

ANNALES DE L'I. H. P., SECTION A

B. DUCOMET

Analytic properties of the resolvent for the Helmholtz operator outside a periodic surface

Annales de l'I. H. P., section A, tome 61, n° 3 (1994), p. 293-314

http://www.numdam.org/item?id=AIHPA_1994__61_3_293_0

© Gauthier-Villars, 1994, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Analytic properties of the resolvent for the Helmholtz operator outside a periodic surface

by

B. DUCOMET

Service de Physique et Techniques Nucléaires,
CEA-Centre d'Études de Bruyères-le-Châtel, BP 12, 91680 Bruyères-le-Châtel, France

ABSTRACT. – In this paper, we study the analytic properties of the resolvent of a “periodic Helmholtz operator” in three dimensions, which is the first step in studying the decay of the local energy for the solutions of the associated wave equation, outside a three-dimensional periodic surface in \mathbf{R}^3 . In particular, we show that the results proven by Gerard in the Schrödinger context also hold for the wave equation.

RÉSUMÉ. – Dans ce travail, nous étudions les propriétés analytiques de la résolvante d'un « opérateur de Helmholtz périodique » en dimension trois, qui constitue la première étape de l'étude de la décroissance de l'énergie locale pour les solutions de l'équation des ondes associées, en dehors d'une surface périodique de \mathbf{R}^3 . En particulier, nous montrons que les résultats de Gérard, dans le contexte de l'équation de Schrödinger, s'appliquent à l'équation des ondes.

1. INTRODUCTION

We consider the following Dirichlet initial boundary problem for the wave equation in an exterior domain Ω^e of \mathbf{R}^3 , with a smooth boundary Γ^e :

$$\begin{cases} \square w = 0, & \text{for } (x, t) \in \Omega^e \times [0, +\infty) \\ w(x, 0) = w_0(x), & \text{for } x \in \Omega^e \\ \partial_t w(x, 0) = w_1(x), & \text{for } x \in \Omega^e \\ w(x, t) = 0, & \text{for } (x, t) \in \Gamma^e \times [0, +\infty). \end{cases} \quad (1)$$

The local energy for any solution w of (1), in a given bounded region $D \subset \Omega^e$, is defined as:

$$E(w, D, t) = \int_{D \cap \Omega^e} (|\partial_t w|^2 + |\nabla w|^2) dx.$$

If $\mathbf{R}^3 \setminus \Omega^e$ is bounded, a lot is known about the decay of the local energy $E(w, D, t)$ (see [8], [16], [17]). It has been shown that the rate of this decay is tightly related to the geometry of Γ^e .

When Ω^e and $\mathbf{R}^3 \setminus \Omega^e$ are both unbounded, apart some scattering results (spectral theory, asymptotic completeness, partial isometry of wave operators), the situation is less clear.

In the case of scattering of classical waves by a *bounded* non-trapping obstacle, Ralston [16] has shown that the analytic properties of the time-independent propagator $R(k) = (\Delta_D + k^2)^{-1}$ for complex k , allow us to derive the decay of the local energy, using a contour deformation in the inverse Fourier-Laplace transform. In particular, in the 3d case, the contour may be pushed in such a way that the decay is exponential. Following these lines, it is natural to try to extend the previous investigations to the unbounded (periodic) case.

In the present paper, we give preliminary results in that direction: we investigate, following the ideas developed by Gerard [1] in the Schrödinger context, the analytic properties of the resolvent $R(k)$, which is a necessary intermediate step to improve the decay.

The plan of the paper is the following: after the definition of the periodic geometry of the problem (section 2), we study in section 3, the analytic continuation for the “unperturbed” (free) reduced operator corresponding to the plane surface, then we study (section 4) the boundary problem in a fundamental domain, via a Fredholm integral equation, and we give, in section 5, the analytic continuation for the “perturbed” reduced resolvent. We end the paper with the analytic continuation for the total resolvent.

2. GEOMETRY OF THE PROBLEM

Let Ω^e be a domain in \mathbf{R}^3 such that:

$$\mathbf{R}^2 \times [0, +\infty) \subset \Omega^e \subset \mathbf{R}^2 \times [h, +\infty) \quad (2)$$

for some $h > 0$.

We write the points of \mathbf{R}^3 as (x', x_3) , with $x' \in \mathbf{R}^2$ and $x_3 \in \mathbf{R}$.

Let us consider, in \mathbf{R}^2 , the lattice defined by:

$$\mathcal{L} = \left\{ l = \sum_{i=1}^2 n_i e_i, n_i \in \mathbf{Z}, i = 1, 2 \right\}, \quad (3)$$

where $\{e_i\}_{i=1,2}$ is a basis for \mathcal{L} .

We also denote by \mathcal{L}^* the dual lattice of \mathcal{L} , well defined by the dual basis $\{e_i^*\}_{i=1,2}$, with the "normalized" scalar product in \mathbf{R}^2 : $e_i^* \cdot e_j = 2\pi\delta_{ij}$.

To \mathcal{L} , we associate the fundamental domain \mathcal{C} :

$$\mathcal{C} = \{x' = (x_1, x_2) : 0 < x_i < d_i, i = 1, 2\}, \quad (4)$$

with $d_i = |e_i|$.

The dual domain \mathcal{C}^* is defined in an analogous way by:

$$\mathcal{C}^* = \left\{ p = (p_1, p_2) : 0 < p_i < \frac{2\pi}{d_i}, i = 1, 2 \right\}, \quad (5)$$

Then the domain Ω^e satisfies the periodicity condition:

For all $l \in \mathcal{L}$,

$$(x', x_3) \in \Omega^e \Rightarrow (x' + l, x_3) \in \Omega^e. \quad (6)$$

This geometric framework allows us to restrict problem 1 to the fundamental domain \mathcal{C} .

2.1 The periodic formulation

In the following, we focus our attention on the simplified Dirichlet initial boundary problem for the wave equation in Ω^e :

$$\begin{cases} \square w = 0, & \text{for } (x, t) \in \Omega^e \times [0, +\infty) \\ w(x, 0) = 0, & \text{for } x \in \Omega^e \\ \partial_t w(x, 0) = f(x), & \text{for } x \in \Omega^e \\ w(x, t) = 0, & \text{for } (x, t) \in \Gamma^e \times [0, +\infty). \end{cases} \quad (7)$$

This unessential restriction is used to clarify the exposition.

