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The Hawking effect

by

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ABSTRACT. – We consider a spherical star, stationary in the past, collapsing to a Black-Hole in the future. Assuming the quantum state of a Klein-Gordon field to be the Fock vacuum in the past, we prove that an observer at rest in the Schwarzschild coordinates, will measure a thermal state with the Hawking temperature, at the last time of the gravitational collapse. © Elsevier, Paris

RÉSUMÉ. – On considère une étoile sphérique, stationnaire dans le passé, s'effondrant en un trou noir dans le futur. Supposant qu'un champ quantique de Klein-Gordon est dans l'état du vide de Fock dans le passé, on prouve qu'un observateur au repos en coordonnées de Schwarzschild, mesurera, en temps infini, un état thermal à la température Hawking. © Elsevier, Paris

I. INTRODUCTION

The aim of this paper is to give a rigorous mathematical proof of the famous result by S. Hawking [21], on the emergence of a thermal state at the last moment of a gravitational collapse. In a previous paper [6], we proved that an observer infalling across the Black-Hole horizon, measures a thermal radiation of particules outgoing from the Black-Hole to infinity. In the present work we consider the case of an observer at rest with respect to the Schwarzschild coordinates.

We recall that the space-time exterior of a spherical star with mass $M > 0$, and radius $z(t)$ in the Regge-Wheeler coordinate, stationary in the past, collapsing to a Black-Hole in the future, is described in Schwarzschild coordinates by the globally hyperbolic manifold

$$\mathcal{M} = \left\{(t, r_*, \omega) \in \mathbb{R}_t \times]z(t), \infty[_{r_*} \times S^2_\omega\right\}, \; r_* = r + 2M \ln(r - 2M), \tag{I.1}$$

$$g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{I.2}$$

Denoting

$$\kappa = \frac{1}{4M} \tag{I.3}$$

the surface gravity, z satisfies (see [5])

$$\begin{gathered} z \in C^2(\mathbb{R}), \;\; \forall t \in \mathbb{R}, \;\; z(t) < -t, \\ -1 < \dot{z}(t) \leq 0, \;\; t \leq 0 \Rightarrow z(t) = z(0) < 0, \end{gathered} \tag{I.4}$$

$$\begin{gathered} z(t) = -t - Ae^{-2\kappa t} + \zeta(t), \;\; A > 0, \\ \mid \zeta(t) \mid + \mid \dot{\zeta}(t) \mid = O(e^{-4\kappa t}), \;\; t \to +\infty. \end{gathered} \tag{I.5}$$

We consider the hyperbolic mixed problem for the Klein-Gordon equation of mass $m \geq 0$, in $\mathcal{M}$:

$$\mid g \mid^{-1} \partial_\mu(\mid g \mid g^{\mu\nu} \partial_\nu \psi) + m^2 \psi = 0,$$

which, in the Schwarzschild metric (I.2), takes the form:

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2} + m^2\right)\right\}\psi = 0 \; in \; \mathcal{M}, \tag{I.6}$$

$$\psi(t, r_* = z(t), \omega) = 0, \;\; t \in \mathbb{R}, \;\; \omega \in S^2, \tag{I.7}$$

$$\Psi(t = s, r_*, \omega) = \Phi(r_*, \omega), \tag{I.8}$$

where we have put:

$$\Psi = {}^t(\psi, \partial_t \psi).$$

We proved in [5] that the solution $\Psi(t, .) = U(t, s)\Phi$ is determined by a propagator $U(t, s)$ which is strongly continuous on the family of Hilbert spaces $\mathcal{H}(t)$ of finite energy fields:

$$\mathcal{H}(t) = \left[D(\mathbb{H}_t^{\frac{1}{2}})\right] \times L_t^2. \tag{I.9}$$

Here $\mathbb{H}_t$ is the selfadjoint operator on the space L_t^2:

$$L_t^2 \equiv L^2(]z(t), \infty[_{r_*} \times S_\omega^2, r^2 dr_* d\omega), \tag{I.10}$$

$$\mathbb{H}_t = -\frac{1}{r^2}\frac{\partial}{\partial r_*}r^2\frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2} + m^2\right), \tag{I.11}$$

with domain

$$D(\mathbb{H}_t) = \{f \in L_t^2,\ \mathbb{H}_t f \in L_t^2,\ f((z(t), \omega) = 0\}. \tag{I.12}$$

and, in this paper, $[D(\mathbb{K})]$ denotes the closure of the domain $D(\mathbb{K})$ of an operator $\mathbb{K}$ on a Hilbert space H for the norm $\| \mathbb{K}(\cdot) \|_H$.

We now give a brief formal description of the Hawking effect. The fundamental space for the Quantum Field Theory turns out to be:

$$\mathcal{H}^{\frac{1}{2}}(t) \equiv \left[D(\mathbb{H}_t^{\frac{1}{4}})\right] \times \left[D(\mathbb{H}_t^{-\frac{1}{4}})\right]. \tag{I.13}$$

The *quantum vacuum state* at time t is defined by the generating functional

$$\Phi_t \in \mathcal{H}^{\frac{1}{2}}(t) \longrightarrow E_t^0(\Phi_t) = \exp\left(-\frac{1}{2} \| \Phi_t \|^2_{\mathcal{H}^{\frac{1}{2}}(t)}\right). \tag{I.14}$$

More generally, a *thermal quantum state* with temperature $\theta > 0$ with respect to a hamiltonian $\mathbb{H} > 0$ is defined (see e.g. [9]) by the generating functional

$$\Phi \longrightarrow E^\theta(\Phi) = \exp\left(-\frac{1}{2}\left\|\sqrt{\coth\left(\frac{1}{2\theta}\mathbb{H}^{\frac{1}{2}}\right)}\Phi\right\|^2_{[D(\mathbb{H}^{\frac{1}{4}})]\times[D(\mathbb{H}^{-\frac{1}{4}})]}\right). \tag{I.15}$$

The *ground quantum state* in $\mathcal{M}$ is defined by the functionals $\mathcal{E}_t$:

$$\Phi_t \in (C_0^2 \times C_0^1)(]z(t), \infty[_{r_*} \times S_\omega^2) \longrightarrow \mathcal{E}_t(\Phi_t) \equiv E_0^0\Big(U(0,t)\Phi_t\Big).$$

Since the star is stationary in the past, $U(0, t)$ is unitary for $t < 0$, and the ground state is just the vacuum state in the past:

$$t \leq 0 \Rightarrow \mathcal{E}_t(\Phi_t) = E_t^0(\Phi_t).$$

In order to investigate the structure of the ground state in the future, we take Φ^0 in a dense subspace $\mathcal{D}$ of $[C_0^\infty(]z(0), \infty[_{r_*} \times S_\omega^2)]^2$. The fundamental problem is then to evaluate

$$\lim_{T\to+\infty} \| U(0,T)\Phi^0 \|_{\mathcal{H}^{\frac{1}{2}}(0)} \,. \tag{I.16}$$

To make clear the origin of the Hawking effect, we now explain this phenomenon on a "toy model" where all the calculations can be explicitely made. This model is defined by

$$\partial_t^2 u - \partial_x^2 u = 0, \;\; t \in \mathbb{R}, \;\; z(t) < x,$$

$$u(t, x = z(t)) = 0.$$

Note that this very simple model is physically relevant: it describes the dynamics of the radial component of the first mode of the electromagnetic tensor field on the Schwarzschild metric [2]. We denote $U_*(t,s)$ the propagator associated with this mixed hyperbolic problem, and we introduce the operators:

$$\mathbb{H}_{\leftarrow/\rightarrow} = -\frac{d^2}{dx^2}$$

with domains

$$D(\mathbb{H}_{\rightarrow}) = \{u \in L^2([-A, \infty[); \; u" \in L^2([-A,\infty[), \; u(-A) = 0\},$$

$$D(\mathbb{H}_{\leftarrow}) = \{u \in L^2(\mathbb{R}); \; u" \in L^2(\mathbb{R})\}.$$

We take:

$$\Phi_* \in \mathcal{D}_* = \left\{ {}^t(f,p); \; f, p \in C_0^\infty(]-A, \infty[), \; \int p = 0 \right\},$$

and we have to study:

$$\lim_{T\to\infty} \| U_*(0,T)\Phi_* \|_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}})]} .$$

First there exist unique $f_{\leftarrow}, f_{\rightarrow} \in C_0^\infty(]-A,\infty[)$ such that:

$$\Phi_* = \Omega_{\leftarrow}^- \Phi_* + \Omega_{\rightarrow}^- \Phi_*,$$

where $\Omega_{\leftarrow}^-$ and $\Omega_{\rightarrow}^-$ play the role of wave operators by splitting the field in leftgoing and rightgoing parts:

$$\Omega_{\leftarrow}^- \Phi_* = {}^t(f_{\leftarrow}, -f'_{\leftarrow}), \;\; \Omega_{\rightarrow}^- \Phi_* = {}^t(f_{\rightarrow}, f'_{\rightarrow}).$$

On the one hand, we have for $T > 0$:

$$\left(U_*(0,T)\Omega_{\rightarrow}^- \Phi_*\right)(x) = \left(\Omega_{\rightarrow}^- \Phi_*\right)(x - T),$$

hence for $\varepsilon \geq 0$:

$$\| U_*(0,T)\Omega^-_{\rightarrow}\Phi_* \|_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}+\varepsilon})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}+\varepsilon})]} \xrightarrow[T\to+\infty]{} \| \Omega^-_{\rightarrow}\Phi_* \|_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}+\varepsilon})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}+\varepsilon})]} . \tag{I.17}$$

On the other hand we easily check (see [5]), that for $T > 0$ large enough:

$$U_*(0,T)\Omega^-_{\leftarrow}\Phi_* = {}^t(f_T, f'_T),$$

$$f_T(x) = -f_{\leftarrow}(T + x - 2\tau(x)),$$

where $\tau(x)$ is defined by:

$$z(\tau(x)) = x - \tau(x), \quad z(0) \leq x < 0.$$

Now we specify function $z(t)$ for t large enough, by choosing:

$$z(t) = -t - Ae^{-2\kappa t} + Ae^{-2\kappa t}\varphi\big(Ae^{-2\kappa t}\big), \quad 0 < A, \quad 0 < \kappa,$$

where $\varphi(x)$ is the local solution of:

$$\frac{\ln(1-\varphi(x))}{1-\varphi(x)} = -\kappa x, \quad \varphi(0) = 0.$$

Then we have:

$$f_T(x) = -f_{\leftarrow}\left(T + \frac{1}{\kappa}\ln(-x) - \frac{1}{\kappa}\ln(A)\right),$$

and an explicit calculation by Fourier transform gives the fundamental estimate:

$$\begin{aligned} &\| U_*(0,T)\Omega^-_{\leftarrow}\Phi_* \|_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}})]} \\ &= \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{\leftarrow}^{\frac{1}{2}}\right)}\Omega^-_{\leftarrow}\Phi_* \right\|_{[D(\mathbb{H}_{\leftarrow}^{\frac{1}{4}})]\times[D(\mathbb{H}_{\leftarrow}^{-\frac{1}{4}})]}, \end{aligned} \tag{I.18}$$

$$0 < \varepsilon \Rightarrow \| U_*(0,T)\Omega^-_{\leftarrow}\Phi_* \|_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}-\varepsilon})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}-\varepsilon})]} \xrightarrow[T\to+\infty]{} 0. \tag{I.19}$$

We immediately conclude from (I.17), (I.18) and (I.19) that:

$$\begin{aligned} &\lim_{T\to\infty} \| U_*(0,T)\Phi_* \|^2_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}})]} \\ &= \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{\leftarrow}^{\frac{1}{2}}\right)}\Omega^-_{\leftarrow}\Phi_* \right\|^2_{[D(\mathbb{H}_{\leftarrow}^{\frac{1}{4}})]\times[D(\mathbb{H}_{\leftarrow}^{-\frac{1}{4}})]} + \| \Omega^-_{\rightarrow}\Phi_* \|^2_{[D(\mathbb{H}_{\rightarrow}^{\frac{1}{4}})]\times[D(\mathbb{H}_{\rightarrow}^{-\frac{1}{4}})]} . \end{aligned}$$

The Hawking effect is expressed by the second term which is characteristic of a "rightgoing" thermal radiation at temperature $\frac{\kappa}{2\pi}$.

In the case of the Klein-Gordon field outside a collapsing star, the situation is much more complicated because of the space curvature, the mass of the field, and the perturbation $\zeta(t)$ in the collapse function z (I.5). But the origine of the Hawking radiation is essentialy the same, and with many subtle technical steps, of Scattering Theory type, the problem is reduced to the previous one. Our proof is based on a sharp study of the backward propagator, and we use some results from our previous works: the functional framework for quantum fields and the study of the quantum state near the Black-Hole horizon [6], the analysis of the infinite Doppler effect [5], the modified wave operators for a long range type interaction [4], the analyticity of the gravitational potential [7]. We briefly describe our approach. To investigate (I.16) we consider the part of the field far from the star, $(1-\chi)U(0,T)\Phi^0$, and the part of the field near the star, $\chi U(0,T)\Phi^0$, where χ is a smooth cut-off function equal to 1 for $r_* \leq \frac{1}{2}$, and equal to 0 for $r_* \geq 1$. Firstly, thanks to the hyperbolicity, $(1-\chi)U(0,T)\Phi^0 = (1-\chi)U(-T)\Phi^0$ where $U(t)$ is the unitary group associated with the Klein-Gordon equation on the whole Schwarzschild space-time. Then we can use the scattering theory for an eternal black-hole developped in [4]: we introduce formally the Wave Operator at infinity:

$$\Omega_\infty^\pm \Phi = \lim_{t\to\pm\infty} U_\infty^D(-t)(1-\chi)U(t)\Phi, \tag{I.20}$$

where $U_\infty^D(t)$ is the Dollard modified propagator associated with the Klein-Gordon equation in the Minkowski space-time ([4]):

$$\partial_t^2 u + \mathbb{H}_\infty u = 0, \tag{I.21}$$

$$\mathbb{H}_\infty = -\Delta_{\mathbb{R}_x^3} + m^2. \tag{I.22}$$

Then we prove that:

$$\lim_{T\to+\infty} \| (1-\chi)U(0,T)\Phi^0 \|_{\mathcal{H}^{\frac{1}{2}}(0)} = \| \Omega_\infty^- \Phi^0 \|_{[D(\mathbb{H}_\infty^{\frac{1}{4}})]\times[D(\mathbb{H}_\infty^{-\frac{1}{4}})]}. \tag{I.23}$$

The second step, the estimate of $\| \chi U(0,T)\Phi^0 \|_{\mathcal{H}^{\frac{1}{2}}(0)}$, is much more delicate. We remark that $\chi U(0,T)\Phi^0$ is entirely determined by the field equation, the boundary condition, a causality condition on the support of $U(0,T)\Phi^0$, and the trace $\varphi_T(s,\omega)$ of $[U(s,T)\Phi^0]_1$ on $\gamma = \{(s, r_* = 1-s, \omega);\ (s,\omega) \in \mathbb{R}\times S^2\}$. Therefore we investigate the Hyperbolic Mixed Characteristic Problem where u is solution of (I.6) in

$\mathcal{O} \equiv \{(t, r_*, \omega);\ (t,\omega) \in [0,\infty[\times S^2,\ \ z(t) \le r_* \le 1-t\}$, $u(t, r_*, \omega) = 0$ in $\mathcal{O}$ for t large enough, $u(t, z(t), \omega) = 0$, and $u = \varphi$ given in $H^1(\gamma)$. Using the tools of [6], we show that the map $\varphi \mapsto (\chi u(t=0), \chi\partial_t u(t=0))$ is well defined from $H^1(\gamma)$ into $\mathcal{H}^{\frac{1}{2}}(0)$. The next crucial point is that

$$\varphi_T(s,\omega) \sim \varphi^-(s-T,\omega),\ \ T \to +\infty. \tag{I.24}$$

To get φ^- we construct the Black-Hole Horizon Wave Operators

$$\Omega^{\pm}_{BH}\Phi = \lim_{t\to\pm\infty} U_{BH}(-t)\chi U(t)\Phi, \tag{I.25}$$

where $U_{BH}(t)$ is the group associated with the asymptotic dynamics at the horizon:

$$\partial_t^2\Psi + \mathbb{H}_{BH}\Psi = 0,\ \ t\in\mathbb{R},\ \ r_*\in\mathbb{R},\ \ \omega\in S^2, \tag{I.26}$$

$$\mathbb{H}_{BH} = -\partial^2_{r_*}. \tag{I.27}$$

The strong asymptotic completeness of these operators is a consequence of the properties of analyticity of the map $r_* \mapsto r$ established in [7] and we have

$$\varphi^-(s,\omega) = \left[\Omega^-_{BH}\Phi^0\right]_1(r_* = -2s+1,\omega). \tag{I.28}$$

Then we reach the key point of the proof of the Hawking effect: this is the following fundamental identity:

$$\lim_{T\to+\infty} \| \chi U(0,T)\Phi^0 \|_{\mathcal{H}^{\frac{1}{2}}(0)} = \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\Omega^-_{BH}\Phi^0 \right\|_{[D(\mathbb{H}^{\frac{1}{4}}_{BH})]\times[D(\mathbb{H}^{-\frac{1}{4}}_{BH})]}. \tag{I.29}$$

From this we conclude that

$$\lim_{T\to+\infty} \| U(0,T)\Phi^0 \|^2_{\mathcal{H}^{\frac{1}{2}}(0)} = \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\Omega^-_{BH}\Phi^0 \right\|^2_{[D(\mathbb{H}^{\frac{1}{4}}_{BH})]\times[D(\mathbb{H}^{-\frac{1}{4}}_{BH})]} + \| \Omega^-_{\infty}\Phi^0 \|^2_{[D(\mathbb{H}^{\frac{1}{4}}_{\infty})]\times[D(\mathbb{H}^{-\frac{1}{4}}_{\infty})]}. \tag{I.30}$$

This is the main mathematical result of this work (Theorem III.3). Therefore, taking account of (I.15), and noting that:

$$\left[\Omega_{BH}^{-}\Phi^{0}\right]_{1}' + \left[\Omega_{BH}^{-}\Phi^{0}\right]_{2} = 0, \tag{I.31}$$

(I.30) means that an observer at rest in the Schwarzschild coordinates measures as $T \to +\infty$, a thermal radiation at temperature $\frac{1}{8\pi M}$ of particules outgoing from the Black-Hole the infinity. It is the exact corroboration of S. Hawking's analysis [21] (among a huge litterature, see e.g. Candelas [10], Fredenhagen und Haag [17], Gibbons and Hawking [19], Sewell [27], [28], Unruh [29], Wald [30], York [32], and see also the references in the classic monographs on quantum field theory in curved space-time by Birrel and Davies [8], DeWitt [12], Fulling [18], Grib, Mamayev and Mostepanenko [20], Wald [31], as well as the volume [1]).

The paper is organized as follows. Taking advantage of the spherical invariance, we can reduce the problem to solving an equation in one space dimension, which we investigate in the second part. Then we get the crucial asymptotic behaviour for the three dimensional problem in the third part, and we prove the Hawking effect in part 4.

