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Maximum entropy principle in relativistic radiation hydrodynamics

by

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ABSTRACT. – The principle of maximum entropy is employed in order to close the set of the moment equations in relativistic radiation hydrodynamics to a finite order. A procedure to obtain explicit expressions for the source terms is given. In particular in the present paper bremsstrahlung and Thomson scattering are considered.

RÉSUMÉ. – Le principe d'entropie maximale est employé pour fermer à un ordre fini l'ensemble des équations des moments en hydrodynamique radiative relativiste. On donne un procédé pour obtenir des expressions explicites des termes de source. En particulier nous regardons dans cet article les phénomènes de bremsstrahlung et de diffusion Thomson.

1. INTRODUCTION

Several problems in astrophysics and cosmology require a careful relativistic treatment of radiative transport which is of crucial dynamical importance [1], [2], [3], [4]. Typical situations, such as gravitational collapse, accretion disk into black hole, perturbation of the cosmic background radiation are usually represented by a two component model: an ideal fluid which interacts with radiation [5].

The transport equation for photons is in general an integro-differential equation in seven independent variables and the effort to solve it directly by numerical methods seems prohibitive also for the present computing resources. In particular cases, for special geometries (e.g. spherical symmetry) some numerical codes exist, but the most common method to tackle the problem is to resort to analytical approximations in order to get a set of reduced equations which gives a description within an acceptable freedom of accuracy and at the same time presents reasonable numerical difficulties of implementation.

The simplified mathematical models of radiative transport describe radiation as a dissipative fluid. The main difficulty concerns the fact that in several problems radiation can cover all the range of the optical depth from the opaque to the transparent one. This implies the possibility that radiation is completely out of the equilibrium and therefore almost all the current fluid theories fail to describe radiation because they have the underlying hypothesis that the thermodynamic state is not too far from equilibrium.

By using the Chapman-Enskog expansion [6], Thomas obtained in the diffusion approximation the analogue of the Eckart theory for radiation. Therefore such an approach suffers the well-known problems of instability and acausality [7].

An improved relativistic theory, the relativistic extended thermodynamics, for dissipative fluids was proposed by Israel and Stewart [8] (see also [9], [11]). Udey and Israel [10] and Schweizer [12] showed how it is possible to cast the constitutive equations for the radiative flux and the stress tensor in the framework of extended thermodynamics. However, that theory predicts that the largest characteristic velocities are smaller than the speed of light even if there is no evident physical reason for this behaviour and the shock structure is not regular. A proposed method of overcoming these problems is to include the non-linear effects in the constitutive equations of extended thermodynamics. A systematic procedure to get this aim is provided in the frame of continuum mechanics by the Rational Extended Thermodynamics [13], [14]. Its application to radiation hydrodynamics has led to an exact nonlinear closure [15] relation of the stress tensor as a function of the radiative energy density and flux which is the same as that obtained by Levermore [16] under the hypothesis that there exists, in the presence of a static medium, a reference frame where radiation is isotropic. The analysis of the shock structure and asymptotic wave solutions shows that the inclusion of nonlinear effects does not solve the problem of the maximum characteristic speed and irregular shock structure.

An alternative approach is the method of moments (see [17] for a review). The distribution function of photons can be expanded as a series of polynomials whose coefficients are the solution of an infinite set of differential equations obtained by considering the moments of the transport equation. For practical calculations one has to truncate the infinite system to a finite order. But in the moment equation of order k, the $k+1$ and the $k+2$ moments appear. Therefore the resulting truncated system is not closed. If one simply neglects the moments of order greater than a prescribed one, the system can have complex eigenvalues. In order to obtain well behaved solutions the structure of the moment equations should provide a hyperbolic system in order to assure finite speeds of propagation for the disturbances. Several *ad hoc* closures appeared in literature. Recently, it was proposed to use the principle of maximum entropy for closing at the wanted order the set of the moment equations (see [18]).

Here we apply this method to relativistic radiation hydrodynamics and obtain an expression of the distribution function of photons which maximizes the entropy functional and has an explicit dependence on the frequency. This latter property allows us to get in a systematic way the moments of the source term as functions of the moments of the distribution function of photons and completely closes the system of the moment equations.

The developed procedure is applicable to all the cases of physical interest where radiation plays a relevant role. Here, in account of the algebraic difficulties, we shall restrict attention to bremsstrahlung and Thomson scattering. The analysis of Compton and double Compton scattering will be considered in a future paper.

In section 2, the basic equations for a radiating gas are presented, in section 3 the method of moments is sketched. Section 4 is devoted to apply the maximum entropy principle and in the subsequent section the case of almost equilibrium situation is analysed in detail. In section 6 the problem of expressing the moments of the source term is solved for bremsstrahlung and Thomson scattering. At last an asymptotic analysis for waves of high frequency is presented.

Units so that $c = \hbar = k = 1$ are used, c, $\hbar$, k being the light velocity, Planck's constant and Boltzmann's constant, respectively. We shall work in the framework of classical General Relativity and adopt for the metric tensor the signature +2 (see [19]).

2. EQUATION OF MOTION FOR A RELATIVISTIC RADIATING GAS

Let us consider a photon which moves in the space-time. By neglecting the electromagnetic field, the equations of motion are,

$$\frac{dx^\alpha}{d\lambda} = k^\alpha, \tag{1}$$

$$\frac{Dk^\alpha}{d\lambda} = 0, \tag{2}$$

where λ is an affine parameter so that k^α is the photon four-momentum and $\frac{D}{d\lambda}$ is the total derivative expressed, in local coordinates, by

$$\frac{Dk^\mu}{d\lambda} = k^\alpha \frac{\partial k^\mu}{\partial x^\alpha} + k^\alpha k^\gamma \Gamma^\mu_{\alpha\gamma}.$$

$\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols associated to the metric tensor.

The state of the photon is instantaneously determined by the four-momentum at the event x. The set of the photon states constitutes the photon phase-space

$$M_{ph} = \{(x, k) : x \in \mathcal{M}, k \in T_x(\mathcal{M}),\ k^\mu k_\mu = 0,\ k \text{ future} - \text{directed}\}$$

$T_x(\mathcal{M})$ being the tangent space to $\mathcal{M}$ in x.