The stationary version of (7) is then obtained by Fourier-Laplace transforming (7) in the t variable:

$$\begin{cases} (\Delta + k^2) u = f & \text{in } \Omega^e \\ u = 0 & \text{for } x \in \Gamma^e, \end{cases} \tag{8}$$

where $u(x, k) = \int_0^{+\infty} e^{-ikt} w(x, t) dt$.

With the classical notation $D_i = \frac{1}{i} \frac{\partial}{\partial x_i}$, we are led to solve the Helmholtz equation:

$$\begin{cases} (D_{x'}^2 + D_{x_3}^2 - K^2) u = f & \text{in } \Omega^e \\ u = 0 & \text{for } x \in \Gamma^e, \end{cases} \tag{9}$$

with

$$D_{x'}^2 = D_{x_1}^2 + D_{x_2}^2.$$

As is well known, to guarantee the uniqueness of the solution, we must add a radiation condition. Due to the periodic character of the problem, this condition is unusual and will be given below.

We consider Γ^e as a perturbation of the plane $\Gamma_0^e \equiv \{x_3 = 0\}$, and the region Ω^e as a perturbation of the half-space $\Omega_0^e \equiv \mathbf{R}^2 \times \mathbf{R}_+$.

To take advantage of the periodicity, we define the cylindrical domains: $\Omega_0 = \mathcal{C} \times \mathbf{R}_+$ (the ‘‘unperturbed’’ cylinder), and $\Omega = \Omega^e \cap \Omega_0$ (the ‘‘perturbed’’ cylinder).

Let us consider the ‘‘unperturbed’’ Dirichlet laplacian:

$$H_0 = D_{x'}^2 + D_{x_3}^2, \tag{10}$$

with domain:

$$D(H_0) = L^2(\Delta; \Omega_0^e) \cap H_0^1(\Omega_0^e), \tag{11}$$

where:

$$L^2(\Delta; \Omega_0^e) = \{u \in L^2(\Omega_0^e) : \Delta u \in L^2(\Omega_0^e)\},$$

with the norm:

$$\|u\|_{L^2(\Delta; \Omega_0^e)} = (\|u\|_{L^2(\Omega_0^e)}^2 + \|\Delta u\|_{L^2(\Omega_0^e)}^2)^{1/2}.$$

Then, the perturbed operator \mathbf{H} can be defined formally, in the same way, by replacing everywhere Ω_0^e by Ω^e .

Following the analysis of Gerard [1], we shall see in the next sections, that the following reconstruction formula for the resolvent of \mathbf{H} is valid:

$$\begin{aligned} & \forall u \in C_0^\infty(\mathbf{R}^3) : \\ & (\mathbf{H} - k^2)^{-1} u \\ = & \frac{1}{(2\pi)^2} \int_{\mathcal{C}^*} e^{ip \cdot x'} (\mathbf{H}_p - k^2)^{-1} \left(\sum_{n \in \mathcal{L}} u(x' + n, x_3) e^{ip \cdot (x' + n)} \right) dp. \end{aligned} \quad (12)$$

As in [1], the p -integration may produce complex singularities in the k -variable, leading to a first kind of obstruction for the decay of energy in time, the second being a bad asymptotic behaviour of $(\mathbf{H} - k^2)^{-1}$ in the k -complex plane, for large $|k|$.

2.2 Functional spaces and boundary value problems

We follow the definitions of Alber [9], to define suitable restrictions of periodic operators, and periodic extensions of operators acting on functions defined in the period.

First, we consider the following truncated domains:

$$\begin{aligned} \Omega_{\rho, \rho'} &= \{x \in \Omega : \rho < x_3 < \rho'\} \\ \Omega_\rho &= \{x \in \Omega : x_3 < \rho\}. \end{aligned}$$

The space of \mathcal{L} -periodic test functions $C_{\mathcal{L}}^\infty(\mathbf{R}^3)$ is the set of smooth functions $C^\infty(\mathbf{R}^3)$ such that:

1. $u(x' + n, x_3) = u(x)$, for $x \in \mathbf{R}^3$,
2. $\text{Supp}(u) \subset \{x \in \mathbf{R}^3 : |x_3| < \rho\}$,

for a suitable $\rho > 0$.

For each open set U in Ω , we denote by $H_\Delta(U)$ the Sobolev space:

$$H_\Delta(U) = \{u \in H^1(U) : \Delta u \in L^2(U)\},$$

and by $H_{\mathcal{L}}^s(\Omega)$, the closure of $C_{\mathcal{L}}^\infty(\mathbf{R}^3)$ in $H^s(U)$.

Now we can define a suitable domain for the reduced operators.

Let $u \in L_{\text{loc}}^2(\Omega)$, and \tilde{u} its unique \mathcal{L} -periodic continuation. We say that $u \in H_{\Delta, \mathcal{L}}(\Omega)$, if:

1. $u \in H_{\mathcal{L}}^1(\Omega) \cap H_\Delta(U)$, and
2. $\tilde{u} \in H_{\Delta, \text{loc}}(\Omega^e)$.

Then we define the reduced operator H_p by:

$$\begin{cases} D(\mathbf{H}_p) = H_{\Delta, \mathcal{L}}(\Omega), \\ \forall u \in H_{\Delta, \mathcal{L}}(\Omega) : \mathbf{H}_p u = ((D_{x'} + p)^2 + D_{x_3}^2) u. \end{cases} \tag{13}$$

Now we introduce a convenient radiation condition, adapted to the periodic situation.