II. ONE DIMENSIONAL COLLAPSE

We consider the hyperbolic mixed problem:

$$\partial_t^2 u - \partial_x^2 u + Vu = 0,\ t \in \mathbb{R}^+,\ z(t) < x, \tag{II.1}$$

$$u(t, x = z(t)) = 0,\ t \in \mathbb{R}^+, \tag{II.2}$$

where the function z satisfies (I.5) and the potential V is such that there exist $m \geq 0$, $\mu \in \mathbb{R}$, $\varepsilon > 0$ with

$$\begin{cases} V \in C^\infty(\mathbb{R}),\ 0 \leq V(x), \\ \lim_{t\to\infty}\left(e^{(\kappa+\varepsilon)t}\sup\{V(x) : x \leq -t\}\right) = 0, \\ \int_0^\infty e^{-2\kappa t}\sup\{\max(-V'(x), 0);\ z(t) \leq x \leq -t\}\,dt < \infty, \\ V(x) = m^2 + \frac{\mu}{x} + O(x^{-1-\varepsilon}),\ x \to +\infty, \end{cases} \tag{II.3}$$

and $\kappa > 0$ is given by (I.5). Given $V \in L^\infty(\mathbb{R})$, $V \geq 0$, and an interval $I \subset \mathbb{R}$, we introduce the positive selfadjoint operator $\mathbb{H}_{V,I}$ on $L^2(I)$ with dense domain $D(\mathbb{H}_{V,I})$, defined by:

$$\mathbb{H}_{V,I} = -\partial_x^2 + V, \tag{II.4}$$

$$D(\mathbb{H}_{V,I}) = \left\{ f \in L^2(I);\ \mathbb{H}_{V,I} f \in L^2(I) \text{ and } x \in \partial I \Rightarrow f(x) = 0 \right\}. \quad \text{(II.5)}$$

We recall that $\left[D(\mathbb{H}^s_{V,I})\right]$ is the closure of $D(\mathbb{H}^s_{V,I})$ for the norm $\| \mathbb{H}^s_{V,I} . \|_{L^2(I)}$ and we define: the Hilbert spaces

$$\mathcal{H}(V,I) = \left[D(\mathbb{H}^{\frac{1}{2}}_{V,I})\right] \times L^2(I), \quad \text{(II.6)}$$

$$\mathcal{H}^{\frac{1}{2}}(V,I) = \left[D(\mathbb{H}^{\frac{1}{4}}_{V,I})\right] \times \left[D(\mathbb{H}^{-\frac{1}{4}}_{V,I})\right], \quad \text{(II.7)}$$

and the group $U_{V,I}(t)$ unitary on $\mathcal{H}(V,I)$ and $\mathcal{H}^{\frac{1}{2}}(V,I)$

$$U_{V,I}(t) = \begin{bmatrix} \cos\left(t\mathbb{H}^{\frac{1}{2}}_{V,I}\right) & \mathbb{H}^{-\frac{1}{2}}_{V,I} \sin\left(t\mathbb{H}^{\frac{1}{2}}_{V,I}\right) \\ -\mathbb{H}^{\frac{1}{2}}_{V,I} \sin\left(t\mathbb{H}^{\frac{1}{2}}_{V,I}\right) & \cos\left(t\mathbb{H}^{\frac{1}{2}}_{V,I}\right) \end{bmatrix}. \quad \text{(II.8)}$$

For simplicity we put:

$$\begin{aligned} \mathbb{H}_{V,t} &= \mathbb{H}_{V,[z(t),\infty[}, \ \mathcal{H}(V,t) \\ &= \mathcal{H}(V,[z(t),\infty[), \ \mathcal{H}^{\frac{1}{2}}(V,t) = \mathcal{H}^{\frac{1}{2}}(V,[z(t),\infty[). \end{aligned} \quad \text{(II.9)}$$

We recall (see [5]) that the mixed problem (II.1), (II.2) is solved by a propagator $U_V(t,s)$:

$$\begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} = U_V(t,s) \begin{pmatrix} u(s) \\ \partial_t u(s) \end{pmatrix},$$

which is well defined and strongly continuous on the family $\mathcal{H}(V,t)$. We denote $H^1(I)$ the Sobolev space defined as the completion of $C_0^\infty(\bar{I})$ for the norm:

$$\| f \|^2_{H^1(I)} = \int_I | f'(x) |^2 + | f(x) |^2 \, dx.$$

Eventually we shall need the subspace:

$$\mathcal{D}_V \equiv \left(D(\mathbb{H}^{-\frac{1}{2}}_{V,\mathbb{R}}) \times D(\mathbb{H}^{-1}_{V,\mathbb{R}}) \right) \cap \left[C_0^\infty(\mathbb{R})\right]^2. \quad \text{(II.10)}$$

This space is in fact sufficiently large:

PROPOSITION II.1. – *For any potential* $V \geq 0$, $V \in C^\infty \cap L^\infty(\mathbb{R})$, $\mathcal{D}_V$ *is a dense subspace of* $\mathcal{H}(V,\mathbb{R}) \cap \mathcal{H}^{\frac{1}{2}}(V,\mathbb{R})$.

In order to investigate the asymptotic behaviour of the dynamics $U_{V,\mathbb{R}}$, we choose a cut-off function $\chi(x)$ such that:

$$\begin{gathered}\chi \in C^\infty(\mathbb{R}),\ \exists a,b;\ 0<a<b<1;\\ x<a \Rightarrow \chi(x)=1,\ b<x \Rightarrow \chi(x)=0,\end{gathered} \tag{II.11}$$

and we introduce the Wave Operators

$$\Omega^{\pm}_{0,\mathbb{R}}F = \lim_{t\to\pm\infty} U_{0,\mathbb{R}}(-t)\boldsymbol{\chi} U_{V,\mathbb{R}}(t)F \ \ in \ \ H^1(\mathbb{R})\times L^2(\mathbb{R}), \tag{II.12}$$

$$\begin{aligned}&\Omega^{\pm}_{V,[z(0),\infty[}F\\ &= \lim_{t\to\pm\infty} U_{V,[z(0),\infty[}(-t)(\mathbf{1}-\boldsymbol{\chi})U_{V,\mathbb{R}}(t)F \ \ in \ \ \mathcal{H}(V,0)\cap\mathcal{H}^{\frac{1}{2}}(V,0),\end{aligned} \tag{II.13}$$

where, given some intervals $I\subset J\subset\mathbb{R}$ and $\varphi\in C^\infty(J)$, $\boldsymbol{\varphi}$ denotes the operator

$$\boldsymbol{\varphi}:\ {}^t(f,p)\in\mathcal{D}'(J)\times\mathcal{D}'(J)\longmapsto{}^t(\varphi f,\varphi p)\in\mathcal{D}'(I)\times\mathcal{D}'(I).$$

The scattering of classic fields is described in our functional framework by the following:

THEOREM II.2. – *Given a potential V satisfying assumption* (II.3), *for any $F={}^t(f,p)\in\mathcal{D}_V$, $F(x)=0$ for $x<R$, $x\in\mathbb{R}$, the strong limits $\Omega^{\pm}_{0,\mathbb{R}}F$, $\Omega^{\pm}_{V,[z(0),\infty[}F$ exist and are independent of χ satisfying* (II.11). *Moreover we have:*

$$x<R\Rightarrow\Omega^{\pm}_{0,\mathbb{R}}F(x)=0, \tag{II.14}$$

$$\left[\Omega^{\pm}_{0,\mathbb{R}}F\right]_1' = \pm\left[\Omega^{\pm}_{0,\mathbb{R}}F\right]_2, \tag{II.15}$$

$$\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\right)}\,\Omega^{\pm}_{0,\mathbb{R}}F\in\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R}), \tag{II.16}$$

$$\|\left[\Omega^{\pm}_{0,\mathbb{R}}F\right]_1\|_{L^2(\mathbb{R})}\leq\|f\|_{L^2(\mathbb{R})}+\|\mathbb{H}^{-\frac{1}{2}}_{V,\mathbb{R}}p\|_{L^2(\mathbb{R})}, \tag{II.17}$$

$$\|\Omega^{\pm}_{0,\mathbb{R}}F\|^2_{\mathcal{H}(0,\mathbb{R})}+\|\Omega^{\pm}_{V,[z(0),\infty[}F\|^2_{\mathcal{H}(V,0)}=\|F\|^2_{\mathcal{H}(V,\mathbb{R})}, \tag{II.18}$$

$$\|\Omega^{\pm}_{0,\mathbb{R}}F\|^2_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})}+\|\Omega^{\pm}_{V,[z(0),\infty[}F\|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)}=\|F\|^2_{\mathcal{H}^{\frac{1}{2}}(V,\mathbb{R})}. \tag{II.19}$$

Now we can state the key result of this part:

THEOREM II.3. – *Given a potential V satisfying assumption* (II.3), *for all F in $\mathcal{D}_V$ we have:*

$$\| U_V(0,T)F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[T\to+\infty]{} \| \Omega^-_{V,[z(0),\infty[} F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} + \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\right)} \Omega^-_{0,\mathbb{R}} F \right\|^2_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})}. \tag{II.20}$$

Remark II.4. – We note that this limit does not depend on the function z.

Proof of Proposition II.1. – Since $V \in C^\infty(\mathbb{R})$, we have $C_0^\infty(\mathbb{R}) \subset D(\mathbb{H}^n_{V,\mathbb{R}})$ for any $n \in \mathbb{N}$, whence by interpolation

$$C_0^\infty(\mathbb{R}) \subset \bigcap_{s\geq 0} D(\mathbb{H}^s_{V,\mathbb{R}}). \tag{II.21}$$

Then $\mathcal{D}_V$ is a subspace of $\mathcal{H}(V,\mathbb{R})$. Moreover, since

$$D(\mathbb{H}^s_{V,\mathbb{R}}) \subset D(\mathbb{H}^t_{V,\mathbb{R}}), \quad s \leq t \leq 0,$$

we have

$$\mathcal{D}_V \subset \bigcap_{s\geq -1} D(\mathbb{H}^{s+\frac{1}{2}}_{V,\mathbb{R}}) \times D(\mathbb{H}^s_{V,\mathbb{R}}) \tag{II.22}$$

and it is also a subset of $\mathcal{H}^{\frac{1}{2}}(V,\mathbb{R})$. On the other hand we have:

$$\mathbb{H}_{V,\mathbb{R}} C_0^\infty(\mathbb{R}) \subset D(\mathbb{H}^{-1}_{V,\mathbb{R}}) \cap C_0^\infty(\mathbb{R}) \subset D(\mathbb{H}^{-\frac{1}{2}}_{V,\mathbb{R}}) \cap C_0^\infty(\mathbb{R}),$$

thus:

$$(\mathbb{H}_{V,\mathbb{R}} C_0^\infty(\mathbb{R}))^2 \subset \mathcal{D}_V.$$

Now we consider the space $\left[\left[D(\mathbb{H}^s_{V,\mathbb{R}})\right]\right]$ completion of $D(\mathbb{H}^s_{V,\mathbb{R}})$ in the graph norm

$$\| \mathbb{H}^s_{V,\mathbb{R}}(.) \|^2_{L^2(\mathbb{R})} + \| \cdot \|^2_{L^2(\mathbb{R})} \cdot$$

For $s = \frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{4}$, let g in $\left[\left[D(\mathbb{H}^s_{V,\mathbb{R}})\right]\right]$ be such that

$$\forall f \in C_0^\infty(\mathbb{R}) \quad < \mathbb{H}_{V,\mathbb{R}} f, g >_{[[D(\mathbb{H}^s_{V,\mathbb{R}})]]} = 0.$$

Then:

$$\left(\mathbb{H}^{2s}_{V,\mathbb{R}} + 1\right)\mathbb{H}_{V,\mathbb{R}} g = 0, \ in\ \mathcal{D}'(\mathbb{R}).$$

Since V is non negative and $g \in L^2(\mathbb{R})$ we conclude that $g = 0$, hence $\mathbb{H}_{V,\mathbb{R}}(C_0^\infty(\mathbb{R}))$ is dense in $\left[\left[D(\mathbb{H}^s_{V,\mathbb{R}})\right]\right]$, and therefore also in $\left[D(\mathbb{H}^s_{V,\mathbb{R}})\right]$.

Q.E.D.

Proof of Theorem II.2. – Given two densely defined self-adjoint operators $\mathbb{K}_1$ and $\mathbb{K}_2$ on $L^2(I)$ we define formally the wave operator

$$\Omega^{\pm}(\mathbb{K}_2, \mathbb{K}_1)f = \lim_{t\to\pm\infty} e^{-it\mathbb{K}_2}e^{it\mathbb{K}_1}P_{ac}(\mathbb{K}_1)f \ \ in \ \ L^2(I), \tag{II.23}$$

where $P_{ac}(\mathbb{K})$ is the projector onto the absolutely continuous subspace of $\mathbb{K}$. We recall that if $f \in D(\mathbb{K}_1^s)$, then $\Omega^{\pm}(\mathbb{K}_2, \mathbb{K}_1)f \in D(\mathbb{K}_2^s)$ and we have the intertwining relation

$$\mathbb{K}_2^s\Omega^{\pm}(\mathbb{K}_2, \mathbb{K}_1)f = \Omega^{\pm}(\mathbb{K}_2, \mathbb{K}_1)\mathbb{K}_1^s f. \tag{II.24}$$

We fix $R \in \mathbb{R}$. First we note that

$$\left(\mathbb{H}_{V,]-\infty,R]} \oplus \mathbb{H}_{V,[R,\infty[} + i\right)^{-1} - \left(\mathbb{H}_{V,\mathbb{R}} + i\right)^{-1}$$

is of finite rank and thus trace class. Hence the Birman-Kuroda theorem ([26], theorem XI.9) and the invariance principle ([26], theorem XI.11) yield that the wave operators

$$\Omega^{\pm}\left(\mathbb{H}^{\frac{1}{2}}_{V,]-\infty,R]} \oplus \mathbb{H}^{\frac{1}{2}}_{V,[R,\infty[}\,,\ \mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}\right)$$

exist and are complete. Secondly, we have

$$\begin{aligned}&\left(\mathbb{H}_{V,]-\infty,R]} + i\right)^{-1} - \left(\mathbb{H}_{0,]-\infty,R]} + i\right)^{-1}\\ &\quad = \left(\mathbb{H}_{V,]-\infty,R]} + i\right)^{-1}\left(\mathbf{1}_{]-\infty,R[}V\right)\left(\mathbb{H}_{0,]-\infty,R]} + i\right)^{-1},\end{aligned}$$

and since the potential V is exponentially decreasing as $x \to -\infty$ Theorem XI.21 in [26] guarantees that $\left(\mathbf{1}_{]-\infty,R[}V\right)\left(\mathbb{H}_{0,]-\infty,R]} + i\right)^{-1}$ is trace class; hence the wave operators

$$\Omega^{\pm}\left(\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,R]} \oplus \mathbb{H}^{\frac{1}{2}}_{V,[R,\infty[}\,,\ \mathbb{H}^{\frac{1}{2}}_{V,]-\infty,R]} \oplus \mathbb{H}^{\frac{1}{2}}_{V,[R,\infty[}\right)$$

exist and are complete. Now, since V is positive, the eigenvalues of $\mathbb{H}_{V,\mathbb{R}}$ are strictly positive if they exist. But V satisfies the hypothesis of the Kato-Agmon-Simon theorem ([26], theorem XIII.58) so the point spectrum is empty. Moreover since V is integrable near $-\infty$, $\mathbb{H}_{V,\mathbb{R}}$ has no singular spectrum as a consequence of the R. Carmona theorem [11]. Finally we conclude that

$$\Omega^{\pm}\left(\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,R]} \oplus \mathbb{H}^{\frac{1}{2}}_{V,[R,\infty[}\,,\ \mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}\right)$$

is an isometry from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. Now given $F =^t (f,p) \in \mathcal{D}_V$ we denote

$$U_{V,\mathbb{R}}(t)F =^t (u(t), \partial_t u(t)),$$

and we have:

$$u(t) = e^{it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}} h_{V,\mathbb{R}}^{+} + e^{-it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}} h_{V,\mathbb{R}}^{-}, \tag{II.25}$$

with

$$h_{V,\mathbb{R}}^{\pm} = \frac{1}{2}\left(f \mp i\mathbb{H}_{V,\mathbb{R}}^{-\frac{1}{2}} p\right) \in \bigcap_{s=-\frac{1}{2}}^{\infty} D(\mathbb{H}_{V,\mathbb{R}}^{s}). \tag{II.26}$$

We denote:

$$\begin{aligned} h_{0,]-\infty,R]}^{\pm} \oplus h_{V,[R,\infty[}^{\pm} &= \Omega^{\pm}\left(\mathbb{H}_{0,]-\infty,R]}^{\frac{1}{2}} \oplus \mathbb{H}_{V,[R,\infty[}^{\frac{1}{2}}, \mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}\right) h_{V,\mathbb{R}}^{\pm} \\ &\in \bigcap_{s=-\frac{1}{2}}^{\infty} D(\mathbb{H}_{0,]-\infty,R]}^{s}) \oplus D(\mathbb{H}_{V,[R,\infty[}^{s}), \end{aligned} \tag{II.27}$$

hence we get for $s \geq -\frac{1}{2}$:

$$\begin{aligned} &\|\mathbb{H}_{V,\mathbb{R}}^{s} e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}} h_{V,\mathbb{R}}^{\pm} - \mathbb{H}_{0,]-\infty,R]}^{s} e^{\pm it\mathbb{H}_{0,]-\infty,R]}^{\frac{1}{2}}} h_{0,]-\infty,R]}^{\pm} \\ &\oplus \mathbb{H}_{V,[R,\infty[}^{s} e^{\pm it\mathbb{H}_{V,[R,\infty[}^{\frac{1}{2}}} h_{V,[R,\infty[}^{\pm}\|_{L^2(\mathbb{R})} \xrightarrow[t\to+\infty]{} 0. \end{aligned} \tag{II.28}$$

Moreover, since $\mathbb{H}_{V,\mathbb{R}}$ and $\mathbb{H}_{0,]-\infty,R]}$ have purely absolutely continuous spectrum, we have:

$$\mathbb{H}_{V,\mathbb{R}}^{s} e^{\pm it\mathbb{H}_{V,I}^{\frac{1}{2}}} h_{V,\mathbb{R}}^{\pm} \xrightarrow[t\to+\infty]{} 0 \ \ in \ \ L^2_{loc}(\mathbb{R}), \ \ s = 0,1, \tag{II.29}$$

$$\begin{gathered} \mathbb{H}_{0,]-\infty,R]}^{s} e^{\pm it\mathbb{H}_{0,]-\infty,R]}^{\frac{1}{2}}} h_{0,]-\infty,R]}^{\pm} \xrightarrow[t\to+\infty]{} 0 \ \ in \ \ L^2_{loc}(]-\infty,R]), \\ s = 0,1. \end{gathered} \tag{II.30}$$

Since R is arbitrary, (II.28) with $s = 0,1$ gives:

$$\mathbb{H}_{V,[R,\infty[}^{s} e^{\pm it\mathbb{H}_{V,[R,\infty[}^{\frac{1}{2}}} h_{V,[R,\infty[}^{\pm} \xrightarrow[t\to+\infty]{} 0 \ \ in \ \ L^2_{loc}([R,\infty[), \ \ s = 0,1. \tag{II.31}$$

Then we deduce that for $s = 0, 1$ we have

$$\| \mathbb{H}_{V,0}^{s}((1-\chi)e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}h_{V,\mathbb{R}}^{\pm} - e^{\pm it\mathbb{H}_{V,[z(0),\infty[}^{\frac{1}{2}}}h_{V,[z(0),\infty[}^{\pm}) \|_{L^2([z(0),\infty[)} \xrightarrow[t\to+\infty]{} 0. \tag{II.32}$$

Thus, by interpolation, (II.32) is still valid for $s = \frac{1}{4}, \frac{1}{2}$. To complete the proof of the existence of the strong limit $\Omega_{V,[z(0),\infty[}^{+}F$, the following result is required:

$$\begin{aligned}\delta_s(t) \equiv \| \mathbb{H}_{V,0}^{s}((1-\chi)\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}h_{V,\mathbb{R}}^{\pm} \\ - \mathbb{H}_{V,0}^{\frac{1}{2}}e^{\pm it\mathbb{H}_{V,[z(0),\infty[}^{\frac{1}{2}}}h_{V,[z(0),\infty[}^{\pm}) \|_{L^2([z(0),\infty[)} \xrightarrow[t\to+\infty]{} 0, \\ s = 0, -\frac{1}{4}.\end{aligned} \tag{II.33}$$

We denote

$$\mathbb{H}_{V,0}^{-\frac{1}{2}}h_{V,[z(0),\infty[}^{\pm} = g_{V,[z(0),\infty[}^{\pm}, \quad \mathbb{H}_{V,\mathbb{R}}^{-\frac{1}{2}}h_{V,\mathbb{R}}^{\pm} = g_{V,\mathbb{R}}^{\pm}.$$

Since

$$(1-\chi)\mathbb{H}_{V,\mathbb{R}} = \mathbb{H}_{V,0}(1-\chi) + 2\chi'\frac{d}{dx},$$

we get

$$\begin{aligned}\delta_s(t) \leq \| \mathbb{H}_{V,0}^{s+1}((1-\chi)e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}g_{V,\mathbb{R}}^{\pm} - e^{\pm it\mathbb{H}_{V,0}^{\frac{1}{2}}}g_{V,[z(0),\infty[}^{\pm}) \|_{L^2([z(0),\infty[)} \\ + 2\left\|\mathbb{H}_{V,0}^{s}\chi'\frac{d}{dx}(e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}g_{V,\mathbb{R}}^{\pm})\right\|_{L^2([z(0),\infty[)}.\end{aligned}$$