In kinetic theory one assumes that the mean number of photons with world lines crossing a space-like hypersurface G of space-time is expressed as the integral of a distribution function $F(x^\mu, k^\mu)$ over the region Σ of the phase space with x in G.

Then, by introducing the following volume element on Σ [20]

$$\sigma_\alpha k^\alpha \wedge \pi_{ph}$$

with

$$\sigma_\alpha = \frac{1}{3!}\eta_{\alpha\beta\gamma\delta} dx^\beta \wedge dx^\gamma \wedge dx^\delta$$

element of hypersurface and

$$\pi_{ph} = \frac{2}{(2\pi)^3}\sqrt{-g}\,\frac{dk^1 \wedge dk^2 \wedge dk^3}{|\ k_4\ |},$$

element of volume in the space of momenta, g being the determinant of the metric tensor, one has that the mean number of photons crossing G is

$$\bar{N} = \int_{\Sigma} F(x^\mu, k^\mu)\sigma_\alpha k^\alpha \wedge \pi_{ph}.$$

Now let D be a region of the space time and $\hat{D}$ the subset of M_{ph} given by the pairs (x^μ, k^μ) of M_{ph} with x^μ in D. By using the Stokes theorem we have that the number of collisions in $\hat{D}$ is given by

$$\int_{\partial \hat{D}} F(x^\mu, k^\mu)\sigma_\alpha k^\alpha \wedge \pi_{ph} = \int_{\hat{D}} d(F(x^\mu, k^\mu)\sigma_\alpha k^\alpha \wedge \pi_{ph}).$$

In the absence of collisions or in the case of a detailed balance between creations and annihilations of photons, we have

$$d(F(x^\mu, k^\mu)\sigma_\alpha k^\alpha \wedge \pi_{ph}) = 0.$$

Whence [20]

$$L[F] = k^\alpha \frac{\partial F}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} k^\beta k^\gamma \frac{\partial F}{\partial k^\alpha} = 0,$$

which is the Liouville equation, $L[F]$ representing the density of collisions in phase-space.

In general the following transport equation holds

$$L[F] = C[F]$$

where $C[F]$ is the collision term.

Photons interact with other species of particles which obey similar transport equations. Just one additional component will be considered. We assume it to be in local thermal equilibrium and described by a Juttner distribution (the relativistic analogue of the Maxwellian distribution)

$$f(x, p^\alpha) = \left[exp\left(\frac{-p_\alpha u^\alpha(x) - \tilde{\mu}(x)}{T(x)} \right) \mp 1 \right]^{-1}$$

where p^α is the four-momentum, $\tilde{\mu}(x)$ the chemical potential, $T(x)$ the temperature and u^μ the mean four-velocity of the particles which constitute the material component, the upper sign being for bosons the lower for fermions.

By calculating the moments associated with F, we get macroscopic quantities as the 4-current density

$$N^{\mu}_{ph} = \int_{P_x} k^{\mu} F(x,k)\pi_{ph}$$

and the energy-momentum tensor of radiation

$$T^{\mu\nu}_{ph} = \int_{P_x} k^{\mu} k^{\nu} F(x,k)\pi_{ph},$$

where P_x is the future-directed part of the light cone of the tangent space at the event x.

In a similar way by integrating the Juttner distribution, one has the particle 4-current density and the energy momentum tensor of the fluid through which radiation propagates. The sum of the energy momentum tensor of matter and radiation is the source term of the Einstein equation. The fundamental equations of motion for the matter plus radiation consist of the field equation for the evolution of the metric tensor $g_{\mu\nu}$, the conservation laws of total energy-momentum and number of particles, the equations of state for the material medium and the transport equation for photons [12]. Solving this system of equations is prohibitive even numerically also in special geometry or special relativity. The main difficulty arises from the integration of the transport equation. This has led to the development of analytic approximation methods in order to simplify the transport equation. One of the most known methods is that of moments which substitutes the original transport equation with a hierarchy of equations for the integrated quantities associated to the distribution function of photons, reducing in this way the dimensions of the problem from seven in the phase-space to four in the space-time.

3. THE METHOD OF MOMENTS

Let x be an event of space-time and u^{μ} the four-velocity field of the world lines of the fluid (or generally, the world lines of "fiducial observers"). We denote by Ω the unit sphere in the projected tangent space orthogonal to u^{μ} and consider a function, $G(\mathbf{n})$, defined in Ω, $\mathbf{n}$ being the unit vector in Ω . One can develop G in series of the projected symmetric trace free (PSTF) polynomials

$$G(\mathbf{n}) = \sum_{k=0}^{\infty} \mathcal{G}_{\alpha_1 \cdots \alpha_k} \Phi^{A_k},$$

where

$$\Phi^{A_k} = 1 \quad \text{for } k = 0,$$
$$\Phi^{A_k} = n^{<\alpha_1} \cdots n^{\alpha_k>} \quad \text{for } k = 1, 2, \cdots\cdots$$

are the PSTF polynomials and

$$\mathcal{G}^{A_k} = \frac{(2k+1)!!}{4\pi k!} \int_\Omega \Phi^{A_k} G(\mathbf{n}) d\Omega,$$

are the PSTF moments of G, with $<>$ denoting the symmetric trace free component. Capital script letters are used to indicate the PSTF-tensors. The convergence of the series is guaranteed under the hypothesis of integrability of the function $G(\mathbf{n})$ and moreover if $G(\mathbf{n})$ is more regular the sum function has the same regularity.