To see this we consider the Fourier expansion of an arbitrary solution u of the equation $(\mathbf{H}_p - k^2)u = 0$:

$$u(x) = \sum_{n \in \mathcal{L}} u_n(x_3) e^{in \cdot x'} \tag{14}$$

where the functions u_n satisfy:

$$\frac{d^2}{dx_3^2} u_n + (k^2 - M_n) u_n = 0, \tag{15}$$

and, for $p \in \mathcal{C}^*$:

$$M_n = M_n(p) = (p + n)^2.$$

Then, we say that:

CONDITION 1. – (Alber) *The function u satisfies Condition \mathcal{R} for $k \in \mathbf{R}$, if:*

$$\begin{cases} \frac{d}{dx_3} u_n = i|(M_n - k^2)^{1/2}|u_n + O(1), & \text{if } M_n < k^2 \\ u_n = O(1), & \text{if } M_n > k^2. \end{cases} \tag{16}$$

Then we consider the problem:

PROBLEM 1. – *Let $k \in \mathbf{R}$, such that $k^2 \neq M_n, \forall n \in \mathcal{L}$.*

Let $g \in L^2_{loc}(\Omega^e)$ with support in the strip $\{x_3 < R\}$, for $R > 0$, large enough.

We look for $u \in L^2_{loc}(\Omega^e)$, \mathcal{L} -periodic, such that:

$$\begin{cases} (\mathbf{H}_p - k^2)u = g & \text{in } \Omega^e \\ u = 0 & \text{for } x \in \Gamma^e, \end{cases} \tag{17}$$

and satisfying the radiation condition \mathcal{R} .

3. ANALYTIC PROPERTIES OF THE “UNPERTURBED” REDUCED RESOLVENT

In order to consider Ω^e as a perturbation of Ω_0^e , we need to extend any operator \mathcal{A} , with domain $D(\mathcal{A}) = D_{\Omega^e}$, acting on functions defined in Ω^e , to an operator acting on functions defined in the unperturbed space Ω_0^e , with domain $D_{\Omega_0^e}$: we extend it, after [2], by zero on $D_{\Omega^e}^\perp = D_{\Omega_0^e}^\perp \ominus D_{\Omega^e}$, the orthogonal of D_{Ω^e} in $D_{\Omega_0^e}$.

The same extension holds for the “reduced” quantities relative to Ω_0 and Ω .

For any $\phi \in D_{\Omega_0^e}$, we write:

$$\phi = \psi + \psi^\perp,$$

where $\psi^\perp \in D_{\Omega_0^e}^\perp$.

We shall use freely this convention, for the reduced operators $\mathbf{H}_{0,p}$, \mathbf{H}_p , or the “complete” operators \mathbf{H}_0 and \mathbf{H} .

3.1 Properties of the “unperturbed” resolvent kernel

Let us consider, for any $(x, y, p) \in \Omega_0 \times \Omega_0 \times \mathcal{C}^*$, the following formal series, representing the kernel of the reduced resolvent $(\mathbf{H}_{0,p} - k^2)^{-1}$:

$$\begin{aligned} & \mathbf{K}_{0,p}(x', x_3, y', y_3, p) \\ &= \sum_{n \in \mathcal{L}^*} \frac{e^{in(x'-y')}}{2\omega_n} [e^{i\omega_n|x_3-y_3|} - e^{i\omega_n|x_3+y_3|}], \end{aligned} \quad (18)$$

where: $\omega_n = \omega_n(k, p) = [k^2 - (n+p)^2]^{1/2}$, with the choice $\text{Im}(\omega_n) > 0$, for the determination of the square root.

The properties of $\mathbf{K}_{0,p}$ are the following:

LEMMA 1. – Let $y = (y', y_3)$ any given point in Ω_0 .

1. The operator $\mathbf{K}_{0,p}$, with kernel $\mathbf{K}_{0,p}(x', x_3, y', y_3, p)$ is bounded from $L^2(\Omega_0)$ into $H^2(\Omega_0)$, if $[k^2 - (n+p)^2]^{1/2} > 0, \forall n \in \mathcal{L}^*$.

2. If $|n+p| \neq k$, the series (18) is convergent and defines an element of $L_{\text{loc}}^2(\Omega_0^e)$, which satisfies, as a distribution in $\mathcal{D}'(\Omega_0^e)$, the equation:

$$\begin{aligned} & (\mathbf{H}_{0,p} - k^2) \mathbf{K}_{0,p}(x', x_3, y', y_3, p) \\ &= \sum_{n \in \mathcal{L}^*} \delta(x' - y' - n) \otimes \delta(x_3 - y_3). \end{aligned} \quad (19)$$

3. The series (18) satisfies, in $(C^\infty_\mathcal{L}(\Omega_0))'$, the equation:

$$(\mathbf{H}_{0,p} - k^2) \mathbf{K}_{0,p}(x', x_3, y', y_3, p) = \delta(x - y) \otimes \delta(x_3 - y_3). \quad (20)$$

Proof. – Relation (19) is verified by a direct computation: (18) is the (periodic) elementary solution of the problem.

Let $\rho > 0$, and $0 < y_3 < \rho$, and consider:

$$\Phi_n^-(x_3, y_3, k) = \frac{1}{2\omega_n} e^{i\omega_n|x_3-y_3|},$$

and:

$$\Phi_n^+(x_3, y_3, k) = \frac{-1}{2\omega_n} e^{i\omega_n(x_3+y_3)}.$$

It is sufficient to check that:

$$\sum_{n \in \mathcal{L}^*} \int_{\Omega_\rho} |\Phi_n^+(x_3, y_3, k) + \Phi_n^-(x_3, y_3, k)|^2 dx' dx_3 < \infty. \quad (21)$$

But an elementary computation gives:

$$\begin{aligned} \int_0^\rho |\Phi_n^+|^2 dx_3 &= \frac{1}{2|\omega_n|^2} \int_0^\rho e^{-2\text{Im}(\omega_n)|x_3-y_3|} dx_3 \\ &\leq \frac{1}{2(\text{Im}(\omega_n))^3} (1 - e^{-2\text{Im}(\omega_n)(\rho+x_3)}), \end{aligned}$$

which is the general term of a convergent series. This shows that $\sum_{n \in \mathcal{L}^*} \|\Phi_n^+\|_{\Omega_\rho}^2 < \infty$. The same result holds clearly for $\sum_{n \in \mathcal{L}^*} \|\Phi_n^-\|_{\Omega_\rho}^2$.