As previously we have

$$\| \mathbb{H}_{V,0}^{s+1}((1-\chi)e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}g_{V,\mathbb{R}}^{\pm} - e^{\pm it\mathbb{H}_{V,0}^{\frac{1}{2}}}g_{V,[z(0),\infty[}^{\pm}) \|_{L^2([z(0),\infty[)} \xrightarrow[t\to+\infty]{} 0,$$

for $s = -1, 0$, and consequently also for $s = -\frac{1}{4}$. On the other hand Lemma II.8 of [6] implies:

$$\begin{aligned}\left\|\mathbb{H}_{V,0}^{-\frac{1}{4}}\chi'\frac{d}{dx}\left(e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}g_{V,\mathbb{R}}^{\pm}\right)\right\|_{L^2([z(0),\infty[)} \\ \leq C\left\|\chi'\frac{d}{dx}(e^{\pm it\mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}}}g_{V,\mathbb{R}}^{\pm})\right\|_{L^2([z(0),\infty[)},\end{aligned}$$

and by (II.29) we have:

$$\left\|\chi'\frac{d}{dx}(e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}g^{\pm}_{V,\mathbb{R}})\right\|_{L^2([z(0),\infty[)}\xrightarrow[t\to+\infty]{}0.$$

Therefore (II.33) is established, and we conclude that the strong limit (II.13) exists and does not depend on χ since

$$\Omega^+_{V,[z(0),\infty[}F = {}^t\!\left(h^+_{V,[z(0),\infty[}+h^-_{V,[z(0),\infty[}, i\mathbb{H}^{\frac{1}{2}}_{V,0}h^+_{V,[z(0),\infty[}-i\mathbb{H}^{\frac{1}{2}}_{V,0}h^-_{V,[z(0),\infty[}\right). \tag{II.34}$$

To prove the existence of $\Omega^+_{0,\mathbb{R}}F$ we first see as consequence of (II.28), (II.29), (II.30), that:

$$\begin{aligned}&\| \mathbb{H}^s_{0,]-\infty,1]}(\chi e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}h^{\pm}_{V,\mathbb{R}}\\ &- e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}}h^{\pm}_{0,]-\infty,1]})\|_{L^2(]-\infty,1])}\xrightarrow[t\to+\infty]{}0,\ s\in[0,1].\end{aligned} \tag{II.35}$$

Then we show:

$$\begin{aligned}\Delta(t)\equiv&\|\chi\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}h^{\pm}_{V,\mathbb{R}}\\ &-\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}}h^{\pm}_{0,]-\infty,1]}\|_{L^2(]-\infty,1])}\xrightarrow[t\to+\infty]{}0.\end{aligned} \tag{II.36}$$

We denote

$$\mathbb{H}^{-\frac{1}{2}}_{0,]-\infty,1]}h^{\pm}_{0,]-\infty,1]}=g^{\pm}_{0,]-\infty,1]}.$$

Since

$$\chi\mathbb{H}_{V,\mathbb{R}}=\mathbb{H}_{0,]-\infty,1]}\chi+2\chi'\frac{d}{dx}+V\chi,$$

we get

$$\begin{aligned}\Delta(t)\le&\|\mathbb{H}_{0,]-\infty,1]}(\chi e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}g^{\pm}_{V,\mathbb{R}}-e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}}g^{\pm}_{0,]-\infty,1]})\|_{L^2(]-\infty,1])}\\ &+2\|\chi'\frac{d}{dx}(e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}g^{\pm}_{V,\mathbb{R}})\|_{L^2(]-\infty,1])}\\ &+\|V\chi e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}g^{\pm}_{V,\mathbb{R}}\|_{L^2(]-\infty,1])}\,.\end{aligned} \tag{II.37}$$

As previously we have

$$\|\mathbb{H}_{0,]-\infty,1]}(\chi e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}}g^{\pm}_{V,\mathbb{R}}-e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}}g^{\pm}_{0,]-\infty,1]})\|_{L^2(]-\infty,1])}\xrightarrow[t\to+\infty]{}0.$$

At last, thanks to (II.29) and the Sobolev embedding:

$$H^1(\mathbb{R}) \subset C^0 \cap L^\infty(\mathbb{R}), \tag{II.38}$$

we have for all $x \in \mathbb{R}$:

$$e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}} g^{\pm}_{V,\mathbb{R}}(x) \xrightarrow[t\to+\infty]{} 0, \tag{II.39}$$

$$\sup_{t,x} \mid e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}} g^{\pm}_{V,\mathbb{R}}(x) \mid < \infty. \tag{II.40}$$

Since $V\chi \in L^2$ we deduce that

$$\| V\chi e^{\pm it\mathbb{H}^{\frac{1}{2}}_{V,\mathbb{R}}} g^{\pm}_{V,\mathbb{R}} \|_{L^2(]-\infty,1])} \xrightarrow[t\to+\infty]{} 0,$$

hence (II.36) is established.

On the other hand, since $\mathbb{H}_{0,\mathbb{R}} - \mathbb{H}_{0,]-\infty,1]} \oplus \mathbb{H}_{0,[1,\infty[}$ is of finite rank, and all these operators have absolutely continuous spectrum, the Kuroda-Birman theorem assures that

$$\Omega^{\pm}(\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\ ,\ \mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]} \oplus \mathbb{H}^{\frac{1}{2}}_{0,[1,\infty[})$$

is an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ and more generally from $D(\mathbb{H}^s_{0,]-\infty,1]}) \oplus D(\mathbb{H}^s_{0,[1,\infty[})$ onto $D(\mathbb{H}^s_{0,\mathbb{R}})$. Then putting

$$h^{\pm}_{0,\mathbb{R}} = \Omega^{\pm}(\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\ ,\ \mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]} \oplus \mathbb{H}^{\frac{1}{2}}_{0,[1,\infty[})(h^{\pm}_{0,]-\infty,1]} \oplus 0) \in \bigcap_{s=-\frac{1}{2}}^{\infty} D(\mathbb{H}^s_{0,\mathbb{R}}), \tag{II.41}$$

we have:

$$\| \mathbb{H}^s_{0,\mathbb{R}} e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}} h^{\pm}_{0,\mathbb{R}} - \mathbb{H}^s_{0,]-\infty,1]} e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}} h^{\pm}_{0,]-\infty,1]} \oplus 0 \|_{L^2(\mathbb{R})} \xrightarrow[t\to+\infty]{} 0, \quad s \geq -\frac{1}{2}. \tag{II.42}$$

Now we prove that

$$\alpha_s(t) \equiv \| \mathbb{H}^s_{0,\mathbb{R}} (e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}} h^{\pm}_{0,\mathbb{R}} - e^{\pm it\mathbb{H}^{\frac{1}{2}}_{0,]-\infty,1]}} h^{\pm}_{0,]-\infty,1]} \oplus 0) \|_{L^2(\mathbb{R})} \xrightarrow[t\to+\infty]{} 0, \quad s \in \left[0, \frac{1}{2}\right]. \tag{II.43}$$

Since this result is true for $s = 0$, it is sufficient to consider the case $s = \frac{1}{2}$. A direct calculation gives:

$$\begin{aligned}
\alpha_{\frac{1}{2}} =& < \mathbb{H}_{0,\mathbb{R}} e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm} - \mathbb{H}_{0,]-\infty,1]} e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm} \\
&\oplus 0 \, , \, e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm} - e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm} \oplus 0 >_{L^2(\mathbb{R})} \\
&+ e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm}(x=1) \frac{d}{dx}\left(e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm}\right)(x=1) \\
\leq& \| \mathbb{H}_{0,\mathbb{R}} e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm} - \mathbb{H}_{0,]-\infty,1]} e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm} \\
&\oplus 0 \, \|_{L^2(\mathbb{R})} \| \, e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm} - e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm} \oplus 0 \, \|_{L^2(\mathbb{R})} \\
&+ \left| \, e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm}(x=1) \right| . \left| \, \frac{d}{dx}\left(e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm}\right)(x=1) \right| .
\end{aligned} \tag{II.44}$$

Because $e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm}$ and $\frac{d}{dx}(e^{\pm it\mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}}} h_{0,]-\infty,1]}^{\pm})$ are bounded in H^1, the Sobolev imbedding (II.62), and the weak convergence to zero of $e^{\pm it\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}} h_{0,\mathbb{R}}^{\pm}$ prove that the last term of (II.44) tends to zero, hence (II.43) is a consequence of (II.42), and we conclude that the strong limit $\Omega_{0,\mathbb{R}}^{+}F$ exists, does not depend on χ, and

$$\Omega_{0,\mathbb{R}}^{+}F = \left(h_{0,\mathbb{R}}^{+} + h_{0,\mathbb{R}}^{-}, i\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}} h_{0,\mathbb{R}}^{+} - i\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}} h_{0,\mathbb{R}}^{-}\right), \tag{II.45}$$

hence the wave operators do not depend on cut-off function χ. Now to prove (II.15) we remark that the definition of $\Omega_{0,\mathbb{R}}^{\pm}$ entails:

$$[\Omega_{0,\mathbb{R}}^{\pm}F]_1'(x \mp t) \mp [\Omega_{0,\mathbb{R}}^{\pm}F]_2(x \mp t) \to 0 \; in \; L^2([1,\infty[), \; t \to \pm\infty.$$

To establish (II.14) we make use of (II.15) to get:

$$\begin{aligned}
&\| \, \Omega_{0,\mathbb{R}}^{\pm}F \, \|_{H^1 \times L^2(]-\infty,R])} \\
&= \| \, U_{0,\mathbb{R}}(t)\Omega_{0,\mathbb{R}}^{\pm}F \, \|_{H^1 \times L^2(]-\infty,R\mp t])} \\
&\leq \| \, U_{V,\mathbb{R}}(t)F \, \|_{H^1 \times L^2(]-\infty,R\mp t])} + \| \, U_{0,\mathbb{R}}(t)\Omega_{0,\mathbb{R}}^{\pm}F - \chi U_{V,\mathbb{R}}(t)F \, \|_{H^1 \times L^2(\mathbb{R})} \\
&\longrightarrow 0, \; t \to \pm\infty.
\end{aligned}$$

To show (II.16) we note that (II.15) and (II.17) imply

$$[\Omega_{0,\mathbb{R}}^{\pm}F]_1 \in D(\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}), \;\; [\Omega_{0,\mathbb{R}}^{\pm}F]_2 \in D(\mathbb{H}_{0,\mathbb{R}}^{-\frac{1}{2}}),$$

and we also have

$$\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}\right)} \leq C(1 + \mathbb{H}_{0,\mathbb{R}}^{-\frac{1}{4}}),$$

whence:

$$\left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}\right)}\, \Omega_{0,\mathbb{R}}^{-} F \right\|_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})}$$
$$\leq C(\| [\Omega_{0,\mathbb{R}}^{\pm}F]_1 \|_{L^2(\mathbb{R})} + \| \mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}[\Omega_{0,\mathbb{R}}^{\pm}F]_1 \|_{L^2(\mathbb{R})}$$
$$+ \| [\Omega_{0,\mathbb{R}}^{\pm}F]_2 \|_{L^2(\mathbb{R})} + \| \mathbb{H}_{0,\mathbb{R}}^{-\frac{1}{2}}[\Omega_{0,\mathbb{R}}^{\pm}F]_2 \|_{L^2(\mathbb{R})}).$$

At last we evaluate:

$$\| [\Omega_{0,\mathbb{R}}^{\pm}F]_1 \|_{L^2(\mathbb{R})}$$
$$\leq \sum_{\pm} \| h_{0,\mathbb{R}}^{\pm} \|_{L^2(\mathbb{R})}$$
$$\leq \sum_{\pm} \| h_{0,]-\infty,1]}^{\pm} \|_{L^2(]-\infty,1])}$$
$$\leq \sum_{\pm} \| h_{V,\mathbb{R}}^{\pm} \|_{L^2(\mathbb{R})}$$
$$\leq \| f \|_{L^2(\mathbb{R})} + \| \mathbb{H}_{V,\mathbb{R}}^{-\frac{1}{2}} p \|_{L^2(\mathbb{R})},$$

$$\| \Omega_{0,\mathbb{R}}^{+}F \|_{\mathcal{H}(0,\mathbb{R})}^2 + \| \Omega_{V,[z(0),\infty[}^{+}F \|_{\mathcal{H}(V,0)}^2$$
$$= 2\sum_{\pm} \| \mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}} h_{0,\mathbb{R}}^{\pm} \|_{L^2(\mathbb{R})}^2 + \| \mathbb{H}_{V,[z(0),\infty[}^{\frac{1}{2}} h_{V,[z(0),\infty[}^{\pm} \|_{L^2([z(0),\infty[)}^2$$
$$= 2\sum_{\pm} \| \mathbb{H}_{0,]-\infty,1]}^{\frac{1}{2}} h_{0,]-\infty,1]}^{\pm} \|_{L^2(]-\infty,1])}^2$$
$$+ \| \mathbb{H}_{V,[z(0),\infty[}^{\frac{1}{2}} h_{V,[z(0),\infty[}^{\pm} \|_{L^2([z(0),\infty[)}^2$$
$$= 2\sum_{\pm} \| \mathbb{H}_{0,]-\infty,z(0)]}^{\frac{1}{2}} h_{0,]-\infty,z(0)]}^{\pm} \|_{L^2(]-\infty,z(0)])}^2$$
$$+ \| \mathbb{H}_{V,0}^{\frac{1}{2}} h_{V,[z(0),\infty[}^{\pm} \|_{L^2([z(0),\infty[)}^2$$
$$= 2\sum_{\pm} \| \mathbb{H}_{V,\mathbb{R}}^{\frac{1}{2}} h_{V,\mathbb{R}}^{\pm} \|_{L^2(\mathbb{R})}^2$$
$$= \| F \|_{\mathcal{H}(V,\mathbb{R})}^2,$$

$$\begin{aligned}
&\| \Omega^+_{0,\mathbb{R}} F \|^2_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})} + \| \Omega^+_{V,[z(0),\infty[} F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \\
&= 2\sum_{\pm} \| \mathbb{H}^{\frac{1}{4}}_{0,\mathbb{R}} h^{\pm}_{0,\mathbb{R}} \|^2_{L^2(\mathbb{R})} + \| \mathbb{H}^{\frac{1}{4}}_{V,0} h^{\pm}_{V,[z(0),\infty[} \|^2_{L^2([z(0),\infty[)} \\
&= 2\sum_{\pm} \| \mathbb{H}^{\frac{1}{4}}_{0,]-\infty,1]} h^{\pm}_{0,]-\infty,1]} \|^2_{L^2(]-\infty,1])} + \| \mathbb{H}^{\frac{1}{4}}_{V,0} h^{\pm}_{V,[z(0),\infty[} \|^2_{L^2([z(0),\infty[)} \\
&= 2\sum_{\pm} \| \mathbb{H}^{\frac{1}{4}}_{0,]-\infty,z(0)]} h^{\pm}_{0,]-\infty,z(0)]} \|^2_{L^2(]-\infty,z(0)])} \\
&\quad + \| \mathbb{H}^{\frac{1}{4}}_{V,0} h^{\pm}_{V,[z(0),\infty[} \|^2_{L^2([z(0),\infty[)} \\
&= 2\sum_{\pm} \| \mathbb{H}^{\frac{1}{4}}_{V,\mathbb{R}} h^{\pm}_{V,\mathbb{R}} \|^2_{L^2(\mathbb{R})} \\
&= \| F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,\mathbb{R})} .
\end{aligned}$$

The existence of the isometry $\Omega^-_{0,\mathbb{R}} \oplus \Omega^-_{V,[z(0),\infty[}$ is established in the same manner.

Q.E.D.

To establish Theorem II.3 we write:

$$U_V(0,T)F = \chi U_V(0,T)F + (1-\chi)U_{V,\mathbb{R}}(-T)F.$$

The above description, in terms of the Wave Operators $\Omega^{\pm}_{0,\mathbb{R}}$, of scattering phenomena induced at infinity by an eternal Black-Hole, allows to tackle the study of the last term, $(1-\chi)U_{V,\mathbb{R}}(-T)F$. At present, we need to analyse very sharply the influence of the gravitational collapse expressed by the term $\chi U_V(0,T)F$.

PROPOSITION II.5. – *Given a potential V satisfying assumption* (II.3), *for all F in $\mathcal{D}_V$ we have:*

$$\| \chi U_V(0,T)F \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[t\to+\infty]{} \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\right)}\, \Omega^-_{0,\mathbb{R}} F \right\|_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})}, \tag{II.46}$$

$$\chi U_V(0,T)F \xrightarrow[t\to+\infty]{} 0 \ \ in\ \ \mathcal{H}^{\frac{1}{2}}(V,0) - weak - *. \tag{II.47}$$

The proof of this Propostion needs several Lemmas, technically delicate.

LEMMA II.6. – *Given $\varphi \in H^1(\mathbb{R}^+)$, $\varphi(t) = 0$ for $t > t_\varphi$, there exists a unique solution u of*

$$\partial_t^2 u - \partial_x^2 u + Vu = 0,\ t \in \mathbb{R}^+,\ z(t) < x < -t+1, \tag{II.48}$$

$$u(t, x = z(t)) = 0,\ t \in \mathbb{R}^+, \tag{II.49}$$

$$u(t,x=-t+1)=\varphi(t),\ t\in\mathbb{R}^+, \tag{II.50}$$

$$u(t,x)=0,\ t>t_\varphi,\ x\in[z(t),-t+1] \tag{II.51}$$

such that there exists $v\in C^0(\mathbb{R}_t,H^1(\mathbb{R}_x))\cap C^1(\mathbb{R}_t,L^2(\mathbb{R}_x))$ *satisfying*

$$u(t,x)=v(t,x),\ t\in\mathbb{R}^+,\ z(t)<x<-t+1. \tag{II.52}$$

Proof of Lemma II.6. – To prove the uniqueness, we consider a solution u for $\varphi=0$, and we put $\tilde{u}(t,x)=u(t,x)$ for $z(t)<x<-t+1$, and $\tilde{u}(t,x)=0$ for $x>-t+1$. Then $\tilde{u}$ is a solution of (II.1), (II.2) and there exists $v\in C^0(\mathbb{R}_t,H^1(\mathbb{R}_x))\cap C^1(\mathbb{R}_t,L^2(\mathbb{R}_x))$ satisfying $\tilde{u}(t,x)=v(t,x)$ for $t\in\mathbb{R}^+$, $z(t)<x$. Thus there exists $F\in\mathcal{H}(V,0)$ such that

$$^t(\tilde{u}(t,.),\partial_t\tilde{u}(t,.))=U_V(t,0)F.$$

Since $\tilde{u}(t,x)=0$ for $t>t_\varphi$, Theorem II-1 of [GravColl] implies $F=0$, hence $u=0$.