We recall two useful formulas which will be used in the following (see [17])

$$\int_\Omega n^{\alpha_1} \cdots n^{\alpha_k} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{4\pi}{k+1} h^{(\alpha_1\alpha_2} \cdots h^{\alpha_{k-1}\alpha_k)} & \text{if } k \text{ is even} \end{cases}$$

and

$$\frac{1}{4\pi} \int_\Omega d\Omega \, (\mathcal{G}_{B_q} \Phi^{B_q}) \Phi^{A_p} = \begin{cases} 0 & \text{if } p \neq q \\ \frac{p!}{(2p+1)!!} \mathcal{G}^{A_p} & \text{if } p = q \end{cases}$$

The photon four-momentum may be resolved into pieces along and orthogonal to u_α

$$k^\lambda = \omega(u^\lambda + n^\lambda)$$

with ω photon frequency as measured in the frame comoving with u^μ, $n^\lambda n_\lambda = 1$ and $u^\lambda n_\lambda = 0$. Therefore the distribution function of photons may be written as

$$F(x^\mu; \omega, \mathbf{n}) = \mathcal{F}(x, \omega) + \mathcal{F}_\mu(x, \omega)\Phi^\mu + \mathcal{F}_{\mu\nu}(x, \omega)\Phi^{\mu\nu} + \cdots\cdots$$

The specific intensity $I_\omega = \frac{2}{(2\pi)^2}\omega^3 F$ can be developed in an analogous way. If the specific intensity is integrated over all the frequencies and expanded in series of PSTF polynomials, one gets

$$I = \int_0^\infty I_\omega d\omega = \sum_{k=0}^\infty \frac{(2k+1)!!}{4\pi k!} \mathcal{M}^{A_k} \Phi_{A_k}.$$

where

$$\mathcal{M}^{A_k}(x) \equiv \int_{P_x} \frac{k^{<\alpha_1} \ldots k^{\alpha_k>}}{(-k_\lambda u^\lambda)^{k-2}} F \pi_{ph} = \int_{P_x} \omega^2 \Phi^{A_k} F \pi_{ph}$$

are the so-called frequency-integrated PSTF-moments of F. If calculated in the local rest frame of the medium where the invariant integration element on the light cone is $\pi_{ph} = \frac{2}{(2\pi)^3} \omega \, d\omega \, d\Omega$, the moments read

$$\mathcal{M}_{A_k} = \frac{1}{\pi^2} \frac{k!}{(2k+1)!!} \int_0^\infty \omega^3 \mathcal{F}^{A_k} d\omega$$

The PSTF moments have to satisfy the infinite set of equations obtained from the transport equation integrated in the momentum space. Explicitly the kth PSTF moment equation is (see [17])

$$\Big\{ \mathcal{M}^{A_k \gamma}_{;\gamma} + \mathcal{M}^{A_k}_{;\gamma} U^\gamma + \frac{k}{2k+1} \mathcal{M}^{A_{k-1};\alpha_k} - (k-1)\mathcal{M}^{A_k \beta\gamma} \sigma_{\beta\gamma} - (k-1)\mathcal{M}^{A_k \beta} a_\beta + \frac{4}{3} \mathcal{M}^{A_k} \Theta + \frac{5k}{(2k+3)} \mathcal{M}^{A_{k-1}\gamma} \sigma^{\alpha_k}_\gamma - k \mathcal{M}^{A_{k-2}\gamma} \omega^{\alpha_k}_\gamma + \frac{k(k+3)}{(2k+1)} \mathcal{M}^{A_{k-1}} a^{\alpha_k} + \frac{k(k-1)(k+2)}{(2k-1)(2k+1)} \mathcal{M}^{A_{k-2}} \sigma^{\alpha_{k-1}\alpha_k} \Big\}^{PSTF} = \mathcal{S}^{A_k}$$

where semicolon indicates covariant derivative, $a_\alpha = u^\beta u_{\alpha;\beta}$ is the four-acceleration, $\Theta = u^\beta_{;\beta}$ the expansion, $\sigma_{\alpha\beta} = (u_{\alpha;\beta})^{PSTF}$ the shear and $\omega = \frac{1}{2}(u_{\alpha;\beta} - u_{\beta;\alpha})^P$ the rotation of the observers, while

$$\mathcal{S}^{A_k}(x) = \int_{P_x} \frac{k^{<\alpha_1} \ldots k^{\alpha_k>}}{(-k_\lambda u^\lambda)^{k-1}} C[F] \pi_{ph}$$

is the kth PSTF-moment of the source function $C[F]$.

With this procedure we substitute the original transport equation in the phase space with the set of moment equations in the space-time. However this approach presents the following problems. On one hand, for practical calculations one has to truncate the expansion to a finite order. But the equation of order k contains the moments of order $k + 1$ and $k + 2$. As a consequence in the first r equations the first $r + 2$ moments appear. Therefore one needs to find two closure relations.

On the other hand, for a general collision kernel, the moments of the source term cannot be expressed as functions of the moments of the distribution function and in principle are to be considered as additional unknown fields. Usually the latter problem is overcome by prescribing phenomenological expressions for $\mathcal{S}^{A_k}$ (e.g. see [17]) or by resorting to approximate procedures based on the Rosseland means (see [12]).

We propose a method which gives the required closure relations and allows to obtain an explicit form of the source moments in terms of the moments of the distribution function in a systematic way.

4. CLOSURE OF THE MOMENT EQUATIONS AND PRINCIPLE OF MAXIMUM ENTROPY

In order to make the PSTF moment formalism computationally viable, we assume that a certain number of macroscopic densities $\mathcal{M}^{A_k}$, with $k = 0, ..., k_{max}$, are sufficient to describe the radiation state satisfactorily. The value of k_{max} must be determined in such a way that theory matches up with experimental results.

Analysing the hierarchy of balance equations for the variables $\mathcal{M}^{A_k}$ with $k = 0, ..., k_{max}$, it is evident that, in order to obtain a closed system of field equations for these variables, it is necessary to relate the additional fields $\mathcal{M}^{A_{k_{max}}\beta}$, $\mathcal{M}^{A_{k_{max}}\beta\gamma}$, $\mathcal{S}^{A_k}$ with $k = 0, ..., k_{max}$ to the aforesaid variables and to those describing the material medium which, in our hypotheses, are the numerical density of material particles n, the temperature T and the four-velocity u^μ.

Such relations are called constitutive equations:

$$\mathcal{M}^{A_{k_{max}}\beta} = \bar{\mathcal{M}}^{A_{k_{max}}\beta}(\mathcal{M}, \cdots\cdots, \mathcal{M}^{A_{k_{max}}}, n, T, u^\mu) \tag{3}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathcal{S}^{A_{k_{max}}} = \bar{\mathcal{S}}^{A_{k_{max}}}(\mathcal{M}, \cdots\cdots, \mathcal{M}^{A_{k_{max}}}, n, T, u^\mu) \tag{4}$$

We shall derive these equations by exploiting the entropy maximum principle.

