\partial\Gamma^{\pm}) \tag{4}$$

$$\xi^{A_1} f_{A_1,\ldots,A_n}(\hat{K}) = 0 \quad \text{for} \quad L_n(\partial\Gamma^{\pm}) \tag{5}$$

where $\hat{K}$ is the hermitian matrix such that:

$det\hat{K} = 0$ and $Tr\hat{K} > 0$ for $\hat{K} \in \partial\Gamma^{+}$ and

$det\hat{K} = 0$ and $Tr\hat{K} < 0$ for $\hat{K} \in \partial\Gamma^{-}$

Every such matrix can be described by the spinor ξ (given up to the phase factor) which appears in the above conditions. The scalar product is:

$$\oint_{\partial\Gamma^{\pm}} (Tr\hat{K})^{-n} < f_1(\hat{K}), f_2(\hat{K}) > d\mu_{\pm}(\hat{K}) =< f_1, f_2 > \tag{6}$$

where

$< f_1(\hat{K}), f_2(\hat{K}) >$ is the scalar product in $\bigodot^n \mathbf{C}^2$ or in $\bigodot^n(\mathbf{C}^2)'$ induced from the canonical scalar product on $\mathbf{C}^2$.

$d\mu_{\pm}(\hat{K})$ mean $SL(2,\mathbf{C})$ invariant measures on $\partial\Gamma^{\pm}$ respectively.

The Hilbert spaces $L'_n(\partial\Gamma^{\pm})$ and $L_n(\partial\Gamma^{\pm})$ carry an irreducible representation of the group $SU(2,2)$, so called "ladder representation" [12]. Their images under the Laplace transform Lap are the spaces $\mathcal{H}'_n(\mathbf{M}^{\pm\pm}) \subset \mathcal{Z}'_n(\mathbf{M}^{\pm\pm})$ and $\mathcal{H}_n(\mathbf{M}^{\pm\pm}) \subset \mathcal{Z}_n(\mathbf{M}^{\pm\pm})$ where the equivalent representation acts. Scalar products in $\mathcal{H}_n$ and $\mathcal{H}'_n$ are given by definition by formulas 6 this means that if $\varphi_1 = Lap(f_1)$ and $\varphi_2 = Lap(f_2)$ then $< \varphi_1, \varphi_2 > \doteq < f_1, f_2 >$.

Let us return to the double fibrations 1, 2 which give connections not only between the manifolds $\mathbf{PT}^{\pm}$ and $\mathbf{M}^{\pm\pm}$ but also between objects defined on them. The transformation I am referring to is the Penrose transform $\mathcal{P}$ that realizes the isomorphisms [16, 18, 20, 21]:

$$\mathcal{P}: \ H^1(\mathbf{PT}^{\pm}, \mathcal{O}(-n-2)) \longrightarrow \mathcal{Z}'_n(\mathbf{M}^{\pm\pm}) \tag{7}$$

$$\mathcal{P}: \ H^1(\mathbf{PT}^{\pm}, \mathcal{O}(n-2)) \longrightarrow \mathcal{Z}_n(\mathbf{M}^{\pm\pm}) \tag{8}$$

where $H^1(\mathbf{X}, \mathcal{O}(k))$ is the first cohomology group of a manifold $\mathbf{X}$ with values in the sheaf of germs of holomorphic sections of the k tensor power of the universal bundle.

Here I wish to make a remark, namely: $H^1(\mathbf{X}, \mathcal{O}(k))$ is a cohomology group but it has also structure of a linear space [4, 5] and this structure will be important in further considerations.

The main goal of this paper is a description of the images of the Hilbert spaces $\mathcal{H}_n$ and $\mathcal{H}'_n$ under the inverse Penrose transform. Because we will be dealing with the manifolds $\mathbf{PT}^{\pm}$ as well, so I will recall some elementary facts about them. They can be identified with coadjoint orbits of the group $SU(2,2)$ [12, 21], and thereby equipped with the symplectic structure. From the physical point of view $\mathbf{PT}^{\pm}$ are phase spaces of classical massless particles with nonvanishing helicities [3, 13, 19]. The standard geometrical quantisation procedure fails in these cases [3, 19]. So quantum states of massless particles can not be described by global sections. The first step to overcome this difficulty is to consider the first cohomology groups as the candidates for Hilbert spaces of quantum states.

By describing the images of the Hilbert spaces $\mathcal{H}_n$ and $\mathcal{H}'_n$ under the inverse Penrose transform we will solve the problem of quantisation of classical massless particles with nonvanishing helicities. Some physical aspects of this will be published later.

3. THE PRELIMINARY OBSERVATIONS

I did not mention a still another realization of the "ladder representation" because here it plays an auxiliary role.