To prove the existence of u, given $\varphi\in C_0^1([0,\infty[)$, we put

$$0\le t\Rightarrow\tilde{\varphi}(t)=\varphi(t),$$

$$-\frac{1}{2}\le t\le 0\Rightarrow\tilde{\varphi}(t)=(2t+1)\varphi(0),$$

$$t\le-\frac{1}{2}\Rightarrow\tilde{\varphi}(t)=0,$$

and we solve the hyperbolic characteristic problem:

$$\partial_t^2w-\partial_x^2w+Vw=0,\ t\in\mathbb{R},\ x>-t+1, \tag{II.53}$$

$$w(t,x=-t+1)=\tilde{\varphi}(t),\ t\in\mathbb{R}, \tag{II.54}$$

$$1-x<t<x-3\Rightarrow w(t,x)=0. \tag{II.55}$$

It is convenient to introduce the characteristic coordinates

$$X=t+x-1,\ \ Y=t-x+3. \tag{II.56}$$

Then $w(t,x)$ is a continuous solution of (II.53), (II.54), (II.55) iff $W(X,Y)=w(t,x)$ is a continuous solution of

$$W(X_0,Y_0)=\tilde{\varphi}\left(\frac{Y_0}{2}-1\right)$$
$$-\frac{1}{4}\int_0^{X_0}\int_0^{Y_0}V\left(\frac{X-Y}{2}+2\right)W(X,Y)dXdY,$$
$$X_0\ge 0,\ Y_0>0, \tag{II.57}$$

$$W(X_0, Y_0) = 0, \quad X_0 \geq 0, \quad Y_0 \leq 0. \qquad \text{(II.58)}$$

The integral equation (II.57) is easily solved by the Picard method putting

$$W^0(X_0, Y_0) = \tilde{\varphi}\left(\frac{Y_0}{2} - 1\right),$$

$$W^{n+1}(X_0, Y_0) = -\frac{1}{4}\int_0^{X_0}\int_0^{Y_0} V\left(\frac{X - Y}{2} + 2\right) W^n(X, Y)dXdY, \quad n \geq 0.$$

Since $\tilde{\varphi}$ and V are bounded we have:

$$\mid W^n(X_0, Y_0) \mid \leq C^n \frac{X_0^n}{n!}\frac{Y_0^n}{n!},$$

hence

$$W(X, Y) = \sum_{n=0}^{\infty} W^n(X, Y) \in C^0([0, \infty[_X \times \mathbb{R}_Y)$$

is a solution of (II.57), (II.58), moreover, $W(X, Y) \in H^1_{loc}([0, \infty[_X \times \mathbb{R}_Y)$ since $V \in C^\infty(\mathbb{R})$. Now we put

$$F_\varphi = {}^t(f, p),$$

where

$$x > -t_\varphi + 1 \Rightarrow f(x) = w(t_\varphi, x), \quad p(x) = \partial_t w(t_\varphi, x)$$

$$z(t_\varphi) < x \leq -t + 1 \Rightarrow f(x) = p(x) = 0.$$

We note that

$$F_\varphi \in H^1(\mathbb{R}) \times L^2(\mathbb{R}),$$

$$F_\varphi(x) \neq 0 \Rightarrow x \in [1 - t_\varphi, t_\varphi + 3].$$

Then $u_\varphi(t, .) = [U_V(t, t_\varphi)F_\varphi]_1$ is a finite energy solution of (II.1), (II.2), (II.50), (II.51), and the Lemma is established for regular φ. To treat the case $\varphi \in H^1(\mathbb{R})$, we remark that the relation

$$\partial_t(\mid \partial_t w \mid^2 + \mid \partial_x w \mid^2 + V \mid w \mid^2) - \partial_x(2\Re(\partial_t w \partial_x \bar{w})) = 0, \qquad \text{(II.59)}$$

implies

$$\| F_\varphi \|^2_{\mathcal{H}(V)} = \int_{-\infty}^{\infty} \mid \tilde{\varphi}'(t) \mid^2 + V(-t + 1) \mid \tilde{\varphi}(t) \mid^2 dt. \qquad \text{(II.60)}$$

Hence using the energy estimate for the propagator $U_V(s,t)$:

$$\sup_{0\leq t\leq T}\left(\int_{z(t)}^{\infty} \mid \partial_t u_\varphi(t,x)\mid^2 + \mid \partial_x u_\varphi(t,x)\mid^2 + V(x)\mid u_\varphi(t,x)\mid^2 dx\right)$$
$$\leq C_T \parallel \tilde{\varphi} \parallel^2_{H^1(\mathbb{R})} . \tag{II.61}$$

At last we remark that the Sobolev embedding

$$\mid \varphi(0) \mid \leq C \parallel \varphi \parallel_{H^1([0,\infty[)} \tag{II.62}$$

implies

$$\parallel \tilde{\varphi} \parallel_{H^1(\mathbb{R})} \leq C \parallel \varphi \parallel_{H^1([0,\infty[)} \tag{II.63}$$

Then we conclude by (II.61), (II.63), and a standard argument of density and continuity.

Q.E.D.

Given χ satisfying (II.11), we introduce the operator P_V defined for $\varphi \in H^1_{compact}([0,\infty[)$ by:

$$P_V\varphi = {}^t(\chi(x)u(t=0,x), \chi(x)\partial_t u(t=0,x)). \tag{II.64}$$

where u is given by Lemma II.6.

We shall establish two main results. The first one, Lemma II.13, assures that P_V is bounded from $H^1(\mathbb{R})$ to $\mathcal{H}^{\frac{1}{2}}(V,0)$. The second one, Lemma II.14, gives a fine analysis of the asymptotic behaviour of $P_V\left(\varphi^{[T]}\right)$ with respect to T, where $\varphi^{[T]}(t) = \varphi(t-T)$.

LEMMA II.7. – *There exists $C>0$ such that for any $\varphi \in H^1(\mathbb{R})$, $\varphi(t)=0$ for $t > t_\varphi$, we have:*

$$\parallel P_V\varphi \parallel_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C\sum_{j=0}^{\infty}(1+j)^{\frac{3}{2}}\left(\int_{j-1}^{j+1} \mid \varphi'(t)\mid^2 + \mid \varphi(t)\mid^2 dt\right)^{\frac{1}{2}}. \tag{II.65}$$

Proof of Lemma II.7. – We suppose $\varphi_j \in H^1(\mathbb{R}^+)$ is compactly supported in $[j-1, j+1]$. With the notations of the proof of the previous Lemma we have $t_{\varphi_j} = j+1$ and:

$$P_V\varphi_j = \boldsymbol{\chi} U_V(0,j+1)F_{\varphi_j} = U_V(0,j+1)F_{\varphi_j} + (\boldsymbol{\chi}-\mathbf{1})U_V(0,j+1)F_{\varphi_j}. \tag{II.66}$$

Since F_{φ_j} is supported in $[-j, 4-j]$, Lemma II-11 of [6] implies

$$\| U_V(0, j+1)F_{\varphi_j} \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C(1+j)^{\frac{3}{2}} \| F_{\varphi_j} \|_{\mathcal{H}(V)}, \tag{II.67}$$

where $C > 0$ is independent of φ_j and $j \in \mathbb{N}$. On the other hand, since $(\chi - \mathbf{1})U_V(0, j+1)F_{\varphi_j}$ is supported in $[z(0), 3]$, we get by Lemma II-8 of [6]:

$$\| (\chi - \mathbf{1})U_V(0, j+1)F_{\varphi_j} \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C' \| (\chi - \mathbf{1})U_V(0, j+1)F_{\varphi_j} \|_{\mathcal{H}(V,0)} . \tag{II.68}$$

We remark that

$$(\chi - \mathbf{1})U_V(0, j+1)F_{\varphi_j} = (\chi - \mathbf{1})U_V(-j-1)F_{\varphi_j}, \tag{II.69}$$

and

$$\| (\chi - \mathbf{1})U_V(-j-1)F_{\varphi_j} \|_{\mathcal{H}(V)} \leq C'' \| F_{\varphi_j} \|_{\mathcal{H}(V)} . \tag{II.70}$$

Then we conclude using (II.60) that

$$\| P_V \varphi_j \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C(1+j)^{\frac{3}{2}} \| \varphi_j \|_{H^1([0,\infty[)} . \tag{II.71}$$

Now given $\varphi \in H^1_{comp}([0, \infty[)$ we define a cut off function θ:

$$| t | > 1 \Rightarrow \theta(t) = 0, \ -1 \leq t \leq 0 \Rightarrow \theta(t) = 1+t, \ 0 \leq t \leq 1 \Rightarrow \theta(t) = 1-t,$$

and we put:

$$\varphi = \sum_{j=0}^{\infty} \varphi_j, \quad \varphi_j(t) = \varphi(t)\theta(t-j). \tag{II.72}$$

Since

$$\| \varphi_j \|^2_{H^1} \leq C \int_{j-1}^{j+1} | \varphi'(t) |^2 + | \varphi(t) |^2 \, dt, \tag{II.73}$$

(II.65) is a consequence of (II.71).

Q.E.D.

The following Lemma gives some useful characterizations of the norm of $[D(\mathbb{H}^s_{0,[R,\infty[})]$. We denote $\hat{f} = \mathcal{F}f$ the Fourier transform of tempered distributions on $\mathbb{R}$:

$$f \in L^1(\mathbb{R}), \quad \hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx.$$

LEMMA II.8. – *Given* $V \in L^\infty(\mathbb{R})$, $V \geq 0$, $R \in \mathbb{R}$, *and* $s > -\frac{3}{4}$, $C_0^\infty(]R,\infty[)$ *is a subspace of* $[D(\mathbb{H}^s_{V,[R,\infty[})]$ *and for* $\phi \in C_0^\infty(]R,\infty[)$ *we have:*

$$-\frac{3}{4} < s \Rightarrow \| \mathbb{H}^s_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} = \frac{1}{4\pi}\int_{\mathbb{R}} | \xi |^{4s} | \hat{\phi}(\xi) - \exp\big(-2i\xi R\big)\hat{\phi}(-\xi) |^2 \, d\xi, \quad \text{(II.74)}$$

$$-\frac{1}{2} < s \Rightarrow \| \mathbb{H}^s_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} = \frac{1}{2\pi}\int_{\mathbb{R}} | \xi |^{4s} \Big[| \hat{\phi}(\xi) |^2 - \exp\big(2i\xi R\big)\hat{\phi}(\xi)\bar{\hat{\phi}}(-\xi)\Big] d\xi, \quad \text{(II.75)}$$

and there exists $C > 0$ *such that*

$$\| \mathbb{H}^{\frac{1}{4}}_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} = C\int_R^\infty \int_R^\infty \big(| x-y |^{-2} + | 2R-x-y |^{-2}\big) | \phi(x) - \phi(y) |^2 \, dxdy, \quad \text{(II.76)}$$

and for any $\phi \in C_0^\infty(]a,b[)$, $R \leq a < b$, *we have:*

$$\| \mathbb{H}^{\frac{1}{4}}_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} \leq 4C\Big(\int_a^b \int_a^b | x-y |^{-2} | \phi(x) - \phi(y) |^2 \, dxdy + \int_a^b \big(| x-b |^{-1} + | x-a |^{-1} - | x-R |^{-1}\big) | \phi(x) |^2 \, dx\Big). \quad \text{(II.77)}$$

Moreover if $-\frac{3}{4} < s \leq \frac{1}{2}$, *for any bounded interval* $]a,b[\subset]R,\infty[$ *there exists* $C_{a,b}$ *such that*

$$\forall \phi \in C_0^\infty(]a,b[) \quad \| \mathbb{H}^s_{V,[R,\infty[}\phi \|_{L^2([R,\infty[)} \leq C_{a,b} \| \mathbb{H}^s_{0,[R,\infty[}\phi \|_{L^2([R,\infty[)} . \quad \text{(II.78)}$$

Proof of Lemma II.8. – Given $\phi \in C_0^\infty(]R,\infty[)$ we put

$$\mathbb{P}\phi(x) = \phi(x) + \phi(2R-x).$$

Since $\frac{1}{\sqrt{2}}\mathbb{P}$ is an isometry from $L^2([R,\infty[)$ into $L^2(\mathbb{R})$ such that:

$$\mathbb{H}_{0,\mathbb{R}}\mathbb{P}\phi = \mathbb{P}\mathbb{H}_{0,[R,\infty[}\phi,$$

we get that $\frac{1}{\sqrt{4\pi}}\mathcal{F}\mathbb{P}$ is an isometry from $L^2([R,\infty[)$ into $L^2(\mathbb{R})$ satisfying

$$\mathcal{F}\mathbb{H}_{0,\mathbb{R}}\mathbb{P}\phi = \mid \xi \mid^2 \mathcal{F}\mathbb{P}\phi.$$

Thus we deduce that $\phi \in D(\mathbb{H}^s_{0,[R,\infty[})$ iff

$$\begin{aligned} &\| \mathbb{H}^s_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} \\ &= \frac{1}{4\pi}\int_{\mathbb{R}} \mid \xi \mid^{4s} \mid \hat{\phi}(\xi) - \exp\big(-2i\xi R\big)\hat{\phi}(-\xi) \mid^2 d\xi < \infty, \end{aligned}$$

then $C_0^\infty([R,\infty[)$ is a subspace of $\left[D(\mathbb{H}^s_{0,[R,\infty[})\right]$ if $4s+2 > -1$ and (II.74 is proved, moreover (II.75) holds for $4s+1 > -1$. To prove (II.76), we note that:

$$\begin{aligned} &\| \mathbb{H}^{\frac{1}{4}}_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \mid \xi \mid\mid \mathcal{F}(\mathbb{P}\phi)(\xi) \mid^2 d\xi \\ &= \frac{1}{2}\Big(\int_{-\infty}^{\infty} \mid \xi \mid^{-2}\mid 1 - e^{i\xi} \mid^2 d\xi\Big)^{-1} \\ &\quad \int_{\mathbb{R}\times\mathbb{R}} \mid x-y \mid^{-2}\mid \mathbb{P}\phi(x) - \mathbb{P}\phi(y) \mid^2 dxdy \\ &= C\int_R^\infty \int_R^\infty \big(\mid x-y \mid^{-2} + \mid 2R-x-y \mid^{-2}\big) \mid \phi(x) - \phi(y) \mid^2 dxdy. \end{aligned}$$

To get (II.77) we deduce from (II.76) that:

$$\begin{aligned} &\| \mathbb{H}^{\frac{1}{4}}_{0,[R,\infty[}\phi \|^2_{L^2([R,\infty[)} \\ &\leq 2C\int_R^\infty \int_R^\infty \mid x-y \mid^{-2}\mid \phi(x) - \phi(y) \mid^2 dxdy \\ &\leq 4C\Big(\int_a^b \int_a^b \mid x-y \mid^{-2}\mid \phi(x) - \phi(y) \mid^2 dxdy \\ &\quad + \int_a^b \big(\mid x-b \mid^{-1} + \mid x-a \mid^{-1} - \mid x-R \mid^{-1}\big) \mid \phi(x) \mid^2 dx\Big). \end{aligned}$$

Now on the one hand we have:

$$\mathbb{H}_{V,[R,\infty[} \leq \mathbb{H}_{0,[R,\infty[} + \| V \|_{L^\infty(\mathbb{R})},$$

hence Heinz's Theorem [22] entails for $s \geq 0$:

$$\| \mathbb{H}^s_{V,[R,\infty[}\phi \|_{L^2([R,\infty[)} \leq C(\| \mathbb{H}^s_{0,[R,\infty[}\phi \|_{L^2([R,\infty[)} + \| \phi \|_{L^2([R,\infty[)}).$$

On the other hand we have for $\phi \in C_0^\infty(]a,b[)$:

$$\| \phi \|_{L^2(]a,b[)} \leq C_{a,b} \| \phi' \|_{L^2(]a,b[)} \leq C_{a,b} \| \mathbb{H}^{\frac{1}{2}}_{0,[R,\infty[}\phi \|_{L^2([R,\infty[)} \cdot$$

Hence (II.78) is proved for $s = \frac{1}{2}$ and the case $0 \leq s \leq \frac{1}{2}$ follows by interpolation. Eventually, since:

$$\mathbb{H}_{0,[R,\infty[} \leq \mathbb{H}_{V,[R,\infty[},$$

the Heinz theorem gives for $s \leq 0$:

$$\mathbb{H}^s_{V,[R,\infty[} \leq \mathbb{H}^s_{0,[R,\infty[},$$

therefore (II.78) holds for $-\frac{3}{4} < s \leq 0$.

Q.E.D.

Now we derive an improvement of Lemma II.7 for $V = 0$.

LEMMA II.9. – *There exists $C > 0$ such that for any $\varphi \in H^1(\mathbb{R})$, $\varphi(t) = 0$ for $t > t_\varphi$, we have:*

$$\| P_0(\varphi) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C \| \varphi \|_{H^1([0,\infty[)} \cdot \tag{II.79}$$

Proof of Lemma II.9. – We choose $\theta_0 \in C^\infty(\mathbb{R})$, $\theta(t) = 0$ for $t < 1$ and $\theta(t) = 1$ for $t > 2$. For $R > \frac{-1-z(0)}{2}$ we put:

$$\theta_R(t) = \theta(t - R).$$

We consider:

$$P_0(\varphi) = P_0(\theta_R\varphi) + P_0((1 - \theta_R)\varphi).$$

By (II.65) we get:

$$\| P_0((1 - \theta_R)\varphi) \|_{\mathcal{H}^{\frac{1}{2}}(0,0)} \leq CR^{\frac{3}{2}} \left(\int_0^{R+2} | \varphi'(t) |^2 + | \varphi(t) |^2 \, dt \right)^{\frac{1}{2}}. \tag{II.80}$$

Now we denote:

$$P_0(\theta_R\varphi) = {}^t(f_R, p_R).$$

According to [5] an explicit calculation gives:

$$f_R(x) = (\theta_R\varphi)\left(t = \tau(x) + \frac{1-x}{2}\right), \tag{II.81}$$

$$p_R(x) = f'_R(x), \tag{II.82}$$

where the function τ is implicitly defined by:

$$\tau(x) = x - z(\tau(x)), \tag{II.83}$$

and we have the following asymptotic behaviours:

$$\tau(x) = -\frac{1}{2\kappa}\ln(-x) + \frac{1}{2\kappa}\ln A + \sigma(x),\ x \to 0^-, \tag{II.84}$$

$$\sigma(x) = O(x),\ x \to 0^-,\ \ \sigma'(x) = O(1),\ x \to 0^-. \tag{II.85}$$

So we infer that:

$$P_0(\theta_R\varphi)(x) \neq 0 \Rightarrow x \in [\alpha(R), 0[, \tag{II.86}$$

where $\alpha(R)$ is the solution of:

$$\alpha(R) - 2z(\tau(\alpha(R))) = 2R + 1. \tag{II.87}$$

We easily show that α is an increasing function of R that satisfies:

$$\alpha'(R) = 2\frac{1 + \dot{z}(\tau(\alpha(R)))}{1 - \dot{z}(\tau(\alpha(R)))} > 0, \tag{II.88}$$

$$z(0) < \alpha(R) \sim e^{-2\kappa R},\ R \to +\infty. \tag{II.89}$$

Because of (II.82) we have

$$\begin{aligned}
&\| \mathbb{H}_{0,0}^{-\frac{1}{4}} p_R \|^2_{L^2([z(0),\infty[)} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\,\xi\,|^{-1} \left(|\,\hat{p}_R(\xi)\,|^2 - \exp\left(2i\xi z(0)\right)\hat{p}_R(\xi)\bar{\hat{p}}_R(-\xi)\right)d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} |\,\xi\,| \left(\,|\,\hat{f}_R(\xi)\,|^2 - \exp(2i\xi z(0))\hat{f}_R(\xi)\bar{\hat{f}}_R(-\xi)\right)d\xi \\
&= \| \mathbb{H}_{0,0}^{\frac{1}{4}} f_R \|^2_{L^2([z(0),\infty[)},
\end{aligned} \tag{II.90}$$

thus, using (II.77) we get:

$$\begin{aligned}
&\| P_0(\theta_R\varphi) \|_{\mathcal{H}^{\frac{1}{2}}(0,0)} \\
&\leq C \int_{\alpha(R/2)}^0 \int_{\alpha(R/2)}^0 | x - y |^{-2} \\
&\quad \left|(\theta_R\varphi)\left(\tau(x) + \frac{1-x}{2}\right) - (\theta_R\varphi)\left(\tau(y) + \frac{1-y}{2}\right)\right|^2 dxdy \\
&\quad + C \int_{\alpha(R/2)}^0 \left(| x |^{-1} + | x - \alpha_R |^{-1} - | x - R |^{-1}\right) \\
&\quad \left|(\theta_R\varphi)\left(\tau(x) + \frac{1-x}{2}\right)\right|^2 dx \\
&\leq C \int_{\alpha(R/2)}^0 \int_{\alpha(R/2)}^0 | x - y |^{-2} \\
&\quad \left|(\theta_R\varphi)\left(\tau(x) + \frac{1-x}{2}\right) - (\theta_R\varphi)\left(\tau(y) + \frac{1-y}{2}\right)\right|^2 dxdy \\
&\quad + C(R) \int_{\alpha(R/2)}^0 | x |^{-1} \left|(\theta_R\varphi)\left(\tau(x) + \frac{1-x}{2}\right)\right|^2 dx \\
&= I(R) + J(R).
\end{aligned} \tag{II.91}$$

On the one hand we put:

$$x' = xe^{\kappa(x-2\sigma(x))}, \quad \beta(R) = x'(x = \alpha(R/2)),$$

$$X = -\frac{1}{2\kappa}\ln(-x') + \frac{1}{2\kappa}\ln(A) + \frac{1}{2}, \quad A(R) = X(x' = \beta(R)),$$

then we have:

$$\frac{| x' - y' |}{| x - y |} \leq \sup_{\alpha(R/2) \leq x < 0} (e^{\kappa(x-2\sigma(x))} \mid 1 + \kappa x - 2\kappa x \sigma'(x) \mid),$$

$$\tau(x) + \frac{1-x}{2} = X(x'),$$

and we evaluate:

$$
\begin{aligned}
&I(R)\\
&\leq k_1(R)\int_{\beta(R)}^{0}\int_{\beta(R))}^{0} \mid x'-y'\mid^{-2}\mid (\theta_R\varphi)(X)-(\theta_R\varphi)(Y)\mid^2 dx'dy'\\
&\leq \kappa^2 k_1(R)\int_{A(R)}^{\infty}\int_{A(R)}^{\infty}\frac{1}{\mid \sinh(X-Y)\mid^2}\left|\int_X^Y(\theta_R\varphi)'(T)dT\right|^2 dXdY\\
&\leq \kappa^2 k_1(R)\int_{A(R)}^{\infty}\int_{A(R)}^{\infty}\frac{\mid X-Y\mid}{\mid \sinh(X-Y)\mid^2}\left|\int_X^Y\mid(\theta_R\varphi)'(T)\mid^2 dT\right|dXdY,
\end{aligned}
$$

where:

$$
k_1(R)=\sup_{\alpha(R/2)\leq x<0}(Ce^{4\kappa(x-2\sigma(x))})\frac{\sup_{\alpha(R/2)\leq x<0}\mid 1+\kappa x-2\kappa x\sigma'(x)\mid^2}{\inf_{\alpha(R/2)\leq x<0}\mid 1+\kappa x-2\kappa x\sigma'(x)\mid^2}.
$$

We put:

$$
U=X+Y,\ \ V=X-Y,
$$

and we get:

$$
\begin{aligned}
I(R)\leq&\ \frac{1}{2}\kappa^2 k_1(R)\int_{-\infty}^{\infty}\frac{\mid V\mid}{\mid\sinh(V)\mid^2}\\
&\left(\int_{-|V|/2}^{|V|/2}\left(\int_{|V|+2A(R)}^{\infty}\mid(\theta_R\varphi)'(\frac{U}{2}+T)\mid^2 dU\right)dT\right)dV\\
\leq&\ \kappa^2 k_1(R)\left(\int_{-\infty}^{\infty}\frac{\mid V\mid^2}{\mid\sinh(V)\mid^2}dV\right)\parallel(\theta_R\varphi)'\parallel^2_{L^2(\mathbb{R})}. \qquad \text{(II.92)}
\end{aligned}
$$

On the other hand we put $S=\tau(x)+\frac{1-x}{2}$, and we get

$$
J(R)\leq k_2(R)\int_{-\infty}^{\infty}\mid(\theta_R\varphi)(S)\mid^2 dS. \qquad \text{(II.93)}
$$

with

$$
k_2(R)=\sup_{\alpha(R/2)\leq x<0}\frac{2\kappa C(R)}{\mid 1+\kappa x-2\kappa x\sigma'(x)\mid}.
$$

We choose $R>0$ large enough to assure that $k_1(R)$ and $k_2(R)$ are finite and the result follows from (II.80), (II.91), (II.92) and (II.93).