The total entropy four-flow is the sum of the material and the photonic part

$$\begin{aligned} h^\mu &= h^\mu_m + h^\mu_{ph} \\ &= -\int_{P_m} p^\mu [f\, lnf \mp (1 \pm f)\, ln(1 \pm f)]\pi_m \\ &\quad - \int_{P_{ph}} k^\mu [F\, lnF - (1+F)\, ln(1+F)]\pi_{ph}. \end{aligned}$$

According to the Boltzmann H-theorem, the entropy production is never negative

$$\sigma(x) = h^\mu_{;\mu} \geq 0.$$

The assumption that the material medium is locally in thermal equilibrium fixes, as we saw, the form of the distribution function of the material particles.

Now we assume that the photon distribution function may be written in the approximated form

$$F(x,k) \approx \bar{F}(\mathcal{M}, \cdots\cdots, \mathcal{M}^{A_{k_{max}}}, k)$$

and we require the distribution function $\bar{F}$ to maximize the radiation entropy density, as measured in the rest frame of the material medium, under the constraints

$$\mathcal{M}^{A_k} = \int_{P_{ph}} \omega^2\, \Phi^{A_k}\, F\, \pi_{ph} \qquad k = 0, 1 \cdots k_{max}. \tag{5}$$

Once we obtain $\bar{F}$, we can calculate the constitutive quantities (3)-(4), by substituting $\bar{F}$ in the expressions of $\mathcal{M}^{A_{k_{max}}\beta}$, $\mathcal{M}^{A_{k_{max}}\beta\gamma}$, $\mathcal{S}^{A_k}$ with $k = 0, ..., k_{max}$.

Let us consider the photon entropy density, as measured by fiducial observers

$$h_{ph} = -u_\mu h^\mu_{ph} = \int_{P_{ph}} u_\mu k^\mu [F\, ln\, F - (1+F)\, ln(1+F)]\pi_{ph}$$

and maximize it with respect to F under the aforesaid constraints.

We take into account of the constraints by introducing the Lagrange multipliers Λ_{A_k} with $k = 0. .... k_{max}$. These depend on the macroscopic densities describing the radiation state.

Therefore

$$h'_{ph} = h_{ph} - \sum_{j=0}^{k_{max}} \Lambda_{A_j} \left(\int_{P_{ph}} \omega^2\, \Phi^{A_j}\, F\, \pi_{ph} - \mathcal{M}^{A_j} \right)$$

has to be maximized without constraints.

Then, we must have

$$\delta h'_{ph} = 0$$

with

$$\delta h'_{ph} = \int_{P_{ph}} \left[u_\mu k^\mu \, ln \, \frac{F}{1+F} - \sum_j \Lambda_{A_j} \frac{\Phi^{A_j}}{(-k_\lambda u^\lambda)^{-2}} \right] \delta F \, \pi_{ph}.$$

This gives

$$F = \frac{1}{exp(-k_\lambda u^\lambda \sum_j \Lambda_{A_j} \Phi^{A_j}) - 1}. \tag{6}$$

When the photon gas is in equilibrium with the material medium, the photon distribution function becomes the Planckian one

$$F_0 = \frac{1}{exp(\frac{\omega}{T}) - 1}.$$

By comparing the last two expressions, we conclude that the equilibrium value of the Lagrange multiplier Λ_0 is equal to the inverse of the absolute temperature T of the material medium, while the remaining Lagrange multipliers Λ_{0A_j}, with $j = 1, ..., k_{max}$, vanish in equilibrium, the index 0 referring to quantities evaluated in the equilibrium state. In situations out of thermodynamic equilibrium, in order to obtain the explicit expressions of the lagrangian multipliers, we have to invert the highly non-linear system of equations consisting of the constraints (5), where F is given by (6). Only for very special situations, exact nonlinear solutions are known. For example in [15] the following nonlinear closure relation involving a variable Eddington factor was found

$$\mathcal{M}_{\alpha\beta} = \mathcal{M}\left(\frac{1-\chi}{2} h_{\alpha\beta} + \frac{3\chi - 1}{2} \frac{\mathcal{M}_\alpha \mathcal{M}_\beta}{\mathcal{M}^\sigma \mathcal{M}_\sigma} \right) - \frac{1}{3} \mathcal{M} h_{\alpha\beta}.$$

with

$$\chi = \frac{5}{3} - \frac{2}{3}\sqrt{4 - 3\frac{\mathcal{M}^\sigma \mathcal{M}_\sigma}{\mathcal{M}^2}},$$

which coincides with the expression obtained by Levermore [16] on the basis of kinematical considerations.

In general in order to invert eqs (5) one has to resort to numerical routines or expansion procedures. In particular if the system is not too far from equilibrium, it is possible to linearize eqs (5) and express the lagrangian multipliers as functions of the first k_{max} moments (see next section). The entropy four-vector satisfies the additional balance law

$$h^{\alpha}_{ph;\alpha} = -\int \ln\left(\frac{F}{F+1}\right) C[F]\pi_{ph}.$$

The existence of the last additional law allows us to show, by using the results presented in [21], [22], [23], [24], [25], that the resulting system of equations for the moments is symmetric time-hyperbolic, after Friedrichs and Lax (see [26] for a review), by assuring a well posedness of the Cauchy data [27] and well-behaved solutions in the sense that the propagation speed of the wave front is always finite. Let us consider the four entropy flux h^{μ}_{ph} and let $h_{ph} = -u_{\mu}h^{\mu}_{ph}$ be the entropy in the rest frame of the observer comoving with the fluid. In order to prove that the system of the moment equations is symmetric-hyperbolic we need to show that the Hessian matrix of the entropy with respect to the fields is negative definite,

$$\sum_{i,j} \frac{\partial^2 h_{ph}}{\partial \mathcal{M}_{A_i} \partial \mathcal{M}_{B_j}} \delta\mathcal{M}_{A_i} \delta\mathcal{M}_{B_j} < o,$$

for arbitrary variations $\delta\mathcal{M}_{A_j}$.