If $\text{Im}(\omega_n) > 0$, we can make $\rho \rightarrow \infty$, which allows the convergence in $L^2(\Omega_0)$: exponential decay in x_3 implies that $\mathbf{K}_{0,p}$ is bounded from L^2 to H^2 \square

Now, we split the kernel $\mathbf{K}_{0,p}(x, y, k)$ into a regular contribution, and a singular part, and we consider the decomposition: $\mathbf{K}_{0,p}(x, y, k) = \sum_{n \in \mathcal{L}^*} \mathbf{K}_{0,p}^{(n)}(x, y, k)$.

LEMMA 2. – Let $\mathbf{G}_0(x, y, k)$ the fundamental outgoing solution for the Helmholtz operator:

$$\mathbf{G}_0(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}. \quad (22)$$

Then the difference: $\mathbf{K}_{0,p}^{\text{reg}}(x, y, k) = \mathbf{K}_{0,p}(x, y, k) - \mathbf{G}_0(x, y, k)$, is a function C^∞ in the x variable, in each open set of $\Omega_0^e \setminus \mathcal{L}_y$, where:

$$\mathcal{L}_y = \{x = (x', x_3) \in \Omega_0^e : x' - y' \in \mathcal{L}, x_3 = y_3\}. \quad (23)$$

Moreover the \mathcal{R} (Alber) radiation condition is satisfied:

$$\begin{cases} \frac{d}{dx_3} \mathbf{K}_{0,p}^{(n)} = i|(M_n - k^2)^{1/2}| \mathbf{K}_{0,p}^{(n)} + O(1), & \text{if } M_n < k^2 \\ \mathbf{K}_{0,p}^{(n)} = O(1), & \text{if } M_n > k^2. \end{cases} \quad (24)$$

Proof. – $\mathbf{K}_{0,p}^{\text{reg}}(x, y, k)$ satisfies, $\forall x \in \Omega_0^c \setminus \mathcal{L}_y$, Helmholtz equation:

$$(\Delta_x + k^2) \mathbf{K}_{0,p}^{\text{reg}}(x, y, k) = 0, \quad (25)$$

in $\mathcal{D}'(\Omega)$.

After lemma 1, $\mathbf{K}_{0,p}(x, y, k)$ is in $L_{\text{loc}}^2(\Omega_0^c)$, the regularized kernel $\mathbf{K}_{0,p}^{\text{reg}}(x, y, k)$ enjoys the same property, and the C^∞ regularity is a consequence of standard elliptic regularity of Δ .

The radiation condition holds because it is clearly satisfied for each term in the series. \square

Finally, we give a further property of $\mathbf{K}_{0,p}(x, y, k)$, which completes the decomposition given in lemma 2:

LEMMA 3. – For $\text{Im}(k) > 0$, $p \in C^*$, and $x \in \mathbf{R}^3$, the following formula holds:

$$\frac{1}{2S^*} \sum_{n \in \mathcal{L}^*} \frac{e^{in(x'-y')}}{2\omega_n} e^{i\omega_n|x_3-y_3|} = \sum_{n \in \mathcal{L}, n+x-y \neq 0} \frac{e^{ik|n+x-y|}}{4\pi|n+x-y|} \quad (26)$$

The proof is just a Poisson transform of the series defining $\mathbf{K}_{0,p}(x, y, k)$.

We just remark that the free Green's function $\mathbf{G}_0(x, y, k)$ is the first term on the left hand side of (26).

3.2 Properties of the “unperturbed” resolvent kernel

Because of the presence of square-roots in the preceding definitions [see equation (18)], the global definition of $\mathbf{K}_{0,p}(x, y, k)$ for complex arguments requires to uniformize the ω_n .

In our periodic situation, the “quasi momentum” p is placed on the same footing as the temporal Fourier variable k . So, following closely Gerard [1], we suppose that k belongs to a bounded open set $\mathcal{U} \subset \mathbf{C}$, and that p remains in a bounded complex neighbourhood \mathcal{W} of \mathcal{L}^* . Then, as k and p move, only a finite number of the M_n change their determination, and

we call $\mathcal{J} \subset \mathcal{L}^*$ the set of the corresponding indices $\mathcal{J} = (n_1, \dots, n_N)$, with $\text{card}(\mathcal{J}) = N$.

We consider the following complex analytic set:

$$\mathcal{G} \subset \mathbf{C}^{(N+3)} = \{(p, k, z_1, \dots, z_N) \in \mathcal{W} \times \mathcal{U} \times \mathbf{C}^N : z_j^2 = k^2 - (n_j + p)^2, j = 1 \dots N\}. \tag{27}$$

The part of \mathcal{G} restricted to the z_j with positive imaginary part, for $j = 1 \dots N$, is a smooth submanifold of \mathcal{G} denoted by \mathcal{G}_∞ . Then $\mathbf{K}_{0,p}(x, y, k)$ can be considered as the restriction to \mathcal{G}_∞ , of the following expression:

$$\begin{aligned} & \mathbf{K}_0(x', x_3, y', y_3, p, k, z_1 \dots z_N) \\ &= \sum_{j=1}^N e^{in_j \cdot (x' - y')} \frac{1}{2z_j} [e^{iz_j|x_3 - y_3|} - e^{iz_j|x_3 + y_3|}] \\ &+ \sum_{n \in \mathcal{L}^* \setminus \mathcal{J}} e^{in \cdot (x' - y')} \frac{1}{2(k^2 - (n + p)^2)^{1/2}} \\ & [e^{i(k^2 - (n+p)^2)^{1/2}|x_3 - y_3|} - e^{i(k^2 - (n+p)^2)^{1/2}|x_3 + y_3|}] \end{aligned} \tag{28}$$

Denoting collectively by z the N -uplet (z_1, \dots, z_N) , we observe that, for $(p, k, z) \in \mathcal{G}_\infty$, the function $\mathbf{K}_0(x, y, p, k, z)$ is the kernel of an operator $\mathbf{K}_0(p, k, z)$ bounded in $L^2(\Omega_0)$.