Q.E.D.

For $\varphi \in H^1(\mathbb{R})$ and $T > 0$ we put:

$$\varphi^{[T]}(t) = \varphi(t - T). \tag{II.94}$$

LEMMA II.10. – *For any $\varphi \in H^1(\mathbb{R})$, $\varphi(t) = 0$ for $t > t_\varphi$, we have:*

$$\begin{aligned} &\| P_0(\varphi^{[T]}) \|^2_{\mathcal{H}^{\frac{1}{2}}(0,0)} \\ &\xrightarrow[T\to+\infty]{} \frac{1}{\pi} \int_{-\infty}^{+\infty} \mid \xi \mid \coth\left(\frac{\pi}{2\kappa} \mid \xi \mid\right) \mid \hat{\varphi}(\xi) \mid^2 d\xi, \end{aligned} \tag{II.95}$$

$$P_0(\varphi^{[T]}) \xrightarrow[T\to+\infty]{} 0 \ \ in \ \ \mathcal{H}^{\frac{1}{2}}(0,0) - weak - *. \tag{II.96}$$

Proof of Lemma II.10. – Thanks to Lemma II.9 we may assume that:

$$\varphi'' \in H^1(\mathbb{R}).$$

We denote:

$$P_0(\varphi^{[T]}) = {}^t(f_T, p_T).$$

We choose $\theta_R \in C^\infty(\mathbb{R})$ as in the previous Lemma and we consider:

$$P_0(\varphi^{[T]}) = P_0(\theta_R\varphi^{[T]}) + P_0((1 - \theta_R)\varphi^{[T]}).$$

By (II.65) we get:

$$\| P_0((1-\theta_R)\varphi^{[T]}) \|_{\mathcal{H}^{\frac{1}{2}}(0,0)} \leq CR^{\frac{3}{2}} \left(\int_{-T}^{R+2-T} \mid \varphi'(t) \mid^2 + \mid \varphi(t) \mid^2 dt \right)^{\frac{1}{2}}. \tag{II.97}$$

Now we denote:

$$P_0(\theta_R\varphi^{[T]}) = {}^t(f_{R,T}, p_{R,T}).$$

According to [5] an explicit calculation gives:

$$\begin{aligned} p_{R,T}(x) &= f'_{R,T}(x) \\ &= \frac{1}{2} \cdot \frac{1 - \dot{z}(\tau(x))}{1 + \dot{z}(\tau(x))} \left(\theta_R\varphi^{[T]}\right)' \left(t = \tau(x) + \frac{1-x}{2}\right), \end{aligned} \tag{II.98}$$

where we have the following asymptotic behaviours:

$$1+\dot{z}(\tau(x)) = -2\kappa x + O(x^2),\ x \to 0^-, \tag{II.99}$$

$$1-\dot{z}(\tau(x)) = 2+2\kappa x + O(x^2),\ x \to 0^-, \tag{II.100}$$

and we have:

$$P_0(\theta_R \varphi^{[T]})(x) \neq 0 \Rightarrow x \in [\alpha_R, 0[, \tag{II.101}$$

with

$$z(0) < \alpha_R = O(e^{-2\kappa R}),\ R \to +\infty. \tag{II.102}$$

We introduce

$$\pi_{R,T}(x) = -\frac{1}{2\kappa x}(\theta_R\varphi^{[T]})'\left(t = -\frac{1}{2\kappa}\ln(-x) + \frac{1}{2\kappa}\ln A + \frac{1}{2}\right). \tag{II.103}$$

We have:

$$\begin{aligned}
&\pi_{R,T}(x) - p_{R,T}(x)\\
&\quad = \frac{1+O(x)}{2\kappa x + O(x^2)}\left\{\left(\theta_R\varphi^{[T]}\right)'\left(-\frac{1}{2\kappa}\ln(-x) + \frac{1}{2\kappa}\ln A + \frac{1}{2} + O(x)\right)\right.\\
&\qquad \left. - \left(\theta_R\varphi^{[T]}\right)'\left(-\frac{1}{2\kappa}\ln(-x) + \frac{1}{2\kappa}\ln A + \frac{1}{2}\right)\right\}\\
&\qquad + \left(\frac{1+O(x)}{2\kappa x + O(x^2)} - \frac{1}{2\kappa x}\right)\left(\theta_R\varphi^{[T]}\right)'\left(-\frac{1}{2\kappa}\ln(-x) + \frac{1}{2\kappa}\ln A + \frac{1}{2}\right)\\
&\quad = I_{R,T}(x) + J_{R,T}(x).
\end{aligned}$$

On the one hand, the Sobolev embedding implies that φ'' is bounded, whence

$$\sup_{R,T} \| I_{R,T} \|_{L^\infty(\mathbb{R})} \leq C\left(\| \varphi \|_{H^1([0,\infty[)} + \| \varphi'' \|_{H^1([0,\infty[)}\right),$$

and since $I_{R,T}$ is compactly supported in $[O(e^{-2\kappa R}), 0]$ we have:

$$\sup_{T} \| I_{R,T} \|_{L^2(\mathbb{R})} \leq Ce^{-\kappa R}\left(\| \varphi \|_{H^1([0,\infty[)} + \| \varphi'' \|_{H^1([0,\infty[)}\right).$$

On the other hand we obtain directly

$$\sup_{T} \| J_{R,T} \|_{L^2(\mathbb{R})} \leq Ce^{-\kappa R} \| \varphi \|_{H^1(\mathbb{R})},$$

hence we deduce that

$$\sup_T \| \pi_{R,T} - p_{R,T} \|_{L^2(\mathbb{R})} \leq Ce^{-\kappa R}\big(\| \varphi \|_{H^1([0,\infty[)} + \| \varphi'' \|_{H^1([0,\infty[)}\big), \tag{II.104}$$

and by Lemma II.8 of [6] we conclude that

$$\sup_T \| \mathbb{H}_{0,0}^{-\frac{1}{4}}(\pi_{R,T} - p_{R,T}) \|_{L^2([z(0),\infty[)} \leq Ce^{-\kappa R}\big(\| \varphi \|_{H^1([0,\infty[)} + \| \varphi'' \|_{H^1([0,\infty[)}\big). \tag{II.105}$$

Now, using (II.74) we evaluate:

$$\| \mathbb{H}_{0,0}^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2([z(0),\infty[)} = \frac{1}{2\pi}\int_{\mathbb{R}} | \zeta |^{-1} (| F_{R-T}(\zeta) |^2 - \exp(2iz(0)A^{-\kappa}e^{2\kappa T}\zeta)F_{R-T}(\zeta)\bar{F}_{R-T}(-\zeta))d\zeta, \tag{II.106}$$

where

$$F_{R-T}(\zeta) = \int e^{i\zeta e^{-2\kappa y}}(\theta_{R-T}\varphi)'(y)dy. \tag{II.107}$$

We recall that according to Lemma II.6 of [6] we have:

$$\int | \zeta |^{-1}| F_{R-T}(\zeta) |^2 \, d\zeta = \int | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \mathcal{F}(\theta_{R-T}\varphi)(\xi) |^2 \, d\xi. \tag{II.108}$$

Since $\theta_{R-T}\varphi$ tends to φ in $H^1(\mathbb{R})$ as $R-T$ tends to $-\infty$, (II.108) implies that for R fixed:

$$\int | \zeta |^{-1}| F_{R-T}(\zeta) |^2 \, d\zeta \xrightarrow[T\to+\infty]{} \int | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi, \tag{II.109}$$

and $| \zeta |^{-1} F_{R-T}(\zeta)\bar{F}_{R-T}(-\zeta)$ converges in $L^1(\mathbb{R})$. We conclude by the nonstationnary phase theorem that for R fixed

$$\| \mathbb{H}_{0,0}^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2([z(0),\infty[)} \xrightarrow[T\to+\infty]{} \frac{1}{2\pi}\int | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi. \tag{II.110}$$

We write:

$$\Big| \| \mathbb{H}_{0,0}^{-\frac{1}{4}} p_T \|_{L^2([z(0),\infty[)} - \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \pi_{R,T} \|_{L^2([z(0),\infty[)} \Big|$$
$$\leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} (p_T - p_{R,T}) \|_{L^2([z(0),\infty[)} + \| \mathbb{H}_{0,0}^{-\frac{1}{4}} (p_{R,T} - \pi_{R,T}) \|_{L^2([z(0),\infty[)},$$

so, taking account of (II.97), (II.105) and (II.110), we get:

$$\limsup_{T\to+\infty} \Bigg| \| \mathbb{H}_{0,0}^{-\frac{1}{4}} p_T \|_{L^2([z(0),\infty[)} - \left(\frac{1}{2\pi} \int \mid \xi \mid \coth\left(\frac{\pi}{2\kappa} \mid \xi \mid\right) \mid \hat{\varphi}(\xi) \mid^2 d\xi \right)^{\frac{1}{2}} \Bigg| \leq Ce^{-\kappa R}$$

hence:

$$\| \mathbb{H}_{0,0}^{-\frac{1}{4}} p_T \|^2_{L^2([z(0),\infty[)} \xrightarrow[T\to+\infty]{} \frac{1}{2\pi} \int \mid \xi \mid \coth\left(\frac{\pi}{2\kappa} \mid \xi \mid\right) \mid \hat{\varphi}(\xi) \mid^2 d\xi. \tag{II.111}$$

Thanks to (II.98) we have

$$\begin{aligned} &\| \mathbb{H}_{0,0}^{-\frac{1}{4}} p_{R,T} \|^2_{L^2([z(0),\infty[)} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \mid \xi \mid^{-1} \Big(\mid \hat{p}_{R,T}(\xi) \mid^2 - \exp\big(2i\xi z(0)\big) \hat{p}_{R,T}(\xi) \bar{\hat{p}}_{R,T}(-\xi) \Big) d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \mid \xi \mid \Big(\mid \hat{f}_{R,T}(\xi) \mid^2 - \exp\big(2i\xi z(0)\big) \hat{f}_{R,T}(\xi) \bar{\hat{f}}_{R,T}(-\xi) \Big) d\xi \\ &= \| \mathbb{H}_{0,0}^{\frac{1}{4}} f_{R,T} \|^2_{L^2([z(0),\infty[)}, \end{aligned} \tag{II.112}$$

thus using (II.97), (II.105) and (II.110) again, we deduce:

$$\| \mathbb{H}_{0,0}^{\frac{1}{4}} f_T \|^2_{L^2([z(0),\infty[)} \xrightarrow[T\to+\infty]{} \frac{1}{2\pi} \int \mid \xi \mid \coth\left(\frac{\pi}{2\kappa} \mid \xi \mid\right) \mid \hat{\varphi}(\xi) \mid^2 d\xi, \tag{II.113}$$

hence (II.95) is established.

As cab be inferred from Proposition II.8, (II.95), (II.97), all that is now required to obtain (II.96), is to prove that, given R, $f_{R,T}$ tends to 0 in the sense of distributions as $T \to \infty$. We introduce:

$$\phi_{R,T}(x) = (\theta_R \varphi^{[T]})\left(t = -\frac{1}{2\kappa} \ln(-x) + \frac{1}{2\kappa} \ln A + \frac{1}{2} \right).$$

Obviously:

$$\| \phi_{R,T} \|_{L^2([z(0),\infty[)} \xrightarrow[T\to+\infty]{} 0.$$

We remark that:

$$\phi_{R,T}(x) - f_{R,T}(x) = \int_{z(0)}^{x} (\pi_{R,T}(y) - p_{R,T}(y))dy.$$

Hence, noting that $\phi_{R,T}$ and $f_{R,T}$ are compactly supported uniformly with respect to T, we deduce from (II.104) that

$$\| f_{R,T} \|_{L^2([z(0),\infty[)} \xrightarrow[T\to+\infty]{} 0. \tag{II.114}$$

Q.E.D.

LEMMA II.11. – *For any $\varphi \in H^1(\mathbb{R})$, $\varphi(t) = 0$ for $t > t_\varphi$, we have:*

$$\| P_0(\varphi^{[T]}) \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[T\to+\infty]{} \frac{1}{\pi}\int_{-\infty}^{+\infty} | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi, \tag{II.115}$$

$$P_0(\varphi^{[T]}) \xrightarrow[T\to+\infty]{} 0 \ \ in \ \ \mathcal{H}^{\frac{1}{2}}(V,0) - weak - *. \tag{II.116}$$

Proof of Lemma II.11. – We get from (II.78) and (II.97):

$$\| P_0((1-\theta_R)\varphi^{[T]}) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq CR^{\frac{3}{2}} \left(\int_{-T}^{R+2-T} | \varphi'(t) |^2 + | \varphi(t) |^2 \, dt \right)^{\frac{1}{2}}. \tag{II.117}$$

Since

$$0 < \mathbb{H}_{0,0} \leq \mathbb{H}_{V,0} \leq \mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbb{R})},$$

the Heinz theorem implies:

$$\begin{aligned} &\| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbb{R})}\right)^{-\frac{1}{4}} \pi_{R,T} \|_{L^2(]z(0),\infty[)} \\ &\quad \leq \| \mathbb{H}_{V,0}^{-\frac{1}{4}} \pi_{R,T} \|_{L^2(]z(0),\infty[)} \leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \pi_{R,T} \|_{L^2(]z(0),\infty[)}, \end{aligned}$$

where $\pi_{R,T}$ is given by (II.103). Hence we have:

$$\begin{aligned} 0 &\leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2(]z(0),\infty[)} - \| \mathbb{H}_{V,0}^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2(]z(0),\infty[)} \\ &\leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2(]z(0),\infty[)} - \| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbb{R})}\right)^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2(]z(0),\infty[)} \\ &\leq 2\int [1 - \left(1 + | \xi |^{-1} \| V \|_{L^\infty(\mathbb{R})} e^{-4\kappa T}\right)^{-\frac{1}{2}}] \, | \xi |^{-1} | F_{R-T}(\xi) |^2 \, d\xi, \end{aligned}$$

where F_{R-T} is defined by (II.107). By the dominated convergence theorem and (II.110) we deduce that

$$\| \mathbb{H}_{V,0}^{-\frac{1}{4}} \pi_{R,T} \|^2_{L^2(]z(0),\infty[)} \xrightarrow[T\to+\infty]{} \frac{1}{2\pi} \int_{-\infty}^{+\infty} | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi. \tag{II.118}$$

We apply Lemma II.8 of [6] with $\alpha = 1$:

$$\| \mathbb{H}_{V,0}^{-\frac{1}{4}} (\pi_{R,T} - p_{R,T}) \|_{L^2(]z(0),\infty[)} \leq C \| \pi_{R,T} - p_{R,T} \|_{L^2(\mathbb{R})}, \tag{II.119}$$

and we get from (II.118) and (II.119) that

$$\| \mathbb{H}_{V,0}^{-\frac{1}{4}} p_{R,T} \|^2_{L^2(]z(0),\infty[)} \xrightarrow[T\to+\infty]{} \frac{1}{2\pi} \int_{-\infty}^{+\infty} | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi. \tag{II.120}$$

The Heinz theorem implies also:

$$\begin{aligned} \| \mathbb{H}_{0,0}^{\frac{1}{4}} f_{R,T} \|_{L^2(]z(0),\infty[)} &\leq \| \mathbb{H}_{V,0}^{\frac{1}{4}} f_{R,T} \|_{L^2(]z(0),\infty[)} \\ &\leq \| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbb{R})}\right)^{\frac{1}{4}} f_{R,T} \|_{L^2(]z(0),\infty[)} . \end{aligned}$$

Hence we have:

$$\begin{aligned} 0 &\leq \| \mathbb{H}_{V,0}^{\frac{1}{4}} f_{R,T} \|^2_{L^2(]z(0),\infty[)} - \| \mathbb{H}_{0,0}^{\frac{1}{4}} f_{R,T} \|^2_{L^2(]z(0),\infty[)} \\ &\leq \| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbb{R})}\right)^{\frac{1}{4}} f_{R,T} \|^2_{L^2(]z(0),\infty[)} - \| \mathbb{H}_{0,0}^{\frac{1}{4}} f_{R,T} \|^2_{L^2(]z(0),\infty[)} \\ &\leq 2 \int \left[1 - \left(1 + | \xi |^{-2} \| V \|_{L^\infty(\mathbb{R})}\right)^{\frac{1}{2}}\right] | \xi | | \hat{f}_{R,T}(\xi) |^2 \, d\xi \\ &\leq C \int | f_{R,T}(x) |^2 \, dx. \end{aligned}$$

hence we conclude by (II.114) that:

$$\| \mathbb{H}_{V,0}^{\frac{1}{4}} f_{R,T} \|^2_{L^2(]z(0),\infty[)} \xrightarrow[T\to+\infty]{} \frac{1}{2\pi} \int_{-\infty}^{+\infty} | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi. \tag{II.121}$$

Finally (II.115) follows from (II.121) and (II.120), and (II.116) follows from (II.115) and (II.96).

Q.E.D.