One has

$$dh_{ph} = \sum_j \Lambda_{A_j} d\mathcal{M}^{A_j}$$

and by introducing the Legendre transformation

$$\overline{h} = h_{ph} - \sum_j \Lambda_{A_j} \mathcal{M}^{A_j},$$

we can write the moments as partial derivatives

$$\mathcal{M}^{A_j} = -\frac{\partial \overline{h}}{\partial \Lambda_{A_j}}.$$

From the definition of $\mathcal{M}$, putting

$$\Sigma = \omega \sum_{j=0}^{kmax} \Lambda_{A_j} \Phi^{A_j},$$

it follows

$$\frac{\partial^2 \overline{h}}{\partial \Lambda_{A_j} \Lambda_{B_k}} = -\frac{\partial \mathcal{M}}{\partial \Lambda_{B_j}} = \frac{2}{(2\pi)^3} \int \omega^3 \frac{\exp \Sigma}{[\exp(\Sigma) - 1]^2} \frac{\partial \Sigma}{\partial \Lambda_{B_k}} \Phi^{A_j} d\omega d\Omega$$
$$= \frac{2}{(2\pi)^3} \int \omega^4 \frac{\exp \Sigma}{[\exp(\Sigma) - 1]^2} \Phi^{A_j} \Phi^{B_k} d\omega d\Omega,$$

that is the Hessian matrix of $\overline{h}$ is positive definite. But we have also

$$\sum_{j,k} \frac{\partial^2 \overline{h}}{\partial \Lambda_{A_j} \partial \Lambda_{B_k}} \delta \Lambda_{A_j} \delta \Lambda_{B_k} = -\sum_{j,k} \delta \mathcal{M}^{A_j} \delta \Lambda_{A_j}$$
$$= -\sum_{j,k} \delta \mathcal{M}^{A_j} \frac{\partial^2 h_{ph}}{\partial \mathcal{M}^{A_j} \partial \mathcal{M}^{B_k}} \delta \mathcal{M}^{B_k},$$

therefore

$$\sum_{j,k} \frac{\partial^2 h_{ph}}{\partial \mathcal{M}_{A_j} \partial \mathcal{M}_{B_k}} \delta \mathcal{M}_{A_j} \delta \mathcal{M}_{B_k} < 0.$$

We observe that the previous result holds for the exact closure relation between the lagrangian multipliers and the moments. When the eqs (5) are inverted with approximations method (e.g. expanding near equilibrium) the hyperbolicity is guaranteed only in a neighbourhood of the equilibrium state.

5. ALMOST THERMAL PHOTON DISTRIBUTION

Now we suppose that radiation is close to the equilibrium with the material medium and set

$$\lambda_{A_j} = \frac{1}{T} \delta_{0j} - \Lambda_{A_j} \quad j = 0, 1, ..., k_{max}.$$

In equilibrium

$$\lambda_0^{A_j} = 0 \quad j = 0, 1, ..., k_{max}.$$

By linearizing F with respect to the λ_{A_j}, we get

$$F \cong F_0 - T^2 \frac{\partial F_0}{\partial T} \sum_{j=0}^{k_{max}} \Phi^{A_j} \lambda_{A_j}$$

Comparing this expression with the expansion of F in series of the PSTF polynomials, at the first order in the λ's we have

$$\begin{aligned}
\mathcal{F} &= F_0 - T^2 \frac{\partial F_0}{\partial T} \lambda \\
\mathcal{F}_\mu &= -T^2 \frac{\partial F_0}{\partial T} \lambda_\mu \\
&\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots \\
\mathcal{F}_{A_{k_{max}}} &= -T^2 \frac{\partial F_0}{\partial T} \lambda_{A_{k_{max}}} \\
\mathcal{F}_{A_{k_{max}+p}} &= 0 \quad \text{for } p > 0.
\end{aligned}$$

Substituting in the formulas of the PSTF moments, we obtain

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_0 - 4\lambda T \mathcal{M}_0 \\
\mathcal{M}^{\alpha_1 \dots \alpha_k} &= \frac{k!}{(2k+1)!!} 4T\mathcal{M}_0 \lambda^{\alpha_1 \dots \alpha_k} \quad \text{for } 0 < k \leq k_{max} \\
\mathcal{M}^{\alpha_1 \dots \alpha_{k_{max}+p}} &= 0 \quad \text{for } p > 0
\end{aligned}$$

where $\mathcal{M}_0$ is the equilibrium energy density of radiation.

Incidentally, we notice that the only PSTF moment of the distribution function which has non-zero value in equilibrium is the energy density, this is due to the fact that the distribution function is isotropic in equilibrium.

Now we are able to find the Lagrange multipliers as functions of the chosen state variables

$$\begin{aligned}
\lambda &= \frac{\mathcal{M}_0 - \mathcal{M}}{4T\mathcal{M}_0} \\
\lambda^{A_j} &= \frac{(2j+1)!!}{-4T\mathcal{M}_0\, j!} \mathcal{M}^{A_j} \quad 0 < j \leq k_{max}.
\end{aligned}$$

Summarizing, we obtained two basic results:

1) the constitutive equations of the $(k_{max}+1)$th and $(k_{max}+2)$th moments of the photon distribution function are:

$$\begin{aligned}
\mathcal{M}^{\alpha_1 \dots \alpha_{k_{max}} \beta} &= 0 \\
\mathcal{M}^{\alpha_1 \dots \alpha_{k_{max}} \beta\gamma} &= 0;
\end{aligned}$$

2) we determined F as a function of the basic state variables by substituting the λ's, we found, into the expressions of the moments of

the distribution function

$$\mathcal{F} = F_0 - T\frac{\partial F_0}{\partial T}\frac{\mathcal{M}_0 - \mathcal{M}}{4\mathcal{M}_0}$$

$$\mathcal{F}^{A_j} = T\frac{\partial F_0}{\partial T}\frac{(2j+1)!!}{4\mathcal{M}_0\, j!}\mathcal{M}^{A_j} \quad 0 < j \leq k_{max}$$

We emphasize that the dependence of F on the frequency ω is completely determined.

6. THE SOURCE TERMS

Besides the closure problem, the method of moments should provide the moments of the source term as functions of the moments of the distribution function. The exact solution of this second problem can be obtained by solving the transport equation directly, but in the frame of an analytic approximation method one needs to find an external procedure to deal with the moments of the source term. The main problem is represented by the fact that in general the cross section depends on the photon frequency.