Let us consider square integrable functions on $\mathbf{C}^2$ with respect to Lebesgue measure. It is the well known Hilbert space $L(\mathbf{C}^2, d\mu)$, under the action of the group $SU(2,2)$ it splits into an infinite direct sum of Hilbert spaces defined by the homogeneity conditions [12]:

$$\mathcal{L}_n = \{ h \in L(\mathbf{C}^2, d\mu) \,:\, h(e^{i\alpha}\xi) = e^{-in\alpha} h(\xi) \} \tag{9}$$

$$\mathcal{L}_{-n} = \{ h \in L(\mathbf{C}^2, d\mu) \,:\, h(e^{i\alpha}\xi) = e^{in\alpha} h(\xi) \} \tag{10}$$

where $\alpha \in R$ (real numbers) and $\xi \in \mathbf{C}^2$.

In this way one obtains the series of harmonic representation of $SU(2,2)$. For example, by identification of $\xi \in \mathbf{C}^2$ with primed indices spinors we have an isomorphism $I \,:\, \mathcal{L}_n \xrightarrow{\xi_{A'}} L'_n(\partial\Gamma^+)$:

$$(If)(\hat{K})_{A'_1,\ldots,A'_n} = f(\xi, \bar{\xi})\xi_{A'_1} \cdots \xi_{A'_n} \tag{11}$$

where bar means complex conjugation. The other splitting operators can be obtained in a similar way.

The following remarks summarize the construction of orthonormal bases for the spaces $\mathcal{L}_{\pm n}$. It is well known that Hermite's functions H_k form a basis for square integrable functions on the space of real numbers $\mathbf{R}$. From the isomorphism $\mathbf{R} \times \mathbf{R} \simeq \mathbf{C}$ one can easily find a basis for the square integrable functions on $\mathbf{C}$:

$$\{ H_k(x)H_l(y) \} \simeq \{ E_{kl}(x+iy) \} \tag{12}$$

In the same way one can construct an orthonormal basis for square integrable functions on $\mathbf{C}^2 \simeq \mathbf{C} \times \mathbf{C}$. The elements of the basis have the form:

$$E_{k_1 l_1 k_2 l_2}(\xi, \bar{\xi}) = \frac{1}{\pi} \frac{e^{\frac{1}{2}|\xi|^2}}{\sqrt{k_1! l_1! k_2! l_2!}} \frac{\partial^{k_1+l_1+k_2+l_2}}{\partial \xi_1^{k_1} \partial \bar{\xi}_1^{l_1} \partial \xi_2^{k_2} \partial \bar{\xi}_2^{l_2}} e^{-|\xi|^2} \tag{13}$$

This construction leads to the following statement:

STATEMENT 1. – *The square integrable functions on* $\mathbf{C}^2$*:*

$$\mathcal{B}^{(n)}_{lk_1k_2}(\xi, \bar{\xi}) = \frac{1}{\pi} \frac{e^{\frac{1}{2}|\xi|^2}}{\sqrt{k_1!(n+l-k_1)!k_2!(l-k_2)!}} \times \frac{\partial^{2l+n}}{\partial \xi_1^{k_1} \partial \xi_2^{n+l-k_1} \partial \bar{\xi}_1^{k_2} \partial \bar{\xi}_2^{l-k_2}} e^{-|\xi|^2} \tag{14}$$

where

$k_1 = 0, 1, \ldots, n+l;\ k_2 = 0, 1, \ldots, l;\ l = 0, 1, \ldots, \infty.$

and

$$\mathcal{B}^{(-n)}_{lk_1k_2}(\xi, \bar{\xi}) = \frac{1}{\pi} \frac{e^{\frac{1}{2}|\xi|^2}}{\sqrt{k_1!(l-k_1)!k_2!(n+l-k_2)!}} \times \frac{\partial^{2l+n}}{\partial \xi_1^{k_1} \partial \xi_2^{l-k_2} \partial \bar{\xi}_1^{k_2} \partial \bar{\xi}_2^{n+l-k_2}} e^{-|\xi|^2} \tag{15}$$

where

$k_1 = 0, 1, \ldots, l;\ k_2 = 0, 1, \ldots, n+l;\ l = 0, 1, \ldots, \infty.$

form orthonormal bases for the Hilbert spaces $\mathcal{L}_n$ *and* $\mathcal{L}_{-n}$ *respectively.*

Proof. – It is trivial because if we require the satisfying homogeneity conditions 9, 10 we obtain some restrictions on the numbers k_1, k_2, l_1, l_2 and the final result is given above.