LEMMA II.12. – *There exists $C > 0$ such that for any $\varphi \in H^1(\mathbb{R})$, $\varphi(t) = 0$ for $t > t_\varphi$, we have:*

$$\| (P_V - P_0)(\varphi) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C \sum_{j=0}^{\infty} e^{-\kappa j} \left(\int_{j-1}^{j+1} | \varphi'(t) |^2 + | \varphi(t) |^2 \, dt \right)^{\frac{1}{2}}. \tag{II.122}$$

Proof of Lemma II.12. – By Lemma II.6, for $\varphi_j \in C_0^\infty(]j-1,j+1[)$ there exists $F_{V,j} \in \mathcal{H}(V,j+1)$ supported in $[-j,-j+4]$, such that:

$$P_V \varphi_j = \chi U_V(0,j+1) F_{V,j}.$$

We write:

$$\begin{aligned}(P_V - P_0)\varphi_j &= U_V(0,j+1)(F_{V,j} - F_{0,j}) \\ &\quad + (\mathbf{1} - \chi) U_V(-j-1)(F_{V,j} - F_{0,j}) \\ &\quad + (U_V(0,j+1) - U_0(0,j+1)) F_{0,j} \\ &\quad + (\mathbf{1} - \chi)(U_V(-j-1) - U_0(-j-1)) F_{0,j} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{II.123}$$

Lemma II.11 of [6] gives:

$$\| I_1 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C(1+j)^{\frac{3}{2}} \| F_{V,j} - F_{0,j} \|_{\mathcal{H}(V,j+1)}, \tag{II.124}$$

and Lemma II.8 of [6] gives:

$$\| I_2 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C \| F_{V,j} - F_{0,j} \|_{\mathcal{H}(V,j+1)} . \tag{II.125}$$

We denote u_V the solution of Lemma II.6 with data φ_j and we put:

$$\begin{aligned}\delta(t) = \int_{-t+1}^{\infty} &| \partial_x (u_V(t,x) - u_0(t,x)) |^2 \\ &+ V(x) \mid (u_V(t,x) - u_0(t,x) |^2 + | \partial_t (u_V(t,x) - u_0(t,x)) |^2 \, dx.\end{aligned}$$

We have the energy estimate for $j-1 \leq T \leq j+1$:

$$\delta(T) = 2\Re \int_{j-1}^{T} \left(\int_{-t+1}^{t+3-2j} V(x)(\partial_t u_V(t,x) - \partial_t u_0(t,x)) \bar{u}_0(t,x) dx \right) dt,$$

and since:

$$u_0(t,x) = \varphi_j\left(\frac{t-x+1}{2}\right), \tag{II.126}$$

we get, using (II.3):

$$\delta(T) \leq C \| \varphi_j \|_{L^2} \, e^{-(\kappa+\varepsilon)j} \int_{j-1}^{T} \delta^{\frac{1}{2}}(t) dt.$$

We remark that:

$$\| \varphi_j \|_{L^2} \leq 2 \| \varphi'_j \|_{L^2},$$

hence we deduce by Gronwall's inequality that

$$\| F_{V,j} - F_{0,j} \|_{\mathcal{H}(V,j+1)} \leq Ce^{-(\kappa+\varepsilon)j} \| \varphi'_j \|_{L^2}, \tag{II.127}$$

and we obtain:

$$\| I_1 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} + \| I_2 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq Ce^{-\kappa j} \| \varphi'_j \|_{L^2} . \tag{II.128}$$

To estimate I_3 we use the Duhamel formula:

$$\begin{aligned} &U_V(0,j+1)F_{0,j} - U_0(0,j+1)F_{0,j} \\ &= - \int_0^{j+1} U_V(0,s) \begin{pmatrix} 0 \\ V[U_0(s,j+1)F_{0,j}]_1 \end{pmatrix} ds. \end{aligned}$$

Since $U_0(s,j+1)F_{0,j}$ is supported in $[z(s), -s+5]$, Lemma II.11 of [6] implies:

$$\| I_3 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C \int_0^{j+1} (1+s)^{\frac{3}{2}} \| V[U_0(s,j+1)F_{0,j}]_1 \|_{L^2} \, ds. \tag{II.129}$$

We introduce θ_j solution of:

$$z(\theta_j) = \theta_j - 2j + 3.$$

By (I.5) we have:

$$\theta_j = j - \frac{3}{2} + O(e^{-2\kappa j}). \tag{II.130}$$

For $j \geq \frac{1}{2}(3 - z(0))$, we have:

$$0 \leq s \leq \theta_j \Rightarrow [U_0(0,j+1)F_{0,j}]_1(x) = f_j(x+s),$$

where

$$f_j = [U_0(0,j+1)F_{0,j}]_1.$$

According to (II.98) we have:

$$f'_j(x) = \frac{1+O(x)}{2\kappa x} \varphi'_j \left(\tau(x) + \frac{1-x}{2} \right).$$

We deduce that:

$$\| f_j \|_{L^\infty} \leq \| f'_j \|_{L^1} \leq C \| \varphi'_j \|_{L^2(\mathbb{R})},$$

and since f_j is supported in $[O(e^{-2\kappa j}), 0]$, we get:

$$\| f_j \|_{L^2} \leq Ce^{-\kappa j} \| \varphi'_j \|_{L^2(\mathbb{R})}, \tag{II.131}$$

and with (II.3)

$$0 \leq s \leq \theta_j \Rightarrow \| V[U_0(s, j+1)F_{0,j}]_1 \|_{L^2} \leq Ce^{-\kappa j} e^{-(\kappa+\varepsilon)s} \| \varphi'_j \|_{L^2(\mathbb{R})} . \tag{II.132}$$

For $s \in [\theta_j, j+1]$ we obtain by (II.3), (II.130), Lemma II.12 and Proposition II.1 of [6]:

$$\| V[U_0(s, j+1)F_{0,j}]_1 \|_{L^2} \leq Ce^{-(\kappa+\varepsilon)j} \| \varphi'_j \|_{L^2(\mathbb{R})} . \tag{II.133}$$

We conclude from (II.133), (II.132) and (II.129) that:

$$\| I_3 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq Ce^{-\kappa j} \| \varphi'_j \|_{L^2(\mathbb{R})} . \tag{II.134}$$

I_4 is estimated using Lemma II.8 of [6] and the Duhamel formula again in the following mannerytnhfefeudd n sqpp-:

$$\begin{aligned} \| I_4 \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} &\leq C \| I_4 \|_{\mathcal{H}(V,0)} \\ &\leq C \int_{-j-1}^{0} \| V(x)\varphi_j \left(\frac{s-x+1}{2} \right) \|_{L^2(\mathbb{R}_x)} ds \\ &\leq Ce^{-2(\kappa+\varepsilon)j} \| \varphi'_j \|_{L^2(\mathbb{R})} . \end{aligned} \tag{II.135}$$

Therefore we conclude from (II.128), (II.134), (II.135), that

$$\| (P_V - P_0)\varphi_j \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq Ce^{-\kappa j} \| \varphi'_j \|_{L^2(\mathbb{R})}, \tag{II.136}$$

and (II.122) follows from the expansion (II.72), (II.73), and (II.136).

Q.E.D.

Now we derive the strong improvement of Lemma II.7, by establishing the continuity of P_V from $H^1(\mathbb{R})$ into $\mathcal{H}^{\frac{1}{2}}(V, 0)$.

LEMMA II.13. – *There exists* $C_V > 0$ *such that for any* $\varphi \in H^1(\mathbb{R})$, $\varphi(t) = 0$ *for* $t > t_\varphi$, *we have:*

$$\| P_V(\varphi) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C_V \| \varphi \|_{H^1([0,\infty[)} . \tag{II.137}$$

Proof of Lemma II.13. – (II.137) is is a direct consequence of (II.122) and (II.79).

Q.E.D.

The main result on the asymptotic behaviour of $P_V\left(\varphi^{[T]}\right)$ is given by the following:

LEMMA II.14. – *For any* $\varphi \in H^1(\mathbb{R})$, $\varphi(t) = 0$ *for* $t > t_\varphi$, *we have:*

$$\| P_V(\varphi^{[T]}) \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[T \to +\infty]{} \frac{1}{\pi} \int_{-\infty}^{+\infty} | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}(\xi) |^2 \, d\xi. \tag{II.138}$$

$$P_V(\varphi^{[T]}) \xrightarrow[T \to +\infty]{} 0 \ \ in \ \ \mathcal{H}^{\frac{1}{2}}(V,0) - weak - *. \tag{II.139}$$

Proof of Lemma II.14. – By (II.122) we have:

$$\| (P_V - P_0)(\varphi^{[T]}) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C \sum_{j=0}^{\infty} e^{-\kappa j} \left(\int_{j-T-1}^{j-T+1} | \varphi'(t) |^2 + | \varphi(t) |^2 \, dt \right)^{\frac{1}{2}} .$$

We apply the dominated convergence theorem and we get

$$\| (P_V - P_0)\left(\varphi^{[T]}\right) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[T \to +\infty]{} 0.$$

Thus (II.138) and (II.139) are consequences of (II.115) and (II.116) respectively.

Q.E.D.

Given $F \in \mathcal{D}_V$, $F(x) = 0$ for $x \leq R$, we denote for $T > 0$ large enough:

$$\varphi_T(t) = [U_V(t, T)F]_1(x = -t + 1) \in H^1_{loc}(\mathbb{R}_t), \tag{II.140}$$

$$\varphi_-(t) = [\Omega^-_{0,\mathbb{R}} F]_1(x = -2t + 1) \in H^1(\mathbb{R}_t). \tag{II.141}$$

φ_T and φ_- satisfy:

$$t \geq \frac{T - R + 1}{2} \Rightarrow \varphi_T(t) = 0, \tag{II.142}$$

$$t \geq \frac{1-R}{2} \Rightarrow \varphi_-(t) = 0. \tag{II.143}$$

LEMMA II.15.

$$\| \varphi_T - \varphi_-^{[\frac{T}{2}]} \|_{H^1([0,+\infty[)} \xrightarrow[T\to+\infty]{} 0. \tag{II.144}$$

Proof of Lemma II.15. – We denote

$$u(t,x) = [U_{V,\mathbb{R}}(t)F]_1(x), \tag{II.145}$$

$$u^-(t,x) = [U_{0,\mathbb{R}}(t)\Omega_{0,\mathbb{R}}^- F]_1(x), \tag{II.146}$$

so we remark that

$$\varphi_T(t) - \varphi_-^{[\frac{T}{2}]}(t) = u(t-T, x=-t+1) - u^-(t-T, x=-t+1). \tag{II.147}$$

Given $s > 0$ we have:

$$\begin{aligned}
&\int_0^\infty \mid \varphi_T(t) - \varphi_-^{[\frac{T}{2}]}(t) \mid^2 dt \\
&\quad \leq s \sup_{\sigma\leq s-T} \| u(\sigma,x) - u^-(\sigma,x) \|^2_{L^\infty(\mathbb{R}_x)} \\
&\qquad + \int_s^{\frac{T+1-R}{2}} \mid \left(u-u^-\right)(t-T,-t+1) \mid^2 dt \\
&\quad \leq Cs \sup_{\sigma\leq s-T} \| u(\sigma,x) - u^-(\sigma,x) \|^2_{H^1(\mathbb{R}_x)} \\
&\qquad + \int_s^{\frac{T+1-R}{2}} \mid \left(u-u^-\right)(t-T,-t+1) \mid^2 dt.
\end{aligned} \tag{II.148}$$

Now we remark that u and u^- are solution of a (wave) equation, hyperbolic *with respect to* x. Hence, using the fact that:

$$\mid t \mid \leq R - x \Rightarrow u(t,x) = u^-(t,x) = 0,$$

and for any x_0 fixed in $\mathbb{R}$

$$u(t, x=-t+x_0) - u^-(t, x=-t+x_0) \to 0,\ t\to -\infty,$$

we conclude that $u(t,x)$ is the unique solution of:

$$\begin{aligned}
&t + x \leq R \Rightarrow u(t,x) \\
&\quad = u^-(t,x) + \frac{1}{2}\int_{-\infty}^{x} V(y)\left(\int_{t-x+y}^{t+x-y} u(s,y)ds\right)dy.
\end{aligned} \tag{II.149}$$

The standard L^2 and H^1 estimates for the one-dimensional wave equation, and the hypothesis on potential V assure that for $x \leq R$,:

$$\| u(t,x) \|_{L^2(]-\infty,R-x]_t)} \leq \| u^- \|_{L^2(\mathbb{R})} + C \int_{-\infty}^{x} e^{y(\kappa+\varepsilon)}(1+ \mid x-y \mid) \| u(t,y) \|_{L^2(]-\infty,R-y]_t)} \, dy, \quad \text{(II.150)}$$

$$\| u(t,x) - u^-(t,x) \|_{H^1(]-\infty,R-x]_t)} \leq C \int_{-\infty}^{x} e^{y(\kappa+\varepsilon)}(1+ \mid x-y \mid) \| u(t,y) \|_{L^2(]-\infty,R-y]_t)} \, dy. \quad \text{(II.151)}$$

Using Gronwall's inequality, (II.150) gives:

$$\sup_{x \leq R} \| u(t,x) \|_{L^2(]-\infty,R-x]_t)} < \infty,$$

hence, with (II.151) we obtain:

$$\| u(t,x) - u^-(t,x) \|_{H^1(]-\infty,R-x]_t)} \leq Ce^{\kappa x},$$

and the Sobolev embedding entails:

$$x \leq R \Rightarrow \sup_{t \leq R-x} \mid u(t,x) - u^-(t,x) \mid \leq C' e^{\kappa x}. \quad \text{(II.152)}$$

(II.148) and (II.152) give for $s, T \geq 1 - R$, $s \geq 0$:

$$\int_0^\infty \mid \varphi_T(t) - \varphi_-^{[\frac{T}{2}]}(t) \mid^2 dt \leq Cs \sup_{\sigma \leq s-T} \| u(\sigma,x) - u^-(\sigma,x) \|^2_{H^1(\mathbb{R}_x)} + C' e^{-2\kappa s}, \quad \text{(II.153)}$$

and since by Theorem II.2 we have:

$$\| u(\sigma,x) - u^-(\sigma,x) \|_{H^1(\mathbb{R}_x)} \xrightarrow[\sigma \to -\infty]{} 0,$$

we conclude that:

$$\| \varphi_T - \varphi_-^{[\frac{T}{2}]} \|_{L^2([0,+\infty[)} \xrightarrow[T \to +\infty]{} 0. \quad \text{(II.154)}$$

Now, using the classic energy estimate for equation:

$$\left(\partial_t^2 - \partial_x^2 + V\right)(u - u^-) = -Vu^-,$$

and taking $T > R - 1$, we can write:

$$\begin{aligned}
&\int_0^{\infty} \mid \varphi_T'(t) - \left(\varphi_-^{[\frac{T}{2}]}\right)'(t) \mid^2 dt \\
&\quad \leq C\Big\{ \| u(-T,x) - u^-(-T,x) \|^2_{H^1(\mathbb{R}_x)} \\
&\quad + \| \partial_t u(-T,x) - \partial_t u^-(-T,x) \|^2_{L^2(\mathbb{R}_x)} \\
&\quad + \int_0^{\frac{T+1-R}{2}} \Big(\int_{-\infty}^{1-t} V(x) \mid u^-(t-T,x) \mid \\
&\qquad\qquad . \mid \partial_t u(t-T,x) - \partial_t u^-(t-T,x) \mid dx \Big) dt \Big\} \\
&\quad \leq C\delta(T) \left\{ \delta(T) + \int_0^{\frac{T+1-R}{2}} \left(\int_{-\infty}^{1-t} e^{2(\kappa+\varepsilon)x} \mid u^-(t-T,x) \mid^2 dx \right)^{\frac{1}{2}} dt \right\} \\
&\quad \leq C'\delta(T)\big(\delta(T) + \| u^- \|_{H^1(\mathbb{R})}\big).
\end{aligned}$$

where

$$\begin{aligned}
\delta(T) = \sup_{\sigma \leq \frac{1-R-T}{2}} \big(&\| u(\sigma,x) - u^-(\sigma,x) \|_{H^1(\mathbb{R}_x)} \\
&+ \| \partial_t u(\sigma,x) - \partial_t u^-(\sigma,x) \|_{L^2(\mathbb{R}_x)} \big).
\end{aligned}$$

Since Theorem II.2 assures that:

$$\delta(T) \xrightarrow[T \to +\infty]{} 0,$$

we deduce that:

$$\| \varphi_T' - (\varphi_-^{[\frac{T}{2}]})' \|_{L^2([0,+\infty[)} \xrightarrow[T \to +\infty]{} 0. \tag{II.155}$$

Then (II.144) follows from (II.154) and (II.155).

Q.E.D.

LEMMA II.16. – *We have:*

$$\| P_V(\varphi_T) \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[T \to +\infty]{} \frac{1}{\pi} \int_{-\infty}^{+\infty} \mid \xi \mid \coth\left(\frac{\pi}{2\kappa} \mid \xi \mid \right) \mid \hat{\varphi}_-(\xi) \mid^2 d\xi. \tag{II.156}$$

$$P_V(\varphi_T) \xrightarrow[T \to +\infty]{} 0 \ \ in \ \ \mathcal{H}^{\frac{1}{2}}(V,0) - weak - *. \tag{II.157}$$

Proof of Lemma II.16. – By Lemmas II.13 and II.15 we have:

$$\| P_V(\varphi_T) - P_V(\varphi_-^{[\frac{T}{2}]}) \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq \| \varphi_T - \varphi_-^{[\frac{T}{2}]} \|_{H^1([0,\infty[)} \xrightarrow[T\to+\infty]{} 0,$$

hence the result is a consequence of Lemma II.14.

Q.E.D.

Proof of Proposition II.5. – We have

$$\chi U_V(0,T)F = P_V(\varphi_T)$$

where φ_T is given by (II.140). On the other hand (II.15) and (II.141) imply the existence of $f \in H^1(\mathbb{R})$, $f(t) = 0$ for t large enough, such that:

$$\Omega_{0,\mathbb{R}}^- F =^t (f, -f')$$

$$\hat{\varphi}_-(\xi) = \frac{1}{2}\hat{f}\left(-\frac{\xi}{2}\right)e^{-i\frac{\xi}{2}}.$$

We deduce that:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} | \xi | \coth\left(\frac{\pi}{2\kappa} | \xi |\right) | \hat{\varphi}_-(\xi) |^2 \, d\xi = \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}\right)} \, \Omega_{0,\mathbb{R}}^- F \right\|^2_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})},$$

therefore the Proposition follows from Lemma II.16.

Q.E.D.

Proof of Theorem II.3. – We have:

$$\begin{aligned} &\| U_V(0,T)F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \\ &= \| \chi U_V(0,T)F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} + \| (1-\chi)U_{V,\mathbb{R}}(-T)F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} \\ &\quad + 2\Re < \chi U_V(0,T)F, (1-\chi)U_{V,\mathbb{R}}(-T)F >_{\mathcal{H}^{\frac{1}{2}}(V,0)}. \end{aligned} \quad \text{(II.158)}$$

According to Theorem II.2

$$\| (1-\chi)U_{V,\mathbb{R}}(-T)F - U_{V,[z(0),\infty[}(-T)\Omega_{V,[z(0),\infty[}^- F \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)\cap\mathcal{H}(V,0)} \xrightarrow[T\to+\infty]{} 0, \quad \text{(II.159)}$$

therefore:

$$\| (1-\chi)U_{V,\mathbb{R}}(-T)F \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \xrightarrow[T\to+\infty]{} \| \Omega^{-}_{V,[z(0),\infty[} F \|_{\mathcal{H}^{\frac{1}{2}}(V,0)}, \tag{II.160}$$

and for $\varepsilon \in [0, \frac{1}{4}]$ we have:

$$\sup_{T\geq 0} \| (1-\chi)U_{V,\mathbb{R}}(-T)F \|_{[D(\mathbb{H}_{V,0}^{\frac{1}{4}+\varepsilon})]\times[D(\mathbb{H}_{V,0}^{-\frac{1}{4}+\varepsilon})]} < \infty. \tag{II.161}$$

We evaluate:

$$\begin{aligned}\delta(T) &= < \chi U_V(0,T)F, (1-\chi)U_{V,\mathbb{R}}(-T)F >_{\mathcal{H}^{\frac{1}{2}}(V,0)} \\ &\leq \| \chi U_V(0,T)F \|_{[D(\mathbb{H}_{V,0}^{\frac{1}{4}-\varepsilon})]\times[D(\mathbb{H}_{V,0}^{-\frac{1}{4}-\varepsilon})]} \\ &\quad \| U_{V,[z(0),\infty[}(T)(1-\chi)U_{V,\mathbb{R}}(-T)F \|_{[D(\mathbb{H}_{V,0}^{\frac{1}{4}+\varepsilon})]\times[D(\mathbb{H}_{V,0}^{-\frac{1}{4}+\varepsilon})]}\end{aligned}$$

Now given a sequence $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}) - weak - *$ as $n \to \infty$, u_n compactly supported in $[z(0), R]$, we have for any $A > 0$:

$$\begin{aligned}\| \mathbb{H}_{V,0}^{-\varepsilon} u_n \|_{L^2([z(0),\infty)} &\leq \| \mathbb{H}_{0,0}^{-\varepsilon} u_n \|_{L^2([z(0),\infty)} \\ &\leq C\big(A^{-2\varepsilon} \| u_n \|_{L^2(\mathbb{R})} + \| | \xi |^{-2\varepsilon} \hat{u}_n(\xi) \|_{L^2(|\xi|\leq A)}\big),\end{aligned}$$

$$\hat{u}_n(\xi) \xrightarrow[n\to\infty]{} 0, \quad \sup_n \| \hat{u}_n(\xi) \|_{L^\infty(|\xi|\leq A)} < \infty.$$

Hence we get that $\mathbb{H}_{V,0}^{-\varepsilon} u_n \to 0$ in $L^2([z(0),\infty[)$ for $0 < \varepsilon < \frac{1}{4}$ and we conclude by (II.47) and (II.161) that $\delta(T) \to 0$. Finally, (II.20) follows from (II.160) and (II.46).

Q.E.D.