Now, we restrict the variations of z to the following product of half-planes, each of them including a small negative neighbourhood of the real axis:

$$\mathcal{Z} = \{z \in \mathbf{C}^N : \text{Im}(z_i) \geq -\varepsilon, i = 1 \dots N\}, \tag{29}$$

for $\varepsilon > 0$.

Now we need to extend the results of lemma 1 in the complex domain, to control the exponential growth of $\mathbf{K}_0(p, k, z)$ in a complex neighbourhood of infinity. So we introduce the weighted spaces [1]:

$$L_\alpha^2(\Omega_0) = \{u \in L_{\text{loc}}^2(\Omega_0) : e^{\alpha \langle x_3 \rangle} u \in L^2(\Omega_0)\}, \tag{30}$$

and:

$$H_\alpha^1(\Omega_0) = \{u \in H_{\text{loc}}^1(\Omega_0) : e^{\alpha \langle x_3 \rangle} u \in H^1(\Omega_0)\}, \tag{31}$$

where $\langle x_3 \rangle = (1 + x_3^2)^{1/2}$ for $\alpha \in \mathbf{R}$, and we put on these spaces the natural associated hilbertian structure.

Then we have:

PROPOSITION 1. – For $a > \varepsilon$, $\mathbf{K}_0(p, k, z)$ can be extended into a bounded operator from $L_a^2(\Omega_0)$ into $H_{-a}^1(\Omega_0)$, meromorphic for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, with polar singularities defined by:

$$\{z_j = 0, j = 1 \dots N\},$$

2. The residues of $\mathbf{K}_0(p, k, z)$ are finite rank operators, and one has the following decomposition, for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$:

$$\forall j = 1 \dots N : \mathbf{K}_0(p, k, z) = \frac{\pi_j}{z_j} + M(p, k, z), \quad (32)$$

where $M(p, k, z)$ is bounded from $L_a^2(\Omega_0)$ into $H_{-a}^1(\Omega_0)$, holomorphic near the singular set $\{z_j = 0, j = 1 \dots N\}$, and the projectors π_j are rank one operators in $L_a^2(\Omega_0)$.

Proof. – The proof of these properties is a straightforward extension of the corresponding proposition of [1].

4. THE BOUNDARY PROBLEM FOR THE “PERTURBED” REDUCED OPERATOR

Let us come back to problem 1, restricted to the fundamental domain Ω :

$$\begin{cases} (\mathbf{H}_p - k^2) u = g, & \text{in } \Omega, \\ u = 0, & \text{for } x \in \Sigma, \end{cases} \quad (33)$$

with the radiation condition \mathcal{R} .

As seen before, we look for the solution u of (33) in the space $H_{\Delta, \mathcal{L}}(\Omega)$. Then, the solution in the total exterior region Ω^e is recovered as the periodic extension \tilde{u} of the solution u .

Let g be the right hand side, in equation (33). It is \mathcal{L} -periodic, and may be written as a function of f :

$$g(x) = \sum_{n \in \mathcal{L}} e^{-ip \cdot (x' + n)} f(x' + n, x_3). \quad (34)$$

In order to use Green's formula, it will be useful to modify slightly the periodic kernel $\mathbf{K}_0(x, y, p, k, z)$, defining the quasi-periodic expression:

$$\mathbf{G}(x, y, p, k, z) = e^{ip \cdot (x' - y')} \mathbf{K}_0(x, y, p, k, z). \quad (35)$$

It is clear that the properties of \mathbf{G} are deduced directly from those of \mathbf{K}_0 : we change \mathbf{K}_0 only by a phase depending only on the tangential variables x' . In particular, proposition 1 applies to \mathbf{G} without any change.

We introduce the volume potential \mathcal{T} in Ω by:

$$\forall v \in C_0^\infty(\bar{\Omega}), \forall (x, y) \in \Omega \times \Omega :$$

$$\mathcal{T} v(x) = -2 \int_{\Omega} \mathbf{G}(x, y, p, k, z) v(y) dy, \quad (36)$$

and the simple-layer potential \mathcal{V} on Σ :

$$\forall \phi \in C^\infty(\Sigma), \forall x \in \Omega :$$

$$\mathcal{V} \phi(x) = -2 \int_{\Sigma} \mathbf{G}(x, y, p, k, z) \phi(y) d\Sigma(y). \quad (37)$$

To give an integral representation for the solution of problem 1 [equations (33)], we call \mathcal{U}_{ad} the set of admissible solutions of (33):

$$\mathcal{U}_{\text{ad}} = \{u \in H_{\Delta, \mathcal{L}}(\Omega) : (\mathbf{H}_p - k^2)u = g \text{ in } \Omega : u|_{\Sigma} = 0\}. \quad (38)$$

To perform limit processes in the potentials, we need a trace result, the proof of which is a direct generalization of the standard Lions-Magenes results [7]:

LEMMA 4. – Let $u \in H_{\Delta, \mathcal{L}}(\Omega)$, and let $v \in H^1(\Omega)$ with bounded support.

Then, the trace $\gamma_0 u = u|_{\Sigma}$ is well defined in $H^{1/2}(\Sigma)$, and the “second” trace $\gamma_1 u = \frac{\partial u}{\partial \nu}|_{\Sigma}$ is defined in $H^{-1/2}(\Sigma)$ by the formula:

$$\int_{\Omega} v \Delta u dx + \int_{\Omega} \nabla u \nabla v dx = \left\langle \frac{\partial u}{\partial \nu} |_{\Sigma}, u|_{\Sigma} \right\rangle_{H^{1/2}, H^{-1/2}}, \quad (39)$$

in the duality between $H^{1/2}(\Sigma)$, and $H^{-1/2}(\Sigma)$.

Then we have the representation formula:

LEMMA 5. – $\forall v \in \mathcal{U}_{\text{ad}}$, we have the following expression:

$$\forall x \in \Omega : v(x) = -\frac{1}{2} \mathcal{V} \psi(x) - \frac{1}{2} \mathcal{T} g, \quad (40)$$

where: $\psi = \frac{\partial v}{\partial \nu}|_{\Sigma}$.

Proof. – Let us consider the truncated domain Ω_R .