One of the advantages of the closure obtained with the maximum entropy principle is that the explicit dependence on the frequency is achieved and, as we shall show, this allows us to evaluate the moments of the collision term as functions of the moments of the photon distribution function. In principle the procedure, we shall outline, is applicable to all the cases of interaction between matter and radiation occurring in the common astrophysical situations.

Essentially, we are interested in the physical situations where the magnetic field is negligible, the frequencies of the photons of interest are large compared with those of the plasma, and in the medium electrons and positrons are in local thermodynamic equilibrium at a temperature T which is in the range $10^5 K(\rho_0/\text{gcm}^{-3})^{2/3} \ll T \ll 6 \times 10^9 K$. The last limitation assures that the medium is non-degenerate and fully ionized and at the same time that the mean electron and positron thermal speeds are non-relativistic. In such a case the dominant processes of interaction are the electron-ion bremsstrahlung and Thomson scattering, with relativistic effects represented by a possible Comptonization of the photon frequency and photon production and destruction by double Compton scattering [28].

In the present article only bremsstrahlung and Thomson scattering are considered. The cases of Compton and double Compton will be treated in a forthcoming article.

6.1. Bremsstrahlung

The invariant collision term is given by

$$C_B[F] = \frac{\pi}{2}\omega^{-2}\rho_0\epsilon_B - \omega\kappa_B F$$

where $\rho_0\epsilon_B = \rho_0\epsilon_B(\omega, n_e(x), T(x))$ is the bremsstrahlung emissivity and $\kappa_B = \kappa_B(\omega, n_e(x), T(x))$ the bremsstrahlung opacity function, n_e and ρ_0 being respectively the electron number density as measured in the comoving frame and rest mass density of the medium.

The moments of the source term for bremsstrahlung read

$$\mathcal{S}_B^{A_k}(x) = \int_{P_x} \frac{k^{<\alpha_1} \ldots k^{\alpha_k>}}{(-k_\lambda u^\lambda)^{k-1}} C_B[F]\pi_{ph} = \frac{2}{(2\pi)^3}\int_0^\infty \int d\omega\, d\Omega\, \omega^2\, \Phi^{A_k} C_B[F]$$

For $k = 0$ we have

$$\mathcal{S}_B = \rho_0\bar{\epsilon}_B - \frac{1}{\pi^2}\int_0^\infty d\omega\,\omega^3\kappa_B F_0 + \frac{1}{\pi^2}T\frac{\mathcal{M}_0 - \mathcal{M}}{4\mathcal{M}_0}\int_0^\infty d\omega\,\omega^3\kappa_B\frac{\partial F_0}{\partial T}$$

where

$$\rho_0\bar{\epsilon}_B = \frac{\rho_0}{2\pi}\int_0^\infty d\omega\epsilon_B$$

is the total emissivity per unit of volume and depends only on the local properties of the matter.

For $k \geq 1$, one has

$$\mathcal{S}_B^{A_k} = -\frac{1}{\pi^2}T\frac{\mathcal{M}^{A_k}}{4\mathcal{M}_0}\int_0^\infty d\omega\,\omega^3\kappa_B\frac{\partial F_0}{\partial T}.$$

Since the medium is in local thermodynamic equilibrium at temperature T, the following relation holds

$$\rho_0\frac{\epsilon_B}{4\pi} = \kappa_B\frac{2}{(2\pi)^2}\omega^3 F_0$$

and therefore

$$\rho_0\bar{\epsilon}_B = \frac{1}{\pi^2}\int_0^\infty d\omega\,\omega^3\kappa_B\, F_0.$$

In the case of electron-ion bremsstrahlung the total emissivity per unit volume is

$$\rho_0\bar{\epsilon}_B = \frac{16}{3}\left(\frac{2\pi}{3}\right)^{1/2} < Z^2 > \alpha^3 m_e n_*^2 T_*^{1/2}\hat{G}(T).$$

where

$$\hat{G}(T) = \int_0^\infty \overline{G} \exp(-x) dx,$$

$\overline{G}(T)$ being the mean Gaunt factor which in the regime of interest for $T \gg 10^5 K$ can be approximated by

$$\overline{G}(T) = \left(\frac{3}{\pi}\right)^{1/2} x^{-1/2} \text{ for } x \gg 1,$$

$$\overline{G}(T) = \left(\frac{\sqrt{3}}{\pi}\right) \ln(2.2/x) \text{ for } x \ll 1.$$

T_* is given by T/m_e,$n_* = \frac{n}{m_e^3}$, $< Z^2 >$ is the mean square ion charge per electron, α is the fine-structure constant and $x = \omega/T$.

If we introduce the bremsstrahlung mean time of photons

$$t_B \equiv \frac{4\pi^2 \mathcal{M}_0}{T \int d\omega\, \omega^3 \kappa_B \frac{\partial F_0}{\partial T}} \tag{7}$$

the moments of the source term can be rewritten as

$$\mathcal{S}_B = -\frac{\mathcal{M} - \mathcal{M}_0}{t_B}, \qquad \mathcal{S}_B^{A_k} = -\frac{\mathcal{M}^{A_k}}{t_B} \qquad k \geq 1.$$

6.2. Thomson scattering

The collision kernel of the Thomson scattering does not depend on the frequency and its moments can be evaluated in an exact way.

The source term for pure Thomson scattering is

$$C_T[F] = n_e \omega \frac{3}{16\pi} \sigma_T \int_\Omega d\Omega(n')\, F(\omega, n')[(n_\alpha n'^\alpha)^2 + 1] - n_e\, \omega \sigma_T\, F$$

with σ_T Thomson cross-section. By rewriting it in the form

$$C_T[F] = -\frac{9}{10} n_e\, \omega \sigma_T \mathcal{F}_{\mu\nu} \Phi^{\mu\nu} - n_e\, \omega \sigma_T \sum_{k=1,3,\cdots,k_{max}} \mathcal{F}_{A_k} \Phi^{A_k},$$

we easily get the moments

$$\mathcal{S}_T = 0$$

$$\mathcal{S}_T^{\alpha\beta} = -\frac{9}{10} \kappa_T \mathcal{M}^{\alpha\beta}$$

$$\mathcal{S}_T^{A_k} = -\kappa_T \mathcal{M}^{A_k} \quad per\ k \neq 0, 2.$$

where $\kappa = n_e \sigma_T$. Also in this case, one can write the moments of the source in the form of relaxation terms by introducing the Thomson mean time

$$t_T = \frac{1}{\kappa_T}. \tag{8}$$

7. ASYMPTOTIC ANALYSIS FOR HIGH FREQUENCY WAVE SOLUTIONS

The number of moments necessary for an adequate description of a physical problem involving radiative transport can change according situations. Essentially, it depends on the range of variability of the optical depth. For almost isotropic radiation only few moments should be sufficient. For problems such as the gravitational collapse, the mean free path is almost zero near the core of the star, but it becomes very large in the outsider stellar atmosphere. In such a case the number of necessary moments can increase considerably.