Performing transformation 11 on the basis functions given in statement 1 we find bases for the spaces $L'_n(\partial\Gamma^{\pm})$ and $L_n(\partial\Gamma^{\pm})$. Next, applying the Laplace transformation *Lap* one obtains basis elements for the Hilbert spaces $\mathcal{H}'_n$, $\mathcal{H}_n$, by definition. Let us write the appropriate expression as an example for $L'_n \xrightarrow{Lap} \mathcal{H}'_n(\mathbf{M}^{++})$:

$$\oint_{\partial\Gamma^+} \xi_1^{n-m}\xi_2^m \, \mathcal{B}^{(n)}_{lk_1k2}(\xi,\bar{\xi})e^{-\frac{i}{2}\xi^+\hat{Z}\xi}d\mu_+(\xi\xi^+) \tag{16}$$

where the collection $\{\, \xi_1^{n-m}\xi_2^m \,\}_{m=0,1,\dots,n}$ can be identified with the value of the section of the bundle $\bigodot^n(\mathbf{C}^2)' \longrightarrow \partial\Gamma^+$ at the point $\xi\xi^+ \in \partial\Gamma^+$, where ξ^+ means Hermitian conjugation.

The matrix $\hat{Z} \in M_{2\times 2}(\mathbf{C})$ describes a point from $\mathbf{M}^{++}$. It satisfies some additional conditions depending on the realization of the twistor form.

Lerner [15] used similiar expressions to realize the inverse Penrose transform noticing that $\partial\Gamma^+$ is $\mathbf{R}_+$ (positive real numbers) bundle over $\mathbf{P}_1\mathbf{C}$ and performing integration over sheaves of this bundle. Applying Lerner's idea to the expression 16 we obtain a closed $(0,1)$ - form on $\mathbf{F}^{+,++}$ which is the pullback of some form on $\mathbf{PT}^+$ representing an element from the first cohomology group on $\mathbf{PT}^+$. Finding its Čech's representative we obtain a cocycle from $H^1(\mathbf{PT}^+, \mathcal{O}(-n-2))$ characterized by:

$$\mathcal{B}^{(n)}_{lk_1k2}(Z) = (i)^{n+1}\sqrt{\frac{k_1!(n+l-k_1)!}{k_2!(l-k_2)!}}\frac{< A^*, Z >^{k_2}< B^*, Z >^{l-k_2}}{< A, Z >^{k_1+1}< B, Z >^{n+l-k_1+1}} \tag{17}$$

where

$k_1 = 0,1,\dots,n+l$; $k_2 = 0,1,\dots,l$; $l = 0,1,\dots,\infty$.

$$A = \tfrac{1}{\sqrt{2}}\begin{pmatrix}-i\\0\\1\\0\end{pmatrix}, B = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\-i\\0\\1\end{pmatrix}, A^* = \tfrac{1}{\sqrt{2}}\begin{pmatrix}i\\0\\1\\0\end{pmatrix}, B^* = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\i\\0\\1\end{pmatrix}$$

$A, B \in \mathbf{PT}^+$ and $Span\{A,B\} \in \mathbf{M}^{++}$,

$A^*, B^* \in \mathbf{PT}^-$ and $Span\{A^*,B^*\} \in \mathbf{M}^{--}$

$Z \in \mathbf{PT}^+$

The matrix of the twistor form is $i\begin{pmatrix}0 & -E\\ E & 0\end{pmatrix}$ where $E = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

The above elements are defined on the intersection of two open subsets:

$$\mathrm{U}_1 = \{ Z \in \mathbf{PT}^+ : < A, Z > \neq 0 \},$$

$$\mathrm{U}_2 = \{ Z \in \mathbf{PT}^+ : < B, Z > \neq 0 \} \quad (18)$$

so each of them represents an element of the first cohomology group of the covering of $\mathbf{PT}^+$ by two open subsets U_1,U_2. We have an affirmative answer to the inverse question: Is every element from the first cohomology group of the covering of $\mathbf{PT}^+$ by two subsets U_1,U_2 spanned by $\mathcal{B}^{(n)}_{lk_1k_2}$? It is easy to prove this by expanding an arbitrary element defined on $\mathrm{U}_1 \cap \mathrm{U}_2$ around the origin of the coordinate system defined by the twistors A, B , A^*, B^* [4, 5]. Let us denote such cohomology group by $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$. We can repeat similar considerations for $L'_n(\partial\Gamma^-) \longrightarrow \mathcal{H}'_n(\mathbf{M}^{--})$ and $L_n(\partial\Gamma^{\pm}) \longrightarrow \mathcal{H}_n(\mathbf{M}^{\pm\pm})$ and the corresponding cohomology groups will be denoted by $H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(-n-2))$, $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(n-2))$ and $H^1_{A^*B^*}(PT^-, \mathcal{O}(n-2))$. Above considerations can be formulated in the statement:

STATEMENT 2. – *Let us take four twistors A, B , A^*, B^* as in 17. They form a basis of $\mathbf{C}^4$, orthonormal with respect to the twistor form represented by matrix $i\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. Open subsets U_1 and U_2 as in 18 and*

$$\begin{aligned} \mathrm{U}_1^* &= \{ Z \in \mathbf{PT}^- : < A^*, Z > \neq 0 \}, \\ \mathrm{U}_2^* &= \{ Z \in \mathbf{PT}^- : < B^*, Z > \neq 0 \} \end{aligned} \quad (19)$$

form covering of $\mathbf{PT}^+$ and $\mathbf{PT}^-$ respectively. Using the notation introduced just before the statement we have the following isomorphisms:

$$\mathcal{H}'_n(\mathbf{M}^{++}) \sim H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$$

$H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$ is spanned by cocycles represented by the sections 17, they are by definition orthonormal.

$$\mathcal{H}_n(\mathbf{M}^{++}) \sim H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(n-2))$$

orhonormal basis is represented by sections:

$$\mathcal{B}^{(-n)}_{lk_1k2}(Z) = (i)^{n+1} \sqrt{\frac{k_1!(l-k_1)!}{k_2!(n+l-k_2)!}} \frac{< A^*, Z >^{k_2} < B^*, Z >^{n+l-k_2}}{< A, Z >^{k_1+1} < B, Z >^{l-k_1+1}} \quad (20)$$

$k_1 = 0, 1, \dots, l;\ k_2 = 0, 1, \dots, n + l;\ l = 0, 1, \dots, \infty.$

$$\mathcal{H}'_n(\mathbf{M}^{--}) \sim H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(-n-2))$$

and the orthonormal basis:

$$\mathcal{B}^{(n)}_{lk_1k2}(Z) = (i)^{n+1}\sqrt{\frac{k_1!(n+l-k_1)!}{k_2!(l-k_2)!}}\frac{< A, Z >^{k_2} < B, Z >^{l-k_2}}{< A^*, Z >^{k_1+1} < B^*, Z >^{n+l-k_1+1}} \tag{21}$$

$k_1 = 0, 1, \dots, n + l;\ k_2 = 0, 1, \dots, l;\ l = 0, 1, \dots, \infty.$

$$\mathcal{H}_n(\mathbf{M}^{--}) \sim H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(n-2))$$

with orthonormal basis:

$$\mathcal{B}^{(-n)}_{lk_1k2}(Z) = (i)^{n+1}\sqrt{\frac{k_1!(l-k_1)!}{k_2!(n+l-k_2)!}}\frac{< A, Z >^{k_2} < B, Z >^{n+l-k_2}}{< A^*, Z >^{k_1+1} < B^*, Z >^{l-k_1+1}} \tag{22}$$

$k_1 = 0, 1, \dots, l;\ k_2 = 0, 1, \dots, n + l;\ l = 0, 1, \dots, \infty.$

Proof. – It is completed by performing computations described for the first isomorphism and taking into account that the Penrose transform is an isomorphism [16].

Remark 1. – All the elements belonging to the first cohomology group of the covering of $\mathbf{PT}^{\pm}$ by two open subsets are extendible to the boundary $\mathbf{PT}^0$. It follows from the construction and the fact that the elements of $\mathcal{H}$ and $\mathcal{H}'$ come from fields defined on real Minkowski space $\mathbf{M}^{00}$ [6, 8, 18]. So we will be considering $\overline{\mathbf{PT}}^{\pm} = \mathbf{PT}^{\pm} \cup \mathbf{PT}^0$ and the coverings by closed subsets $\bar{\mathtt{U}}_1$, $\bar{\mathtt{U}}_2$, $\bar{\mathtt{U}}_1^*$, $\bar{\mathtt{U}}_2^*$ to prove many facts appearing in the further considerations.

STATEMENT 3. – *The following pairs of spaces are dual in the sense of linear algebra (see for example [6, 8]):*

$$H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2)) \text{ and } H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(n-2)) \tag{23}$$

$$H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(n-2)) \text{ and } H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(-n-2)) \tag{24}$$

Proof. – The dot product $\bullet$ gives the natural map [6, 8, 9]:

$$\bullet\ :\ H^p(\mathtt{U}, \mathcal{S}) \otimes H^q(\mathtt{V}, \mathcal{T}) \longrightarrow H^{p+q+1}(\mathtt{U} \cup \mathtt{V}, \mathcal{S} \otimes \mathcal{T})$$

which is induced by multiplication of Cech's representatives. In our case we have the sequence of maps [6, 8] (taking closures as in the remark):

$$\begin{aligned} &H^1_{AB}(\bar{\mathbf{PT}}^+, \mathcal{O}(-n-2)) \otimes H^1_{A^*B^*}(\bar{\mathbf{PT}}^-, \mathcal{O}(n-2)) \\ &\stackrel{\bullet}{\longrightarrow} H^3(\mathbf{PT}, \mathcal{O}(-4)) \stackrel{Serre}{\longrightarrow} \mathbf{C} \end{aligned}$$

The last map is induced by the Serre duality, which is just an integration with the canonical Leray's form:

$$\mathcal{D}Z = det[Z, dZ, dZ, dZ] \tag{25}$$

where $Z \in \mathbf{PT}$. There is an analogous sequence for the second pair of spaces.