Its boundary is composed of three pieces. The lateral surface of the cylinder: $S_{\text{lat}} = \{x \in \Omega : x' \in \partial \mathcal{C}, x_3 < R\}$, the upper plane region: $\Sigma_R = \{x_3 = R\} \cap \bar{\Omega}$, and the portion of the periodic surface, situated in the fundamental domain: $\Sigma = \partial \Omega^e \cap \Omega$.

If we transform the equation:

$$(\mathbf{H}_p - k^2) u = g, \tag{41}$$

with: $h = e^{ip \cdot x'} g$, and: $v = e^{ip \cdot x'} u$, then, using: $\mathbf{H}_p u = -e^{-ip \cdot x'} \Delta (e^{ip \cdot x'} u)$, we get:

$$(\Delta + k^2) v = -h.$$

Using this transformation, we have also for the Green function \mathbf{G} :

$$(\Delta_x + k^2) \mathbf{G}(x, y, p, k, z) = -\delta(x - y). \tag{42}$$

Green formula in Ω_R gives:

$$\begin{aligned} \int_{\partial\Omega_R} \left(v(y) \frac{\partial}{\partial\nu_y} \mathbf{G}(x, y, p, k, z) - \mathbf{G}(x, y, p, k, z) \frac{\partial}{\partial\nu_y} v(y) \right) d\sigma(y) \\ = v(x) - \int_{\Omega_R} \mathbf{G}(x, y, p, k, z) h(y) dy. \end{aligned} \tag{43}$$

Decomposing the surface term in the left hand side, into three parts according to the three contributions of $\partial\Omega_R$, we check first that, by the quasi-periodicity of v and \mathbf{G} , the S_{lat} -contribution is zero. Secondly, we see that the integral on Σ_R tends to zero, as t increases.

To see that, we Fourier expand v and \mathbf{G} on Σ_R . Denoting by v_n and \mathbf{G}_n , their Fourier coefficients, we find, after an elementary computation the integral:

$$\int_{\Sigma_R} dx' \sum_{n \in \mathcal{L}^*} g_n(x) e^{i\omega_n R} \left(\frac{dv_n}{dx_3} - i\omega_n v_n \right)_{x_3=R}. \tag{44}$$

Then, by the radiation condition \mathcal{R} , we find that this integral is zero as $R \rightarrow +\infty$, which ends the proof. \square

Now, as it is classical in potential theory, we have to perform a limit process $\Omega \ni x \rightarrow \tilde{x} \in \Sigma$, to obtain an integral equation on Σ , satisfied by the normal derivative of v .

To this purpose, we consider the following limits for \mathcal{T} , and \mathcal{V} , defined $\forall v \in C_0^\infty(\Omega)$, and $\forall \xi \in \Sigma$, by:

$$\mathbf{U} v(x) = -2 \int_{\Omega} \mathbf{G}(x, y, p, k, z) v(y) dy, \tag{45}$$

and: $\forall \phi \in C^\infty(\Sigma)$, and $\forall \xi \in \Sigma$, by:

$$\mathbf{V} \phi(x) = -2 \int_{\Sigma} \mathbf{G}(x, y, p, k, z) \phi(y) d\Sigma(y). \tag{46}$$

We can extend these operators:

LEMMA 6. – *The volume potential \mathcal{T} extends to a continuous operator from $H^s_{\text{comp}}(\Omega)$ into $H^{s+2}(\Omega)$, for any $s \in \mathbf{R}$, and its restriction \mathbf{U} extends to a continuous operator from $H^s_{\text{comp}}(\Omega)$ into $H^{s+2}(\Sigma)$, for any $s \in \mathbf{R}$.*

2. *The simple-layer potential \mathcal{V} extends to a continuous operator, from $H^s(\Sigma)$ into $H^{s+1}_{\text{loc}}(\Omega)$, for any $s \in \mathbf{R}$, and its restriction \mathbf{V} extends to a continuous operator from $H^s(\Sigma)$ into $H^{s+1}(\Sigma)$, for any $s \in \mathbf{R}$.*

3. *If $\psi \in H^{-1/2}(\Sigma)$, then:*

- $\mathcal{V} \in \mathcal{U}_{\text{ad}}$
- $\mathcal{V}_{\psi|_{\Sigma}} = \mathbf{V} \psi$.

Proof. – 1) After lemma 3, and relation (35), we have the following expression for \mathbf{G} : $\forall p \in C^*$, $\forall (x, y) \in \Omega$:

$$\mathbf{G}(x, y, p, k, z) = \sum_{n \in \mathcal{L}_{x, y}, n+x-y \neq 0} \frac{e^{ik|n+x-y|}}{4\pi|n+x-y|} e^{-i(n+x'-y') \cdot p}, \quad (47)$$

where

$$\mathcal{L}_{x, y} = \{n \in \mathcal{L} : n+x-y \neq 0\}.$$

Then the operator \mathbf{V} can be clearly decomposed into $\mathbf{V}^0 + \mathbf{V}^{\text{reg}}$ with kernel: $\tilde{\mathbf{G}}_0(x, y, p, k, z) = \theta(x) \frac{e^{ik|x-y|}}{4\pi|x-y|}$ where θ is a C^∞ function, and $\mathbf{V}^{\text{reg}}(x, y, p, k, z)$ is a smooth kernel.

Then we are going to apply a result of Seeley given by Giroire [8].

Let us first recall the definition:

DEFINITION 1. – *Let $\mathbf{K}(x, y)$ be an integral kernel, defined in Ω , for $x \neq y$.*

We say that $\mathbf{K}(x, y)$ is “pseudo homogeneous with degree m ” if there exist C^∞ functions $\mathbf{K}_{m+j}(x, z)$, homogeneous in z with degree $m+j$ for $z \neq 0$, such that:

$$\forall J > 0 : \mathbf{K}(x, y-x) - \sum_{j < J} \mathbf{K}_{m+j}(x, y-x) \quad (48)$$

is of class C^d , for each $d < m+J$.