III. CLASSICAL FIELDS

We recall that the solution of the 4-D hyperbolic mixed problem (I.6), (I.7), (I.8), is given by the propagator $U(t,s)$ which is strongly continuous on the family of finite energy spaces $\mathcal{H}(t)$ defined by (I.9). Moreover denoting $\mathcal{E}'$ the space of compactly supported distributions on $\mathbb{R}_{r_*} \times S^2_\omega$, Proposition III.1 of [6] assures that $U(t,s)$ is strongly continuous from $\mathcal{H}(s) \cap \mathcal{E}'$ into $\mathcal{H}^{\frac{1}{2}}(t) \cap \mathcal{E}'$.

Now we consider the Klein-Gordon equation on the whole Schwarzschild space-time:

$$\left\{\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2} + m^2\right)\right\}\psi = 0 \quad in\ \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega. \tag{III.1}$$

The solution of the Cauchy problem is given by the group $U_{\mathcal{S}}(t)$, ${}^t(\psi(t), \partial_t\psi(t)) = U_{\mathcal{S}}(t){}^t(\psi(0), \partial_t\psi(0))$, which is unitary on the Hilbert space of the finite energy data:

$$\mathcal{H}_{\mathcal{S}} \equiv [D(\mathbb{H}_{\mathcal{S}}^{\frac{1}{2}})] \times L^2_{\mathcal{S}}, \tag{III.2}$$

and the Hilbert space of the quantum field theory:

$$\mathcal{H}_{\mathcal{S}}^{\frac{1}{2}} \equiv [D(\mathbb{H}_{\mathcal{S}}^{\frac{1}{4}})] \times [D(\mathbb{H}_{\mathcal{S}}^{-\frac{1}{4}})], \tag{III.3}$$

where $\mathbb{H}_{\mathcal{S}}$ is the selfadjoint operator on the space $L^2_{\mathcal{S}}$:

$$L^2_{\mathcal{S}} \equiv L^2(\mathbb{R}_{r_*} \times S^2_{\omega}, r^2 dr_* d\omega), \tag{III.4}$$

$$\mathbb{H}_{\mathcal{S}} = -\frac{1}{r^2}\frac{\partial}{\partial r_*}r^2\frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2} + m^2\right), \tag{III.5}$$

with domain

$$D(\mathbb{H}_{\mathcal{S}}) = \{f \in L^2_{\mathcal{S}};\ \mathbb{H}_{\mathcal{S}} f \in L^2_{\mathcal{S}}\}. \tag{III.6}$$

We shall use a dense subspace of these spaces:

$$\begin{aligned}\mathcal{D}_{\mathcal{S}} \equiv \{{}^t(f_1, f_2);\ & f_j(r_*, \omega) \\ &= \mathbb{H}_{\mathcal{S}}(\sum_{finite} f_{j,l,m}(r_*) \otimes Y_{l,m}(\omega));\ f_{j,l,m} \in C_0^\infty(\mathbb{R})\},\end{aligned} \tag{III.7}$$

where $\{Y_{l,m};\ l \in \mathbb{N},\ m \in \mathbb{Z},\ | m | \leq l\}$ is the spherical harmonics basis of $L^2(S^2)$.

PROPOSITION III.1. – *$\mathcal{D}_{\mathcal{S}}$ is a dense subspace of $\mathcal{H}_{\mathcal{S}} \cap \mathcal{H}_{\mathcal{S}}^{\frac{1}{2}}$.*

We compare the fields near the Black-Hole, with the plane wave solutions of

$$\partial_t^2\psi_{BH} + \mathbb{H}_{BH}\psi_{BH} = 0,\ \ t \in \mathbb{R},\ \ r_* \in \mathbb{R},\ \ \omega \in S^2, \tag{III.8}$$

where $\mathbb{H}_{BH}$ is the operator

$$\mathbb{H}_{BH} = -\partial^2_{r_*}, \tag{III.9}$$

selfadjoint on the Hilbert space

$$L^2_{BH} \equiv L^2\left(\mathbb{R}_{r_*} \times S^2_{\omega},\ 4M^2 dr_* d\omega\right), \tag{III.10}$$

with domain

$$D(\mathbb{H}_{BH}) \equiv \left\{ f \in L^2_{BH};\ \partial^2_{r_*} f \in L^2_{BH} \right\}. \tag{III.11}$$

The solutions of (III.8) are given by a group $U_{BH}(t)$ unitary on the Hilbert spaces:

$$\mathcal{H}_{BH} \equiv [D(\mathbb{H}^{\frac{1}{2}}_{BH})] \times L^2_{BH}, \tag{III.12}$$

$$\mathcal{H}^{\frac{1}{2}}_{BH} \equiv [D(\mathbb{H}^{\frac{1}{4}}_{BH})] \times [D(\mathbb{H}^{-\frac{1}{4}}_{BH})]. \tag{III.13}$$

To investigate the behaviour of fields near the horizon we choose a cut-off function $\chi(r_*)$ satisfying:

$$\begin{gathered} \chi \in C^\infty(\mathbb{R}),\ \exists a, b;\ 0 < a < b < 1; \\ r_* < a \Rightarrow \chi(r_*) = 1,\ b < r_* \Rightarrow \chi(r_*) = 0, \end{gathered} \tag{III.14}$$

and we introduce the Black-Hole Horizon Wave Operators

$$\Omega^{\pm}_{BH}\Phi = \lim_{t\to\pm\infty} U_{BH}(-t)\boldsymbol{\Theta}_{BH}U_S(t)\Phi \ \ in\ \ \mathcal{H}_{BH} \tag{III.15}$$

where:

$$\boldsymbol{\Theta}_{BH}\begin{pmatrix} f \\ p \end{pmatrix}(r_*, \omega) = \begin{pmatrix} \chi(r_*)f(r_*, \omega) \\ \chi(r_*)p(r_*, \omega) \end{pmatrix}. \tag{III.16}$$

In fact, we have to distinguish between the fields, outgoing from the Black-Hole to infinity (-), and the fields infalling into the Black-Hole from infinity (+). Then we put:

$$\mathcal{H}_{\pm} \equiv \{{}^t(f, \pm\partial_{r_*} f);\ \ f \in [D(\mathbb{H}^{\frac{1}{2}}_{BH})]\},$$

$$\mathcal{H}^{\frac{1}{2}}_{\pm} \equiv \{{}^t(f, \pm\partial_{r_*} f);\ \ f \in [D(\mathbb{H}^{\frac{1}{4}}_{BH})]\}.$$

Here $\partial_{r_*} f$ is well defined by the spectral theory although $[D(\mathbb{H}^s_{BH})]$ is *not* a space of distributions for some s. As regards the asymptotic behaviour of fields at infinity, since the Schwarzschild metric is asymptotically flat, we compare the fields near space-like infinity, with scalar fields in the Minkowski space-time, solutions of the Klein-Gordon equation:

$$\frac{\partial^2}{\partial t^2}\psi_\infty - \Delta_{\mathbb{R}^3_x}\psi_\infty + m^2\psi_\infty = 0,\ in\ \mathbb{R}_t \times \mathbb{R}^3_x. \tag{III.17}$$

hence we introduce the free hamiltonian:

$$\mathbb{H}_\infty = -\Delta_{\mathbb{R}^3_x} + m^2 \tag{III.18}$$

with domain:

$$D(\mathbb{H}_\infty) = \left\{ f \in L^2(\mathbb{R}^3_x, dx);\ -\Delta_{\mathbb{R}^3_x} f \in L^2(\mathbb{R}^3_x, dx) \right\}, \tag{III.19}$$

and the Hilbert spaces:

$$\mathcal{H}_\infty \equiv [D(\mathbb{H}_\infty^{\frac{1}{2}})] \times L^2(\mathbb{R}^3), \tag{III.20}$$

$$\mathcal{H}_\infty^{\frac{1}{2}} \equiv [D(\mathbb{H}_\infty^{\frac{1}{4}})] \times \left[D(\mathbb{H}_\infty^{-\frac{1}{4}}) \right]. \tag{III.21}$$

The solutions of (III.17) are given by the free propagator

$$^t(\psi_\infty(t), \partial_t \psi_\infty(t)) = U_\infty(t)^t(\psi_\infty(t), \partial_t \psi_\infty(t))$$

which is a unitary group on these spaces. Nevertheless, according to [4], because the term $-2Mm^2r^{-1}$ in (III.1) is long range type, we must introduce the Dollard modified propagator, which is unitary on $\mathcal{H}_\infty$ and $\mathcal{H}_\infty^{\frac{1}{2}}$:

$$U_\infty^D(t) = \begin{bmatrix} \cos(t\mathbb{H}_\infty^{\frac{1}{2}} + \ln(t)\mathbb{D}_\infty) & \mathbb{H}_\infty^{-\frac{1}{2}} \sin(t\mathbb{H}_\infty^{\frac{1}{2}} + \ln(t)\mathbb{D}_\infty) \\ -\mathbb{H}_\infty^{\frac{1}{2}} \sin(t\mathbb{H}_\infty^{\frac{1}{2}} + \ln(t)\mathbb{D}_\infty) & \cos(t\mathbb{H}_\infty^{\frac{1}{2}} + \ln(t)\mathbb{D}_\infty) \end{bmatrix}. \tag{III.22}$$

Here for $f \in C_0^\infty(\mathbb{R}^3_x)$ we have:

$$(\mathbb{D}_\infty f)(x) = -\frac{Mm^2}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix.\xi} \mid \xi \mid^{-1} \left(\int_{\mathbb{R}^3} e^{-iy.\xi} f(y) dy \right) d\xi, \tag{III.23}$$

and for $t \in \mathbb{R}^*$

$$\ln(t) \equiv \frac{t}{\mid t \mid} \ln \mid t \mid. \tag{III.24}$$

We introduce an identifying operator Θ_∞ putting:

$$\begin{gathered} \Phi(r_*, \omega) \in [C_0^\infty(\mathbb{R}_{r_*} \times S^2_\omega)]^2,\ x \in \mathbb{R}^3, \\ (\Theta_\infty(\Phi))(x) = \left(1 - \chi(r_* = \mid x \mid) \Phi(r_* = \mid x \mid, \omega = \frac{x}{\mid x \mid} \right), \end{gathered} \tag{III.25}$$

and we define the Flat Infinity Wave Operators:

$$\Omega_\infty^\pm \Phi = \lim_{t \to \pm\infty} U_\infty^D(-t) \boldsymbol{\Theta}_\infty U_{\mathcal{S}}(t) \Phi \ \ in\ \ \mathcal{H}_\infty. \tag{III.26}$$

The scattering of classical fields on the whole Schwarzschild space-time is described by the following:

THEOREM III.2. – *The strong limits* (III.15) *and* (III.26) *exist for any* Φ *in* $\mathcal{H}_S$, *and do not depend on the function* χ *satisfying* (III.14). *Moreover* $\Omega_{BH}^{\pm} \oplus \Omega_{\infty}^{\pm}$ *is an isometry from* $\mathcal{H}_S$ *onto*

$\mathcal{H}_{\pm} \oplus \mathcal{H}_{\infty}$ *and can be extended as an isometry from* $\mathcal{H}_S{}^{\frac{1}{2}}$ *onto* $\mathcal{H}_{\pm}^{\frac{1}{2}} \oplus \mathcal{H}_{\infty}^{\frac{1}{2}}$ *satisfying for any* $t \in \mathbb{R}$:

$$(U_{BH}(t) \oplus U_{\infty}(t))\big(\Omega_{BH}^{\pm} \oplus \Omega_{\infty}\big) = \big(\Omega_{BH}^{\pm} \oplus \Omega_{\infty}\big)U_S(t). \tag{III.27}$$

Furthermore if $\Phi \in \mathcal{D}_S$ *we have:*

$$\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{BH}^{\frac{1}{2}}\right)}\,\Omega_{BH}^{\pm}\Phi \in \mathcal{H}_{\pm}^{\frac{1}{2}}. \tag{III.28}$$

Now we can state the fundamental result concerning the backward propagator $U(0,T)$:

THEOREM III.3 (Main Result). – *For all* Φ *in* $\mathcal{D}_S$ *we have:*

$$\| U(0,T)\Phi \|^2_{\mathcal{H}^{\frac{1}{2}}(0)}$$

$$\xrightarrow[T\to+\infty]{} \| \Omega_{\infty}^{-}\Phi \|^2_{\mathcal{H}_{\infty}^{\frac{1}{2}}} + \| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{BH}^{\frac{1}{2}}\right)}\,\Omega_{BH}^{-}\Phi \|^2_{\mathcal{H}_{-}^{\frac{1}{2}}}. \tag{III.29}$$

Remark III.4. – We note that the limit (III.29) does not depend of the history of the collapse of the star, described by the function z. It is a "à la Wheeler", "No Hair type" result.

Proof of Proposition III.1. – We shall use the basis of spherical harmonics of $L^2(S^2_\omega)$, $\{Y_{l,m};\ l \in \mathbb{N},\ m \in \mathbb{Z},\ | m |\leq l\}$. For $f \in L^1_{loc}(\mathbb{R}_x)$ we define:

$$\mathcal{R}^{-1}(f \otimes Y_{l,m})(r_*, \omega) = r^{-1}f(x) \otimes Y_{l,m}(\omega),$$

with

$$x = r_* = r + 2M\ln(r - 2M).$$

We have:

$$\mathbb{H}_S\mathcal{R}^{-1}(f \otimes Y_{l,m}) = \mathcal{R}^{-1}(\mathbb{H}_{V_l,\mathbb{R}}f \otimes Y_{l,m}), \tag{III.30}$$

with:

$$V_l(x) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right),\ \ l \in \mathbb{N}. \tag{III.31}$$

Obviously this potential satisfies assumption (II.3). Now we have:

$$\mathcal{H}_{\mathcal{S}} = \mathcal{R}^{-1}\left(\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l}\mathcal{H}(V_l,\mathbb{R})\otimes Y_{l,m}\right),$$

$$\mathcal{H}_{\mathcal{S}}^{\frac{1}{2}} = \mathcal{R}^{-1}\left(\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l}\mathcal{H}^{\frac{1}{2}}(V_l,\mathbb{R})\otimes Y_{l,m}\right),$$

and for

$$\Phi(r_*,\omega) = \sum_{finite}\mathcal{R}^{-1}(F_{l,m}(x)\otimes Y_{l,m}(\omega)), \tag{III.32}$$

we have:

$$\| \Phi \|^2_{\mathcal{H}_{\mathcal{S}}} = \sum \| F_{l,m} \|^2_{\mathcal{H}(V_l,\mathbb{R})}, \quad \| \Phi \|^2_{\mathcal{H}_{\mathcal{S}}^{\frac{1}{2}}} = \sum \| F_{l,m} \|^2_{\mathcal{H}^{\frac{1}{2}}(V_l,\mathbb{R})}.$$

At last we remark that:

$$\mathcal{D}_{\mathcal{S}} = \{\sum_{finite}\mathcal{R}^{-1}(\mathbb{H}_{V_l,\mathbb{R}}F_{l,m}\otimes Y_{l,m});\ F_{l,m}\in C_0^\infty(\mathbb{R}_x)\times C_0^\infty(\mathbb{R}_x)\}.$$

Since we have shown in the proof of Proposition II.1 that $\mathbb{H}_{V_l,\mathbb{R}}C_0^\infty(\mathbb{R}_x)$ is dense in $\left[D(\mathbb{H}^s_{V_l,\mathbb{R}})\right]$, $s = -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}$, we conclude that $\mathcal{D}_{\mathcal{S}}$ is dense in $\mathcal{H}_{\mathcal{S}}\cap\mathcal{H}_{\mathcal{S}}^{\frac{1}{2}}$.

Q.E.D.

Proof of Theorem III.2. – Lemma IV.1 in [5] states that $\Omega^{\pm}_{BH}\oplus\Omega^{\pm}_{\infty}$ exists and is an isometry from $\mathcal{H}_{\mathcal{S}}$ onto $\mathcal{H}_{\pm}\oplus\mathcal{H}_{\infty}$. Moreover Theorem 1 in [4] assures the intertwining relation (III.27). We deduce from (III.27) that $\Omega^{\pm}_{BH}\oplus\Omega^{\pm}_{\infty}$ can be extended as an isometry from $\mathcal{H}_{\mathcal{S}}^{\frac{1}{2}}$ onto $\mathcal{H}^{\frac{1}{2}}_{\pm}\oplus\mathcal{H}^{\frac{1}{2}}_{\infty}$. In fact it will be useful to recover these results in a manner which makes explicit the relation between the wave operators of the three-dimensional and the one-dimensional problems:

LEMMA III.5. – *For $F_l\in\mathcal{H}(V_l,\mathbb{R})$ we have:*

$$\Omega^{\pm}_{BH}\mathcal{R}^{-1}(F_l\otimes Y_{l,m}) = \frac{1}{2M}\Omega^{\pm}_{0,\mathbb{R}}F_l\otimes Y_{l,m} \tag{III.33}$$

Proof of Lemma III.5. – We deduce from (III.30) that:

$$U_{\mathcal{S}}(t)\mathcal{R}^{-1}(F_l\otimes Y_{l,m}) = \mathcal{R}^{-1}(U_{V_l,\mathbb{R}}(t)F_l\otimes Y_{l,m}). \tag{III.34}$$

Thus:

$$U_{BH}(-t)\boldsymbol{\Theta}_{BH}U_{\mathcal{S}}(t)\mathcal{R}^{-1}(F_l \otimes Y_{l,m})$$
$$= U_{BH}(-t)\mathcal{R}^{-1}(U_{0,\mathbb{R}}(t)U_{0,\mathbb{R}}(-t)\chi U_{V_l,\mathbb{R}}(t)F_l \otimes Y_{l,m}).$$

We easily check that $U_{BH}(-t)\mathcal{R}^{-1}U_{0,\mathbb{R}}(t)$ is equibounded with respect to t on $H^1(\mathbb{R}_{r_*}) \times L^2(\mathbb{R}_{r_*}) \otimes Y_{l,m}$ hence we get:

$$\| U_{BH}(-t)\boldsymbol{\Theta}_{BH}U_{\mathcal{S}}(t)\mathcal{R}^{-1}(F_l \otimes Y_{l,m})$$
$$- U_{BH}(-t)\mathcal{R}^{-1}(U_{0,\mathbb{R}}(t)\Omega^{\pm}_{0,\mathbb{R}}F_l \otimes Y_{l,m}) \|_{H^1(\mathbb{R}_{r_*} \times L^2(\mathbb{R}_{r_*})\otimes Y_{l,m}} \xrightarrow[t\to\pm]{} 0.$$

Now given $f \in H^1(\mathbb{R})$ we put $F^{\pm} = {}^t(f, \pm f')$, and we verify that:

$$\left\| U_{BH}(-t)\mathcal{R}^{-1}\big(U_{0,\mathbb{R}}(t)F^{\pm}\otimes Y_{l,m}\big) - \frac{1}{2M}F^{\pm}\otimes Y_{l,m} \right\|_{H^1(\mathbb{R}_{r_*} \times L^2(\mathbb{R}_{r_*})\otimes Y_{l,m}}$$
$$\xrightarrow[t\to\pm]{} 0.$$

Therefore (III.33) is a consequence of (II.15).

Q.E.D.

We could get a part of the intertwining relation (III.27) noting that:

$$\begin{aligned} U_{BH}(t)\Omega^{\pm}_{BH}\mathcal{R}^{-1}(F_l \otimes Y_{l,m}) &= \frac{1}{2M}U_{0,\mathbb{R}}(t)\Omega^{\pm}_{0,\mathbb{R}}F_l \otimes Y_{l,m} \\ &= \frac{1}{2M}\Omega^{\pm}_{0,\mathbb{R}}U_{V_l,\mathbb{R}}(t)F_l \otimes Y_{l,m} \\ &= \Omega^{\pm}_{BH}\mathcal{R}^{-1}(U_{V_l,\mathbb{R}}(t)F_l \otimes Y_{l,m}) \\ &= \Omega^{\pm}_{BH}U_{\mathcal{S}}(t)\mathcal{R}^{-1}(F_l \otimes Y_{l,m}). \end{aligned}$$

At last for Φ given by (III.32) we infer from (III.33) and (II.16):

$$\begin{aligned} &\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\,\Omega^{-}_{BH}\Phi \\ &= \frac{1}{2M}\left(\sum_{finite}\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\right)}\,\Omega^{-}_{0,\mathbb{R}}F_{l,m} \otimes Y_{l,m}\right) \\ &\in \bigoplus_{finite}\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R}) \otimes Y_{l,m} \subset \mathcal{H}^{\frac{1}{2}}_{BH}. \end{aligned} \tag{III.35}$$

Q.E.D.