I mentioned earlier that my intention is to describe the image of the spaces $\mathcal{H}_n$ and $\mathcal{H}'_n$ under the inverse Penrose transform. It has not been done yet in a satisfactory way. The results we have obtained depend on a very specific choice of the twistors A, B, A^*, B^*. In the next chapter I will get rid of this restriction, not only in the definitions of the spaces H^1 but also in the definitions of the orthonormal bases.

4. HILBERT SPACES FOR MASSLESS PARTICLES

The complex conjugation induces a linear isomorphism:

$$H^1(\mathbf{PT}^{\pm}, \mathcal{O}(k)) \longrightarrow H^1(\mathbf{PT}^{*\pm}, \mathcal{O}(k)) \tag{26}$$

where $\mathbf{PT}^{*\pm}$ mean dual spaces, so together with the spaces $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(\pm n-2))$ and $H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(\pm n-2))$ we have to take into consideration their images under the above isomorphism, namely: $H^1_{AB}(\mathbf{PT}^{*+}, \mathcal{O}(\pm n-2))$ and $H^1_{A^*B^*}(\mathbf{PT}^{*-}, \mathcal{O}(\pm n-2))$. It looks very simple on basis elements because they depend on the twistors by the twistor form only so the action of the linear isomorphism reduces to the change $< C, D > \to < D, C >$ and taking complex conjugation for coefficients.

For further detailed considerations let us take the space $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$ as an example. Applying the isomorphism 26 to the elements 17 gives us:

$$\left[\mathcal{B}^{*(n)}_{lk_1k_2}\right] \in H^1_{AB}(\mathbf{PT}^{*+}, \mathcal{O}(-n-2))$$

represented by the cocycles:

$$\mathcal{B}^{*(n)}_{lk_1k_2}(\bar{Z}) = (i)^{n+1}\sqrt{\frac{k_1!(n+l-k_1)!}{k_2!(l-k_2)!}}\frac{< Z, A^* >^{k_2} < Z, B^* >^{l-k_2}}{< Z, A >^{k_1+1} < Z, B >^{n+l-k_1+1}} \tag{27}$$

where

$k_1 = 0, 1, \ldots, n+l;\ k_2 = 0, 1, \ldots, l;\ l = 0, 1, \ldots \infty.$

Imitating methods of classical complex analysis developed by Bergman leads to the expression:

$$\sum_{l=0}^{\infty}\sum_{k_1=0}^{n+l}\sum_{k_2=0}^{l}\left[\mathcal{B}^{*(n)}_{lk_1k_2}\right] \times \left[\mathcal{B}^{(n)}_{lk_1k_2}\right] \tag{28}$$

Here $\times$ is the cross-product known from the theory of cohomology. By its definition, every term of the above sum is a cocycle from $H^2(\mathbf{PT}^{*+} \times \mathbf{PT}^{+}, \mathcal{O}(-n-2, -n-2))$ (strictly speaking from the second cohomology group of the covering) where $\times$ is the cartesian product of the manifolds and $\mathcal{O}(-n-2, -n-2)$ is the tensor product of sheaves $\mathcal{O}(-n-2)$ over $\mathbf{PT}^{*+}$ and $\mathcal{O}(-n-2)$ over $\mathbf{PT}^{+}$.

Performing the above summation is just a technical task so, omitting details, the final result is given by the following expression for the representative from H^2:

$$\Phi^{(++)}_n(\bar{W}, Z) = \frac{d^{n+1}}{d\lambda^{n+1}}\left[\frac{1}{\lambda}\ln\frac{(x-\lambda)(y-\lambda)}{xy}\right] \tag{29}$$

where

$W, Z \in PT^+$, $\lambda =< W, Z >$, $x =< W, A >< A, Z >$,

$y =< W, B >< B, Z >$.

The twistors A, B are the same as in the previous section. From such construction and from the final result we conclude the following properties of $\Phi^{(++)}_n$:

1. it depends only on the twistors A and B which define the covering of $\mathbf{PT}^+$,
2. it does not depend on the auxiliary twistors A^*, B^*,
3. dependence on twistors is by the twistor form only, so its particular realization does not matter,
4. it does not have any singularity when $\lambda \to 0$, which can be seen by expanding logarithm arround the submanifold $\lambda = 0$.