Then, we have:

THEOREM 1 (Seeley). – *Let Ω be an open set in \mathbf{R}^n , and t a positive real number. Then the operator \mathbf{A} defined, for each $f \in C^\infty_0(\Omega)$, and given by:*

$$\mathbf{A} f(y) = \int_{\Omega} \mathbf{K}(y, y-x) f(x) dx, \quad (49)$$

where \mathbf{K} is a pseudo-homogeneous kernel of degree $t - n$, extends to a continuous operator from $H_{\text{comp}}^s(\Omega)$ into $H_{\text{loc}}^{s+t}(\Omega)$.

If we denote by \tilde{p} the three-vector $(p, 0)$, we can write the relation (47) as:

$$\mathbf{G}(x, y, p, k, z) = \sum_{n \in \mathcal{L}_{x, y}, n+x-y \neq 0} e^{-i(x-y) \cdot \tilde{p}} \frac{e^{ik|n+x-y|}}{4\pi|n+x-y|} e^{-i(n+y) \cdot p}. \tag{50}$$

By the regularity of \mathbf{G}^{reg} , it is sufficient to verify the hypothesis of the preceding theorem for the only singular term, corresponding to $n = 0$:

$$\mathbf{K}(y, y-x) = e^{-i(x-y) \cdot \tilde{p}} \frac{e^{ik|x-y|}}{4\pi|x-y|} e^{-i(n+y) \cdot p}. \tag{51}$$

Expanding the exponentials near zero shows that $\mathbf{K}(y, y-x)$ has degree -1 .

2) It is clear, using local charts for the compact C^∞ manifold Σ , that the theorem of Seeley holds if we replace Ω by Σ (in that case we have $n = 2$), so the preceding arguments simply the result.

3) In this case it is sufficient, as before, to consider the singular part $n = 0$ in the series.

First, we notice that the first part of 3) holds, due to standard properties of simple-layer potentials [6]. On the other hand \mathcal{V} maps continuously $H^{-s}(\Sigma)$ into $H^{-s+3/2}(\Omega)$, for each $s \in \mathbf{R}$, just because it holds for the part \mathcal{V}^0 , corresponding to the singular part \mathbf{G}_0 , after the results of Eskin (see [9], p. 106). Then, the second part of 3) is proven by taking $s = 1/2$. \square

Then we can solve problem 1, using:

THEOREM 2. – 1. For $\text{Im}(k) > 0$, the solution v of (42) is given by:

$$\forall x \in \Omega : v(x) = -\frac{1}{2} \mathcal{V} \psi(x) - \frac{1}{2} \mathcal{T} h, \tag{52}$$

where ψ is the solution of the first kind integral equation:

$$\mathbf{V} \psi + \psi_0 = 0, \tag{53}$$

and $\psi_0 = \mathbf{U} h$.

2. If $h \in H_{\text{comp}}^s(\Omega)$, then the solution u belongs to $H^{s+2}(\Sigma)$.

Proof. – Just take the limit in the representation formula of lemma 5, as $x \in \Omega$ tends toward a point of Σ . Due to the formulae of lemma 6, we obtain formula (53), denoting by ψ the normal derivative of v on Σ :

$$\mathbf{V} \psi + \psi_0 = 0 \tag{54}$$

We decompose \mathbf{V} into:

$$\mathbf{V} = \mathbf{V}_0 + \tilde{\mathbf{V}}, \quad (55)$$

where \mathbf{V}_0 is the operator with kernel \mathbf{G}_0 (see lemma 2).

After lemma 6, we see that $\tilde{\mathbf{V}}$ is a compact perturbation of \mathbf{V}_0 . As it is well known (see [8]), \mathbf{V}_0 can be also decomposed as $J + K$, where J is the integral operator with kernel $\frac{1}{|x - y|}$, and K is the integral operator with kernel $\frac{e^{ik|x-y|} - 1}{|x - y|}$. J is an isomorphism from $H^s(\Sigma)$ in $H^{s+1}(\Sigma)$, and K acts from $H^s(\Sigma)$ into $H^{s+3}(\Sigma)$, by the Seeley theorem. As $H^{s+3}(\Sigma)$ is compactly embedded into $H^{s+1}(\Sigma)$, K is a compact perturbation of J .

As $\text{Im}(k^2) \neq 0$, we deduce from part 1 of proposition 3 that $J + K$ is an isomorphism. The same conclusion holds for \mathbf{V} , then the solution of (53) is unique in $H^s(\Sigma)$, as the operator \mathbf{U} maps $H_{\text{comp}}^s(\Omega)$ in $H^{s+2}(\Sigma)$. \square

5. ANALYTIC CONTINUATION FOR THE REDUCED RESOLVENT

To analytically extend the reduced “free” resolvent, we need to control the exponential growth of the various series defining the integral kernels \mathcal{T} and \mathcal{V} together with their restrictions \mathbf{U} , and \mathbf{V} .

PROPOSITION 2. – 1. For $a > \varepsilon$, the operator \mathcal{T} extends, for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, in a bounded operator from $L_a^2(\Omega)$ into $H_{-a}^1(\Omega)$, meromorphic in the variables (p, k, z) , with polar singularities on $\{z_j = 0, j = 1 \dots N\}$.

2. For $a > \varepsilon$, the operator \mathcal{V} extends, for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, in a bounded operator from $H^{-1/2}(\Sigma)$ into $H_{-a}^1(\Omega)$, meromorphic in the same variables, with the same polar singularities.

3. For $a > \varepsilon$, the operator \mathbf{U} extends, for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, in a bounded operator from $L_a^2(\Omega)$ into $H^{1/2}(\Sigma)$, meromorphic in the variables (p, k, z) , with polar singularities on $\{z_j = 0, j = 1 \dots N\}$.

4. The operator \mathbf{V} extends, for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, in a bounded operator from $H^{-1/2}(\Sigma)$ into $H^{1/2}(\Sigma)$, meromorphic in the same variables, with the same polar singularities.

Proof. – It is a consequence of the following property of the common kernel of the four operators \mathcal{T} , \mathcal{V} , \mathbf{U} , and \mathbf{V} : this kernel, \mathbf{G} , enjoys the same properties (boundedness, and analyticity) as $\mathbf{K}_{o,p}$, because the tangential variations of x and y (x' and y') are bounded. Then it is a simple check to

see that proposition 8 applies as well, when the unperturbed cylinder Ω_0 is replaced by the perturbed one Ω , because the boundary Σ is smooth.