Here, in a simple physical situation, we show how it is possible to fix a minimum number of moments by analysing the characteristic velocities. Let us consider a radiative field which is in almost thermal equilibrium with a static medium. Let us suppose, moreover, that we can neglect the effect of the gravitational field and consider a Minkowski space-time. The moment equations read

$$\frac{\partial \mathcal{M}_{i_1\cdots i_k}}{\partial t} + \frac{k}{2k+1}\frac{\partial \mathcal{M}_{<i_1\cdots i_{k-1}}}{\partial x^{i_k>}} + \frac{\partial \mathcal{M}_{i_1\cdots i_k j}}{\partial x^j} = S_{i_1\cdots i_k} \qquad (9)$$
$$\mathrm{k} = 0, 1, \cdots, k_{max}$$

where

$$\mathcal{M}_{i_1\cdots i_{kmax} j} = 0$$

and the source terms $S_{i_1\cdots i_k}$ are modelled by means of relaxation times

$$S_{i_1\cdots i_k} = -\frac{1}{\tau_k}(\mathcal{M}_{i_1\cdots i_k} - \delta_{0k}\mathcal{M}_0),$$

with τ_k defined in eqs (7) and (8).

For the sake of simplicity, we restrict attention to a one-dimensional case. By choosing the x axis as direction of propagation, the equations (9) read

$$\frac{\partial \mathcal{M}_A}{\partial t} + \sum_{B=0}^{k_{max}} C_{AB}\frac{\partial \mathcal{M}_B}{\partial x} = -\frac{1}{\tau_{AB}}(\mathcal{M}_B - \delta_{0B}\mathcal{M}_0) \qquad (10)$$
$$A = 0, 1, \cdots, k_{max},$$

with

$$\mathcal{M}_A = (\mathcal{M}, \mathcal{M}_1, \mathcal{M}_{11}, \cdots)^T,$$

$1/\tau_{AB}$ the diagonal matrix

$$\frac{1}{\tau_{AB}} = \mathrm{diag}\left(\frac{1}{\tau_0}, \frac{1}{\tau_1}, \frac{1}{\tau_2}, \cdots\right)$$

and C_{AB} a tridiagonal matrix

$$C_{AB} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots \\ \alpha_1 & 0 & 1 & \cdots & \cdots \\ 0 & \alpha_2 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & \cdots & & \alpha_{kmax} & 0 \end{pmatrix} \qquad \text{with } \alpha_r = \frac{r^2}{4r^2-1}.$$

We look for solutions of the system (10) represented by wave types of high frequency and small amplitude. By introducing formally a small parameter ε, we seek solutions in the following form

$$\mathcal{U}_0 + \varepsilon \mathcal{U} \exp \frac{i}{\varepsilon}(\Omega t - qx). \tag{11}$$

q is the wave number and Ω is the frequency. $\mathcal{U}_0$ is the unpertubed constant state of thermal equilibrium,

$$\mathcal{U}_0 = (B, 0, 0, \cdots),$$

with B black-body energy density. We recall that the phase velocity is given by

$$v_{ph} = \frac{\Omega}{\mathrm{Re}(q)}.$$

By substituting the expression (11) in (10), one gets

$$i\Omega \mathcal{U}_A - i\, q \sum_B^{kmax} C_{AB} \mathcal{U}_B = -\frac{\varepsilon}{\tau_{AB}} \mathcal{U}_B.$$

If the period of the waves is very short $\varepsilon/\tau \ll 1$, one can neglect, for a small time analysis, the right hand side of eq. (10). Therefore, we obtain the same equations as in the free-streaming case, in the sense that the response time of the medium is much longer than the period of the wave.

In the limit of high frequency one expects that the phase speed is c, *i.e.* the velocity of light.

The phase speeds are given in the limit of high frequency by the eigenvalues of the matrix C_{AB} and these in turn depend on the number n of the considered moments. In table 1 we show the eigenvalues for $n = 2$, 3, 6, 10. By increasing the number of the moments, they tend to assume values between–1 and 1: for $n = 2$, the maximum absolute value of the eigenvalues λ_{max} is $\sqrt{3}/3$, for $n = 3$ $\lambda_{max} = \sqrt{3/5}$. In table 2 the value of λ_{max} is reported as a function of n. One sees that $\lambda_{max} \approx 1$ if $n \geq 30$ with an error smaller than 0.1%. However, with only 5 momenta $\lambda_{max} \approx 1$ with an error of about 10% which can be acceptable in many problems.

TABLE 1

Characteristic velocity			
for several numbers of moments			
$n = 2$	$n = 3$	$n = 6$	$n = 10$
0.577	0.774	0.932	0.974
-0.577	0	0.661	0.865
–	-0.774	0.238	0.679
–	–	-0.238	0.433
–	–	-0.661	0.149
–	–	-0.932	-0.149
–	–	–	-0.433
–	–	–	-0.679
–	–	–	-0.865
–	–	–	-0.974

TABLE 2

Characteristic velocity					
with maximum absolute value					
for several numbers of moments					
n	$\lambda_{\max}$	n	$\lambda_{\max}$	n	$\lambda_{\max}$
–	–	11	0.978	21	0.993
2	0.577	12	0.982	22	0.994
3	0.774	13	0.984	23	0.994
4	0.861	14	0.986	24	0.995
5	0.906	15	0.987	25	0.995
6	0.932	16	0.989	26	0.995
7	0.949	17	0.991	27	0.996
8	0.960	18	0.992	28	0.996
9	0.968	19	0.992	29	0.996
10	0.974	20	0.993	30	0.997

We notice that for a given n, the solutions which correspond to the eigenvalues with absolute value smaller than λ_{max} have not a clear physical meaning. They seem spurious results deriving from the approximation of the solution of the moment equations to the solution of the transport equation. It is interesting to observe that the results do not change if nonlinear terms are included in the closure relation (e.g. by inverting the constraint equations keeping also the second order term). Indeed, in account of the fact that the system is near thermodynamic equilibrium, only the first order term can be retained. As a consequence, the present asymptotic analysis is intrinsically related to the number of moments. However, the nonlinearity in the closure relation can play a crucial role in assuring the hyperbolicity of the moment equations because the results of section 5 strictly hold only for the exact solutions of eqs (5).