Let us introduce notation:

$$\Phi_{-1}(\lambda) = \frac{1}{\lambda} \ln \frac{(x-\lambda)(y-\lambda)}{xy} \tag{30}$$

with λ, x, y as in 29. For futher purposes let us introduce such an expression:

$$\Phi^*_{-1}(\lambda) = \frac{1}{\lambda} \ln \frac{(x^*+\lambda)(y^*+\lambda)}{x^* y^*} \tag{31}$$

$W, Z \in \mathbf{PT}^-$, $\lambda =< W, Z >$,
$x^* =< W, A^* >< A^*, Z >$, $y =< W, B^* >< B^*, Z >$.
with A^*,B^* as in the previous chapter.

For the remaining cases of the spaces H^1 we can find similar elements of the second cohomology groups with analogous properties. Their significance for our considerations is summarized in the theorem:

THEOREM 1. – *Let us choose four twistors orthonormal with respect to the twistor form such that:*

A, $B \in \mathbf{PT}^+$ *and* $Span\{A, B\} \in \mathbf{M}^{++}$,

A^*, $B^* \in \mathbf{PT}^-$ *and* $Span\{A^*, B^*\} \in \mathbf{M}^{--}$

For $W, Z \in \mathbf{PT}^+$ *we define* x, y *as in 29 and for* $W, Z \in \mathbf{PT}^-$, x^*, y^* *as in 31. There exist cocycles from the second cohomology groups represented by:*

$$\Phi_n^{(++)} \in H^2(\mathbf{PT}^{*+} \times \mathbf{PT}^+, \mathcal{O}(-n-2, -n-2))$$

$$\Phi_{-n}^{(++)} \in H^2(\mathbf{PT}^{*+} \times \mathbf{PT}^+, \mathcal{O}(n-2, n-2))$$

$$\Phi_n^{(--)} \in H^2(\mathbf{PT}^{*-} \times \mathbf{PT}^-, \mathcal{O}(-n-2, -n-2))$$

$$\Phi_{-n}^{(--)} \in H^2(\mathbf{PT}^{*-} \times \mathbf{PT}^-, \mathcal{O}(n-2, n-2))$$

if $\lambda =< W, Z >$ *they satisfy equations:*

$$\Phi_n^{(++)}(\bar{W}, Z) = \frac{d^{n+1}}{d\lambda^{n+1}} \Phi_{-1}(\lambda) \text{ and } \Phi_n^{(--)}(\bar{W}, Z) = \frac{d^{n+1}}{d\lambda^{n+1}} \Phi^*_{-1}(\lambda)$$

$$\tfrac{d^{n-1}}{d\lambda^{n-1}} \Phi_{-n}^{(++)}(\bar{W}, Z) = \Phi_{-1}(\lambda) \text{ and } \tfrac{d^{n-1}}{d\lambda^{n-1}} \Phi_{-n}^{(--)}(\bar{W}, Z) = \Phi^*_{-1}(\lambda)$$

The above cocycles define scalar products for the first cohomology groups of the covering of the twistor spaces $PT^{\pm}$ *by two open subsets, namely:*

1. $\Phi_{-n}^{(--)}$ *for* $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$
2. $\Phi_n^{(--)}$ *for* $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(n-2))$

3. $\Phi_{-n}^{(++)}$ *for* $H^1_{A^*B^*}(\mathbf{PT}^-, \mathcal{O}(-n-2))$
4. $\Phi_{n}^{(++)}$ *for* $H^1_{A^*B^*}(\mathbf{PT}^+, \mathcal{O}(n-2))$

The examples of the orthonormal bases for the spaces being considered are given by formulas 17, 20, 21, 22 respectively but twistors A, B, A^*, B^* are defined at the beginning of the theorem.

Proof. – It will be outlined only for the space $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$, for the other ones it is very similar. The twistors A, B, A^*, B^* define a coordinate system for $\mathbf{PT}^+$, the condition $Span\{A, B\} \in \mathbf{M}^{++}$ guarantees that the subsets U_1 and U_2 form a covering of $\mathbf{PT}^+$. The same considerations as in the previous chapter show that the elements 17 form basis for $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$ with twistors as in the assumption. The scalar product is given by the sequences of the cohomology groups and the maps described below [6, 8]. Let us identify:

$$H^1_{AB}(\bar{\mathbf{PT}}^+, \mathcal{O}(-n-2)) \sim H^1_{AB}(\bar{\mathbf{PT}}^{*-} \times \bar{\mathbf{PT}}^+, \mathcal{O}(0, -n-2)) \quad (32)$$