Proof of Theorem III.3. – We consider

$$\Phi = \mathbb{H}_{\mathcal{S}}(\sum_{finite} F_{l,m} \otimes Y_{l,m}) \in \mathcal{D}_{\mathcal{S}}.$$

We have:

$$\Phi = \mathcal{R}^{-1}(\sum_{finite} G_{l,m} \otimes Y_{l,m}), \quad G_{l,m}(x) = \mathbb{H}_{V_l,\mathbb{R}}(rF_{l,m}(x)) \in \mathcal{D}_{V_l},$$

$$U(0,T)\Phi = \mathcal{R}^{-1}(\sum_{finite} U_{V_l}(0,T)G_{l,m} \otimes Y_{l,m}).$$

Since:

$$\mathbb{H}_t \mathcal{R}^{-1}(F \otimes Y_{l,m}) = \mathcal{R}^{-1}\mathbb{H}_{V_l,\mathbb{R}} F \otimes Y_{l,m},$$

we deduce that

$$\| U(0,T)\Phi \|^2_{\mathcal{H}^{\frac{1}{2}}(0)} = \sum_{finite} \| U_{V_l}(0,T)G_{l,m} \|^2_{\mathcal{H}^{\frac{1}{2}}(V_l,0)}.$$

Theorem II.3 implies:

$$\| U(0,T)\Phi \|^2_{\mathcal{H}^{\frac{1}{2}}(0)} \xrightarrow[T\to+\infty]{} \sum_{finite} \| \Omega^-_{V_l,[z(0),\infty[} G_{l,m} \|^2_{\mathcal{H}^{\frac{1}{2}}(V_l,0)}$$
$$+ \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{0,\mathbb{R}}\right)}\, \Omega^-_{0,\mathbb{R}} G_{l,m} \right\|^2_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})}. \tag{III.36}$$

For $\Phi_\infty \in \mathcal{H}_\infty$, we put:

$$\Omega^{in}_\infty \Phi_\infty = s - \lim_{t\to-\infty} U_{in}(-t)\Theta_\infty U^D_\infty(t)\Phi_\infty \ \ in \ \ \mathcal{H}(0),$$

where

$$U_{in}(t) = U(t,0) \ \ if \ \ t \le 0,$$
$$U_{in}(t) = U(0,-t) \ \ if \ \ t \ge 0.$$

According to Lemma IV.2 in [5], Ω^{in}_∞ is a well defined operator, which is unitary from $\mathcal{H}_\infty$ onto $\mathcal{H}(0)$ and since the intertwining relation

$$U_{in}(t)\Omega^{in}_\infty = \Omega^{in}_\infty U_\infty(t)$$

holds, this wave operator is unitary from $\mathcal{H}^{\frac{1}{2}}_\infty$ onto $\mathcal{H}^{\frac{1}{2}}(0)$. Now we remark that:

$$U_{in}(t)\mathcal{R}^{-1}(G \otimes Y_{l,m}) = \mathcal{R}^{-1}U_{V_l,\mathbb{R}}(t)G \otimes Y_{l,m}.$$

Thus we deduce that:

$$\Omega_{\infty}^{in}\Omega_{\infty}^{-}\mathcal{R}^{-1}(G\otimes Y_{l,m})=\mathcal{R}^{-1}(\Omega_{V_l,[z(0),\infty[}^{-}G\otimes Y_{l,m});$$

Therefore we get:

$$\sum_{finite}\|\,\Omega_{V_l,[z(0),\infty[}^{-}G_{l,m}\,\|_{\mathcal{H}^{\frac{1}{2}}(V_l,0)}^{2}=\|\,\Omega_{\infty}^{-}\Phi\,\|_{\mathcal{H}_{\infty}^{\frac{1}{2}}}^{2}. \tag{III.37}$$

On the other hand (III.35) implies

$$\sum_{finite}\left\|\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{0,\mathbb{R}}^{\frac{1}{2}}\right)}\,\Omega_{0,\mathbb{R}}^{-}G_{l,m}\right\|_{\mathcal{H}^{\frac{1}{2}}(0,\mathbb{R})}^{2}$$
$$=\left\|\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{BH}^{\frac{1}{2}}\right)}\,\Omega_{BH}^{-}\Phi\right\|_{\mathcal{H}_{-}^{\frac{1}{2}}}^{2}. \tag{III.38}$$

Now (III.29) follows from (III.36), (III.37), and (III.38).

Q.E.D.

IV. THE HAWKING EFFECT

We describe the emergence of the Hawking state in two frameworks of the Quantum Field Theory: the point of view of the quantization at time t according to [6] or [14], and the approach of the algebras of local observables on $\mathcal{M}$, in the spirit of [13] and [17]. Then the Hawking effect is a direct consequence of the main Theorem III.3.

IV.1. Time Dependent Quantum Fields

We consider a one parameter family of *real* vector spaces, $(\mathcal{D}_t)_{t\in\mathbb{R}}$, endowed with a skew-symetric, non degenerate, bilinear form, $\sigma_t(.,.)$, and a propagator $U(t,s)$, which is a symplectic isomorphism from $(\mathcal{D}_s,\sigma_s)$ onto $(\mathcal{D}_t,\sigma_t)$.

A *Weyl Quantization* of $(\mathcal{D},\sigma,U)$ is a family of maps $\mathfrak{W}_t:\ \Phi_t\in\mathcal{D}_t\to\mathfrak{W}_t(\Phi_t)$ from $\mathcal{D}_t$ into the space $\mathfrak{U}(\mathfrak{H})$ of unitary operators on some complex Hilbert space $\mathfrak{H}$, satisfying:

$$\begin{gathered}\forall t\in\mathbb{R},\ \forall\Phi_t,\Psi_t\in\mathcal{D}_t,\\ \mathfrak{W}_t(\Phi_t+\Psi_t)=e^{-\frac{i}{2}\sigma_t(\Phi_t,\Psi_t)}\mathfrak{W}_t(\Phi_t)\mathfrak{W}_t(\Psi_t),\end{gathered} \tag{IV.1}$$

$$\forall t \in \mathbb{R},\ \forall \Phi_t \in \mathcal{D}_t,\ \forall X \in \mathfrak{H},$$
$$(\lambda \in \mathbb{R} \to [\mathfrak{W}_t(\lambda\Phi_t)](X)) \in C^0(\mathbb{R}_\lambda, \mathfrak{H}), \tag{IV.2}$$

$$\forall s,t \in \mathbb{R},\ \ \mathfrak{W}_t \circ U(t,s) = \mathfrak{W}_s. \tag{IV.3}$$

The *Quantum Observables Algebra* $\mathfrak{A}$, is the minimal $\mathbb{C}^* - subalgebra$ in the space $\mathcal{L}(\mathfrak{H})$ of bounded linear maps on $\mathfrak{H}$, containing all the operators $\mathfrak{W}_t(\Phi_t)$. In fact, up to a norm-preserving and an involution-preserving isomorphism, $\mathfrak{A}$ does not depend either on t, or on the choice of the Weyl Quantization.

A *Quantum State* ω is a positive, normalised, linear form on $\mathfrak{A}$. The quantum state is characterized at time t by the *Generating Functional* E_t^ω defined on $\mathcal{D}_t$ by:

$$\Phi_t \in \mathcal{D}_t \mapsto E_t^\omega(\Phi_t) = \omega(\mathfrak{W}_t(\Phi_t)).$$

As a consequence of (IV.3) these functionals satisfy:

$$\forall s,t \in \mathbb{R},\ \ E_t^\omega \circ U(t,s) = E_s^\omega. \tag{IV.4}$$

Given a quantum state ω defined at time zero by E_0^ω, the fundamental problem is to describe the quantum state at time t. Thanks to (IV.4) this *quantum* problem is reduced to the study of the *classical* propagator $U(0,t)$.

We apply these considerations to the second order evolution equation:

$$\frac{d^2}{dt^2}\psi + \mathbb{H}_t\psi = 0,$$

where for any t fixed in $\mathbb{R}$, $\mathbb{H}(t)$ is a densely defined, selfadjoint operator on some Hilbert space $L^2(\mathcal{M}_t)$. We assume that the required hypotheses on the time-dependence of $\mathbb{H}(t)$ are satisfied so that the propagator $U(t,s)$ associated with this equation, exists and is an isomorphism on a family $\mathcal{D}_t$ of spaces of real Cauchy data, such that:

$$\mathcal{D}_t \subset [D(\mathbb{H}_t^{\frac{1}{4}})] \times [D(\mathbb{H}_t^{-\frac{1}{4}})].$$

The Fock quantization is defined for $\Phi_t =^t (f_t, p_t) \in \mathcal{D}_t$ by:

$$\sigma_t(\Phi_t, \Psi_t) = 2\Im < \mathbb{K}_t\Phi_t, \mathbb{K}_t\Psi_t >_{L^2(\mathcal{M}_t)}, \tag{IV.5}$$

$$\Phi_0 \in \mathcal{D}_0 \mapsto \mathfrak{W}_0(\Phi_0) = \exp\left[a^*(\mathbb{K}_0\Phi_0) - (a^*(\mathbb{K}_0\Phi_0))^*\right] \in \mathfrak{U}(\mathfrak{H}), \tag{IV.6}$$

$$\mathbb{K}_t\Phi_t = \frac{1}{\sqrt{2}}(\mathbb{H}_t^{\frac{1}{4}} f_t + i\mathbb{H}_t^{-\frac{1}{4}} p_t), \tag{IV.7}$$

where $a^*(h_0)$ is the creation operator on the Fock space

$$\mathfrak{H} = \bigoplus_{n=0}^{\infty} \left[L^2_{\mathbb{C}}(\mathcal{M}_0)\right]_s^{\otimes n}. \qquad \text{(IV.8)}$$

Here $[\mathfrak{h}]_s^{\otimes n}$ stands for the n-fold symmetric tensor product of $\mathfrak{h}$. The *Fock vacuum state* at time t is defined by the functional:

$$\Phi_t \in \mathcal{D}_t \mapsto E_t^0(\Phi_t) = \exp\left(-\frac{1}{2} \parallel \Phi_t \parallel^2_{[D(\mathbb{H}_t^{\frac{1}{4}})]\times[D(\mathbb{H}_t^{-\frac{1}{4}})]}\right), \qquad \text{(IV.9)}$$

and the *thermal state* at time t with temperature $\theta > 0$ is defined by the functional:

$$\Phi_t \in \mathcal{D}_t \mapsto E_t^\theta(\Phi_t) = \exp\left(-\frac{1}{2}\left\|\sqrt{\coth\left(\frac{1}{2\theta}\mathbb{H}_t^{\frac{1}{2}}\right)}\Phi_t\right\|^2_{[D(\mathbb{H}_t^{\frac{1}{4}})]\times[D(\mathbb{H}_t^{-\frac{1}{4}})]}\right). \qquad \text{(IV.10)}$$

These previous tools allow us to define the Fock quantization of a spin-0 field outside the collapsing star, described by (I.10), (I.11), (I.12), putting:

$$\mathcal{D}_t = \left\{\Re\left(\sum_{finite}\begin{pmatrix} f_{l,m}(r_*) \\ p_{l,m}(r_*)\end{pmatrix} \otimes Y_{l,m}(\omega)\right); \right.$$
$$(f_{l,m}, p_{l,m}) \in C_0^2 \times C_0^1([z(t), \infty[),$$
$$\left. f_{l,m}(z(t)) = p_{l,m}(z(t)) + \dot{z}(t) f'_{l,m}(z(t)) = 0 \right\}. \qquad \text{(IV.11)}$$

Since the star is stationnary in the past, we define the *Ground Quantum State* ω_0 by the Fock vacuum in the past, i.e.:

$$E_0^{\omega_0} = E_0^0. \qquad \text{(IV.12)}$$

In the same manner we can construct the quantum fields and quantum states on the Future Black-Hole Horizon, and on the Asymptotic Minkowski Space-Time (see [6]).

We investigate the asymptotic behaviour of this state as t tends to infinity, on the subalgebra of observables generated by $\mathcal{D}_{\mathcal{S}}$.

THEOREM IV.1. – *Let Φ be a real valued element of $\mathcal{D}_{\mathcal{S}}$. Then we have:*

$$\omega_0(\mathfrak{W}_t(\Phi)) \xrightarrow[t\to+\infty]{} \exp\left(-\frac{1}{2} \parallel \Omega_\infty^- \Phi \parallel^2_{\mathcal{H}_\infty^{\frac{1}{2}}}\right)$$
$$\exp\left(-\frac{1}{2}\left\|\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{BH}^{\frac{1}{2}}\right)}\Omega_{BH}^- \Phi\right\|^2_{\mathcal{H}_-^{\frac{1}{2}}}\right). \qquad \text{(IV.13)}$$

Proof of Theorem IV.1. – We get by (IV.4), (IV.9), (IV.12) and Theorem III.3:

$$\omega_0(\mathfrak{W}_t(\Phi)) = \exp\left(-\frac{1}{2} \parallel U(0,t)\Phi \parallel^2_{\mathcal{H}^{\frac{1}{2}}(0)}\right)$$
$$\xrightarrow[t\to+\infty]{} \exp\left(-\frac{1}{2} \parallel \Omega^-_\infty \Phi \parallel^2_{\mathcal{H}^{\frac{1}{2}}_\infty}\right)$$
$$\exp\left(-\frac{1}{2}\left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\,\Omega^-_{BH}\Phi\right\|^2_{\mathcal{H}^{\frac{1}{2}}_-}\right).$$

Q.E.D.

IV.2. Hawking State on $\mathcal{M}$

We construct the *Algebra of Local Observables*, $\mathfrak{A}(\mathcal{M})$, associated with the Klein-Gordon equation (I.6) on $\mathcal{M}$, in a similar way as in [13]. For $\Phi \in C_0^\infty(\mathcal{M},\mathbb{R}) \times C_0^\infty(\mathcal{M},\mathbb{R})$ we put:

$$\mathbb{E}\Phi = \int_{-\infty}^{+\infty} U(0,t)\Phi(t)dt. \tag{IV.14}$$

Following Proposition III.1 of [6], we have

$$\mathbb{E}\Phi \in \mathcal{H}^{\frac{1}{2}}(0).$$

Then we define $\mathfrak{A}(\mathcal{M})$ as the $\mathbb{C}^* - algebra$ generated by all the $\mathfrak{W}_0(\mathbb{E}\Phi)$, where $\mathfrak{W}_0$ is the Fock quantization (IV.6). The *Ground Quantum State* on $\mathfrak{A}(\mathcal{M})$, ω_0, is characterized by the functional:

$$\omega_0(\mathfrak{W}_0(\mathbb{E}\Phi)) = \exp\left(-\frac{1}{2} \parallel \mathbb{E}\Phi \parallel^2_{\mathcal{H}^{\frac{1}{2}}(0)}\right).$$

Puting:

$$\Phi^T(t,r_*,\omega) = \Phi(t-T,r_*,\omega) \tag{IV.15}$$

the fundamental problem is to evaluate:

$$\lim_{T\to+\infty} \omega_0(\mathfrak{W}_0(\mathbb{E}\Phi^T)). \tag{IV.16}$$

In fact, as in [17], we consider a subalgebra of observables, generated by the space of test functions:

$$\mathcal{D}_\mathcal{O} = \{\Phi(t,r_*,\omega) = \Re \sum_{finite} \partial_t^2 \Phi_{l,m}(t,r_*) \otimes Y_{l,m}(\omega);$$
$$\Phi_{l,m} \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)\}. \tag{IV.17}$$

To express our result, we introduce similar operators $\mathbb{E}$ on the Future Black-Hole Horizon, and on the Asymptotic Minkowski Space-Time:

$$\Phi_{BH} \in C_0^\infty(\mathbb{R}_t;\mathcal{H}_{BH}) \mapsto \mathbb{E}_{BH}\Phi_{BH} = \int_{-\infty}^{+\infty} U_{BH}(-t)\Phi_{BH}(t)dt, \qquad \text{(IV.18)}$$

$$\Phi_\infty \in C_0^\infty(\mathbb{R}_t;\mathcal{H}_\infty) \mapsto \mathbb{E}_\infty\Phi_\infty = \int_{-\infty}^{+\infty} U_\infty(-t)\Phi_\infty(t)dt. \qquad \text{(IV.19)}$$

THEOREM IV.2. – *Let Φ^0 be in $\mathcal{D}_\mathcal{O}$. Then we have:*

$$\omega_0\big(\mathfrak{W}_0(\Phi^T)\big) \xrightarrow[T\to+\infty]{} \exp\left(-\frac{1}{2}\parallel \mathbb{E}_\infty\big(\Omega_\infty^-\Phi^0\big)\parallel^2_{\mathcal{H}_\infty^{\frac{1}{2}}}\right) \exp\left(-\frac{1}{2}\left\|\sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}_{BH}^{\frac{1}{2}}\right)}\,\mathbb{E}_{BH}\big(\Omega_{BH}^-\Phi^0\big)\right\|^2_{\mathcal{H}_-^{\frac{1}{2}}}\right). \qquad \text{(IV.20)}$$

Proof of Theorem IV.2. – We evaluate:

$$\omega_0\big(\mathfrak{W}_0(\Phi^T)\big) = \exp\left(-\frac{1}{2}\parallel \mathbb{E}\Phi^T\parallel^2_{\mathcal{H}^{\frac{1}{2}}(0)}\right).$$

We have:

$$\mathbb{E}\Phi^T = U(0,T)\mathbb{E}_\mathcal{S}\Phi^0$$

where

$$\mathbb{E}_\mathcal{S}\Phi^0 = \int_{-\infty}^{+\infty} U_\mathcal{S}(-t)\Phi^0(t)dt.$$

Now if

$$\Phi^0(t,r_*,\omega) = \Re\sum_{finite}\partial_t^2\Phi_{l,m}(t,r_*)\otimes Y_{l,m}(\omega),$$

we have:

$$\mathbb{E}_\mathcal{S}\Phi^0 = -\mathbb{H}_\mathcal{S}\sum_{finite}\Re\int_{-\infty}^{+\infty} U_\mathcal{S}(-t)(\Phi_{l,m}(t)\otimes Y_{l,m})dt,$$

hence

$$\mathbb{E}_{\mathcal{S}}\Phi^0 \in \mathcal{D}_{\mathcal{S}}.$$

Then (III.27) and Theorem III.3 give:

$$\begin{aligned}
&\| \mathbb{E}\Phi^T \|^2_{\mathcal{H}^{\frac{1}{2}}(0)} \\
&\xrightarrow[T\to+\infty]{} \| \Omega^-_\infty \mathbb{E}_{\mathcal{S}}\Phi^0 \|^2_{\mathcal{H}^{\frac{1}{2}}_\infty} + \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\, \Omega^-_{BH}\mathbb{E}_{\mathcal{S}}\Phi^0 \right\|^2_{\mathcal{H}^{\frac{1}{2}}_-} \\
&= \| \mathbb{E}_\infty \Omega^-_\infty \Phi^0 \|^2_{\mathcal{H}^{\frac{1}{2}}_\infty} + \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\, \mathbb{E}_{BH}\Omega^-_{BH}\Phi^0 \right\|^2_{\mathcal{H}^{\frac{1}{2}}_-}.
\end{aligned}$$

Q.E.D.

IV.3. Comments

The physical meaning of Theorem IV.1 and Theorem IV.2, is the following: observers at rest with respect to the Schwarzschild coordinates, measure at their own infinite proper time, a thermal state at temperature $\frac{1}{8\pi M}$, of particules outgoing from the Black-Hole, to infinity. The asymptotic state does *not* depend on the history of the collapse (*No Hair* result), but only on the mass of the star; the presence of the wave operators Ω^-_{BH}, Ω^-_∞ is only related to the curvature to the underlying eternal Black-Hole. We have established in [6] a similar result for an observer falling into the Black-Hole, across the future horizon.

For simplicity we have considered only the case of a (massive or massless) scalar field outside a spherical collapsing star, without electric charge, but our approach, especially the study of the one dimensional collapse in Part II, could be easily applied to treat various kinds of fields, such as the electromagnetic tensor field [2], the Dirac field [24], the $\frac{3}{2}$-spin field [25], in geometrical frameworks associated with charged star or/and space-times with cosmological constant (De Sitter-Schwarzschild-Reissner-Nordström metrics, see e.g. [3], [19]).

Therefore, we may consider that the emergence of a thermal state with the Hawking temperature, at the last time of a spherical gravitational collapse, is at present mathematically well understood, in the framework of semi-classical approximation (weak quantum fields, fixed classical metric). The investigation of the back reaction of the Hawking radiation on the metric (Black-Hole evaporation) is another story.

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