CONCLUSIONS AND ACKNOWLEDGMENTS

We have presented a method which allows us to close at the wanted order the set of the moment equations associated to the transport equation of photons. In order to get an explicit closure relation we have considered an almost termal equilibrium state, but in principle the procedure can be extended by including non linear terms (e.g. quadratic ones). Moreover, the inclusion of Compton and double Compton does not introduce additional conceptual difficulties. We defer to a subsequent paper the study of these further questions.

The proposed model can be usefully employed in the numerical simulation of a wide class of astrophysics problems or laboratory experiments with high speeds where special relativity is required.

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REFERENCES

[1] L. S. Shapiro and S. A. Teukolsky, Black holes, White Dwarfs and Neutron Star, John Wiley and Sons, 1983.

[2] S. W. Bruenn, Stellar Core Collapse: Numerical Model and Infall Epoch, *Ap. J.*, Vol. **58**, 1985, pp. 771-841.

[3] A. Mezzacappa and R. Matzner, Computer Simulation of time-dependent, spherically symmetric space-time containing radiating fluids: formalism and code tests, *Ap. J.*, Vol. **343**, 1989, pp. 853-873.

[4] A. MEZZACAPPA and S. BRUENN, A numerical method for solving the neutrino Boltzmann equation coupled to spherically symmetric stellar core collaps, *Ap. J.*, Vol. **405**, 1993, pp. 669-684.

[5] D. MIHALAS and B. W. MIHALAS, Foundation of Radiation Hydrodynamics, Oxford University Press, 1984.

[6] S. R. DE GROOT, W. A. VAN LEEUVEN and Ch. G. VAN WEERT, Relativistic Kinetic theory, North-Holland, 1980.

[7] W. ISRAEL, Covariant fluid mechanics and thermodynamics: an introduction, in Relativistic Fluid Dynamics, pp. 152-210, edited by A. M. Anile and Y. Choquet-Bruhat, Springer-Verlag, 1987.

[8] W. ISRAEL and J. M. STEWART, Transient relativistic Thermodynamics and kinetic theory, *Ann. Phys.*, Vol. **118**, 1979, pp. 341-372.

[9] W. A. HISCOCK and L. LINDBLOM, Stability and causality in dissipative relativistic fluids, *Ann. Phys.*, Vol. **151**, 1983, pp. 466-496.

[10] N. UDEY and W. ISRAEL, General relativistic radiative transfer: The 14 moment approximation, *Mon. Not. R. Astr. Soc.*, Vol. **199**, 1982, pp. 1137-1147.

[11] D. JOU, J. CASAZ-VÁZQUEZ and G. LEBON, Extended Irreversible Thermodynamics, Springer-Verlag, Berlin, 1993.

[12] M. A. SCHWEIZER, Relativistic radiative hydrodinamics, *Ann. Phys.*, Vol. **183**, 1988, pp. 80-121.

[13] I. MÜLLER, Thermodynamics, Pitman, Boston, 1985.

[14] I. MÜLLER and T. RUGGERI, Extended Thermodynamics, Springer, Berlin, 1993.

[15] A. M. ANILE, S. PENNISI and M. SAMMARTINO, Covariant radiation hydrodinamics, *Ann. Inst. H. Poin.*, Vol. **56**, 1992, pp. 49-74.

[16] C. D. LEVERMORE, Relating Eddington factors to flux limiters, *J.Q.S.R.T.*, Vol. **31**, 1984, pp. 149-160.

[17] S. K. THORNE, Relativistic radiative transfer: moment formalism, *Mon. Not. R. Soc.*, Vol. **194**, 1981, pp. 439-473.

[18] W. DREYER and H. STRUCHTRUP, heat pulse experiments revisited, *Con. Mech. Thermdyn.*, Vol. **5**, 1993, pp. 3-50.

[19] C. W. MISNER, K. S. THORNE and J. A. WHEELER, Gravitation, Freeman, S. Francisco, 1973.

[20] J. EHLERS, General Relativity and kinetic theory in Rendiconti della Scuola Internazionale di Fisica Enrico Fermi, XLVII Corso, pp. 1-70, Academic Press, New York, 1971.

[21] K. O. FRIEDRICHS and P. D. LAX, System of conservation equations with a convex extension, *Proceedings of the National Academy of Science U. S. A.*, Vol. **68**, 1971, pp. 1686-8.

[22] K. O. FRIEDRICHS, Symmetric hyperbolic linear differential equations, *Communication on Pure and Applied Mathematics*, Vol. **31**, 1978, pp. 123-131.

[23] G. BOILLAT, Sur l'existence et la recherche d'équations de conservation supplémentaires pour les systemes hyperbolique, *C. R. Acad. Sc., Paris*, Vol. **278 A**, 1974, pp. 909-912.

[24] G. BOILLAT, Sur une fonction croissante comme l' entropie et génératrice des chocs dans les systèmes hyperbolique, *C. R. Acad. Sc., Paris*, Vol. **283A** , 1976, pp. 409-412.

[25] T. RUGGERI and A. STRUMIA, Main field and convex covariant density for quasilinear hyperbolic systems, *Annales de l'Institut Henri Poincaré, Physique theorique*, Vol. **34**, 1981, pp. 65-84.

[26] A. M. ANILE, Relativistic Fluids and Magneto-Fluids, Cambridge University Press, Cambridge, 1989.

[27] A. FISHER and D. P. MARSDEN, The Einstein evolution equation as a first order quasilinear symmetric hyperbolic system, *Communication on Mathematical Physics*, Vol. **28**, 1972, pp. 1-38.

[28] J. OXENIUS, Kinetic Theory of Particles and Photons, Springer-Verlag, Berlin, 1986.

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