by means of the projection

$$\bar{\mathbf{PT}}^{*-} \times \bar{\mathbf{PT}}^+ \longrightarrow \bar{\mathbf{PT}}^+$$

and taking constant sheaf over $\bar{\mathbf{PT}}^+$. Then we have:

$$\begin{aligned} &H^2(\mathbf{PT}^{*-} \times \mathbf{PT}^-, \mathcal{O}(n-2, n-2)) \times H^1_{AB}(\bar{\mathbf{PT}}^{*-} \times \bar{\mathbf{PT}}^+, \mathcal{O}(0, -n-2)) \\ &\xrightarrow{\bullet} H^4(\bar{\mathbf{PT}}^{*-} \times \mathbf{PT}, \mathcal{O}(n-2, -4)) \\ &\xrightarrow{Kunneth} H^1(\bar{\mathbf{PT}}^{*-}, \mathcal{O}(-n-2)) \otimes H^3(\mathbf{PT}, \mathcal{O}(-4)) \\ &\xrightarrow{Serre} H^1(\bar{\mathbf{PT}}^{*-}, \mathcal{O}(n-2)) \end{aligned}$$

The first arrow means dot product $\bullet$, the second the Künneth formula and the last one means the Serre duality applied to the second term in the tensor product. The next sequence was described in the proof of statement 3. In an explicit realisation scalar product is given by the integral [6, 8, 9]:

$$\oint f(\bar{W}) \Phi_{-n}^{(--)}(\bar{W}, Z) g(Z) \mathcal{D}\bar{W} \wedge \mathcal{D}Z \quad (33)$$

where $\mathcal{D}Z$ and $\mathcal{D}\bar{W}$ are defined in 25 and integration is along an appropriate contour surrounding singularities. It is obvious that proof needs checking the above formula for the basis elements only. In this case we have:

$$\begin{aligned} \left(\mathcal{B}^{(n)}_{lk_1k_2}, \mathcal{B}^{(n)}_{l'k'_1k'_2}\right) &= \oint \mathcal{B}^{*(n)}_{lk_1k_2}(\bar{W}) \Phi_{-n}^{(--)}(\bar{W}, Z) \mathcal{B}^{(n)}_{l'k'_1k'_2}(Z) \mathcal{D}\bar{W} \wedge \mathcal{D}Z \\ &= \delta_{ll'} \delta_{k_1k'_1} \delta_{k_2k'_2} \end{aligned} \quad (34)$$

The integration can be easily accomplished after introducing new variables defined by the twistors A, B, A^*, B^*. The contour has the topology of $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$. Let me leave further details for patient reader who should notice that the intricate definitions of $\Phi_{-n}^{(++)}$ and $\Phi_{-n}^{(--)}$ do not complicate integrations because only derivatives of Φ_{-1} and Φ_{-1}^* will occur.

Remark 2. – 1. If we consider the inductive limit of cohomology groups of the covering of $\mathbf{PT}^{\pm}$ by two open subsets we see that the choice of the particular twistors which define covering does not matter [18].

2. Orthogonality of the twistors A, B, A^*, B^* is important only for the construction of the orthonormal bases.
3. From 1 and the previous investigations we see that the images of the Hilbert spaces $\mathcal{H}'_n$, $\mathcal{H}_n$ under the inverse Penrose transform are the first cohomology groups of the covering $\mathbf{PT}^{\pm}$ by two open subsets with scalar products given by $\Phi_n^{(++)}$, $\Phi_{-n}^{(++)}$, $\Phi_n^{(--)}$, $\Phi_{-n}^{(--)}$ for appropriate spaces.

5. THE REPRODUCING KERNELS FOR HILBERT SPACES OF MASSLES PARTICLES

If we return to the construction of scalar products we see that the reproducing kernels have been found as well. As usual let us restrict our considerations to the space $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$. Because we know the example of an orthonormal basis we will deal with the basis elements $\mathcal{B}_{lk_1k_2}^{(n)}$ only. I claim that $\Phi_n^{(++)}$ is the reproducing kernel for the Hilbert space $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$, where the scalar product is given by $\Phi_{-n}^{(--)}$. To prove this one has to show that:

$$\left(\Phi_n^{(++)}(\bar{Z}, \cdot), \mathcal{B}_{lk_1k_2}^{(n)}\right) = \mathcal{B}_{lk_1k_2}^{(n)} \tag{35}$$

which means just the reproducing property of $\Phi_n^{(++)}$. Explicitly we have:

$$\oint \oint \Phi_n^{(++)}(\bar{W}, Z)\Phi_{-n}^{(--)}(\bar{W}, S)\mathcal{B}_{lk_1k_2}^{(n)}(S)\mathcal{D}S \wedge \mathcal{D}\bar{W} =$$
$$= \oint \Phi_n^{(++)}(\bar{W}, Z)\left[\oint \Phi_{-n}^{(--)}(\bar{W}, S)\mathcal{B}_{lk_1k_2}^{(n)}(S)\mathcal{D}S\right]\mathcal{D}\bar{W}$$

performing the inmost integration leads to the expression:

$$\oint \Phi_n^{(++)}(\bar{W}, Z)\mathcal{B}_{lk_1k_2}^{*(-n)}(\bar{W})\mathcal{D}\bar{W}$$

which equals $\mathcal{B}^{(n)}_{lk_1k_2}$ and proves the claim. All calculations are easy to perform and the choice of the contour follows from the form of singularities of the basis elements. Moreover, the inmost integration shows that bases $\left\{\mathcal{B}^{(n)}_{lk_1k_2}\right\}$ and $\left\{\mathcal{B}^{(-n)}_{lk_1k_2}\right\}$ are dual to each other in the sense of linear algebra which coincides with remark 1. The cohomological interpretation of the above integration is described by the sequences analogous to the ones given on page 10.

Let us introduce the notation: $H^1_c(\mathbf{PT}^+, \mathcal{O}(-n-2))$ for the image of $H^1_{AB}(\mathbf{PT}^+, \mathcal{O}(-n-2))$ in the inductive limit of the covering of $\mathbf{PT}^+$ by two open subsets. For the remaining cohomology groups the mark "c" will mean exactly the same. In this place we can summarize previous considerations:

THEOREM 2. – *The following gives a table of isomorphisms induced by the Penrose transform*

Hilbert space	Scalar product	Reprod. kernel	Dual space	Isomorphic to
$H^1_c(\mathbf{PT}^+, \mathcal{O}(-n-2))$	$\Phi^{(--)}_{-n}$	$\Phi^{(++)}_{n}$	$H^1_c(\mathbf{PT}^-, \mathcal{O}(n-2))$	$\mathcal{H}'_n(\mathbf{M}^{++})$
$H^1_c(\mathbf{PT}^-, \mathcal{O}(-n-2))$	$\Phi^{(++)}_{-n}$	$\Phi^{(--)}_{n}$	$H^1_c(\mathbf{PT}^+, \mathcal{O}(n-2))$	$\mathcal{H}'_n(\mathbf{M}^{--})$
$H^1_c(\mathbf{PT}^+, \mathcal{O}(n-2))$	$\Phi^{(--)}_{n}$	$\Phi^{(++)}_{-n}$	$H^1_c(\mathbf{PT}^-, \mathcal{O}(-n-2))$	$\mathcal{H}_n(\mathbf{M}^{++})$
$H^1_c(\mathbf{PT}^-, \mathcal{O}(n-2))$	$\Phi^{(++)}_{n}$	$\Phi^{(--)}_{-n}$	$H^1_c(\mathbf{PT}^+, \mathcal{O}(-n-2))$	$\mathcal{H}_n(\mathbf{M}^{++})$

Proof. – It is obvious in the context of the previous considerations, where all sufficient calculations were described.

The Hilbert spaces from the last column possess reproducing kernels which were investigated in [12]. How to obtain them from the reproduction kernels for the Hilbert spaces H^1_c by contour integration (or alternatively by means of the Penrose transform) will be shown in a forthcoming publication. There will be given some applications of the obtained results to the quantisation of classical systems of massless particles. Some preliminary suggestions may be found find in [14].

6. CONCLUDING REMARKS

In the previous chapter I have described reproducing kernels for H^1_c spaces. Now it is time to compare them with twistor propagators which were investigated in [6, 8]. Their authors characterized them (up to a scale) by equations:

$$Z^{\alpha}\frac{\partial\Phi_n}{\partial Z^{\alpha}} = \lambda\Phi_{n+1}$$

$$\bar{W}^{\alpha}\frac{\partial\Phi_n}{\partial\bar{Z}^{\alpha}} = \lambda\Phi_{n+1}$$

where left and right sides are the representatives of the cohomology classes. Do the reproducing kernels presented here satisfy the above conditions? The answer is affirmative. Indeed, elementary calculation gives the result:

$$Z^{\alpha}\frac{\partial\Phi_n}{\partial Z^{\alpha}} = \lambda\Phi_{n+1} + \frac{(n+1)!}{(x-\lambda)^{n+2}} + \frac{(n+1)!}{(y-\lambda)^{n+2}}$$

where λ, x, y are given in 29 for A, B from theorem 1. The last two terms are cohomologicaly trivial. For the remaining kernels calculations are exactly the same. I have approached twistor propagators in a different way and my motivation for finding them was different from that presented in the quoted papers, so the representatives must be different as well. I admit that, in order to achieve my goal I used the cohomologiacal methods presented in [6, 8, 9].

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