The remaining assertions come from the fact that the trace process conserves the analyticity. \square

We have also the corresponding “polar” decompositions:

LEMMA 7. – *The residues of the four operators \mathcal{T} , \mathcal{V} , \mathbf{U} , and \mathbf{V} are finite rank operators, and for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, we have the local decompositions:*

$$\forall j = 1 \dots N : \mathcal{T}(p, k, z) = \frac{\pi_j^1}{z_j} + M^1(p, k, z), \quad (56)$$

where $M^1(p, k, z)$ is bounded from $L_a^2(\Omega)$ into $H_{-a}^1(\Omega)$, holomorphic near the singular set $\{z_j = 0, j = 1 \dots N\}$, and the projector π_j^1 is a rank one operator in $L_a^2(\Omega)$,

$$\forall j = 1 \dots N : \mathcal{V}(p, k, z) = \frac{\pi_j^2}{z_j} + M^2(p, k, z), \quad (57)$$

where $M^2(p, k, z)$ is bounded from $H^{-1/2}(\Sigma)$ into $H_{-a}^1(\Omega)$, holomorphic near the singular set $\{z_j = 0, j = 1 \dots N\}$, and the projector π_j^2 is a rank one operator in $H^{-1/2}(\Sigma)$,

$$\forall j = 1 \dots N : \mathbf{U}(p, k, z) = \frac{\pi_j^3}{z_j} + M^3(p, k, z), \quad (58)$$

where $M^3(p, k, z)$ is bounded from $L_a^2(\Omega)$ into $H^{1/2}(\Sigma)$, holomorphic near the singular set $\{z_j = 0, j = 1 \dots N\}$, and the projector π_j^3 is a rank one operator in $L_a^2(\Omega)$,

$$\forall j = 1 \dots N : \mathbf{V}(p, k, z) = \frac{\pi_j^4}{z_j} + M^4(p, k, z), \quad (59)$$

where $M^4(p, k, z)$ is bounded from $H^{-1/2}(\Sigma)$ into $H^{1/2}(\Sigma)$, holomorphic near the singular set $\{z_j = 0, j = 1 \dots N\}$, and the projector π_j^4 is a rank one operator in $H^{-1/2}(\Sigma)$.

The proof is analogous to that of the second part of proposition 8. \square

Let us consider equation (53):

$$\mathbf{V} \psi + \mathbf{U} h = 0. \quad (60)$$

As we have shown, the two operators \mathbf{U} and \mathbf{V} have the same analytical properties, then we can isolate their singularities, and invert “explicitly” the relation (60).

PROPOSITION 3. – For $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, one can write:

$$(\mathbf{V}(p, k, z))^{-1} \mathbf{U}(p, k, z) = \frac{\mathbf{D}(p, k, z)}{\mathbf{f}(p, k, z)}, \quad (61)$$

where \mathbf{D} (resp. \mathbf{f}) is holomorphic in (p, k, z) as a bounded operator from $L_a^2(\Omega)$ into $H^{-1/2}(\Sigma)$ (resp. as a function), for $a > -\inf_{z \in \mathcal{Z}} (\text{Im}(z_j))$.

Proof. – First, we notice that the same Fredholm arguments used in theorem 3 hold, insuring the invertibility of \mathbf{V} , for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$. Then, it remains to check the “cancellation” of polar singularities. In fact, using the decompositions of \mathbf{V} and \mathbf{U} given in the preceding proposition, we can write:

$$\mathbf{V}(p, k, z) = \frac{\pi(z) D_1(p, k, z)}{g(z) f_1(p, k, z)}, \quad (62)$$

and

$$\mathbf{U}(p, k, z) = \frac{\pi(z) D_2(p, k, z)}{g(z) f_2(p, k, z)}, \quad (63)$$

where $g(z) = \prod_{j=1}^N z_j$ and $\pi(z) = \sum_{j=1}^N z_1 z_2 \dots z_{j-1} z_{j+1} \dots z_N$.

The denominators f_1 and f_2 are holomorphic functions for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, D_1 is a bounded operator from $H^{-1/2}(\Sigma)$ into $H^{1/2}(\Sigma)$, and D_2 is a bounded operator from $L_a^2(\Omega)$ into $H^{1/2}(\Sigma)$.

Then, the singular parts are the same on the two sides of (60), and we have:

$$\psi = \frac{D_1^{-1} D_2}{f} h \quad (64)$$

This ends the proof. \square

We are now in position to study the total resolvent $(\mathbf{H} - k^2)^{-1}$.

6. ANALYTIC CONTINUATION FOR THE COMPLETE RESOLVENT

We have the following formula, which “reconstructs” the complete resolvent, by integrating on the periodic variables:

LEMMA 8. – For $\text{Im}(k) > 0$, and $u \in C_0^\infty(\Omega^e)$, the resolvent of \mathbf{H} is given by the following formula:

$$\begin{aligned} (\mathbf{H} - k^2)^{-1} u &= \frac{1}{(2\pi)^2} \int_{\mathcal{C}^*} e^{ip \cdot x'} (\mathbf{H}_p - k^2)^{-1} \\ &\quad \times \left(\sum_{n \in \mathcal{L}} u(x' + n, x_3) e^{ip \cdot (x' + n)} \right) dp. \end{aligned} \quad (65)$$

Proof. – It is a direct consequence of Theorem XIII.85 of [10]: the mapping $p \rightarrow \mathbf{H}_p$ is measurable, the operators \mathbf{H}_p are self-adjoint for each $p \in \mathcal{C}^*$ and, as $\text{Im}(k) > 0$, the borelian mapping $t \rightarrow (t - k^2)^{-1}$ is bounded on \mathbf{R} . \square

After the representation formula of lemma 5, the solution of (33) is given by:

$$v = \frac{1}{2} (\mathcal{V} \mathbf{V}^{-1} \mathbf{U} - \mathcal{T}) g. \quad (66)$$

Then we have the formula:

$$(\mathbf{H}_p - k^2)^{-1} = \frac{1}{2} (\mathcal{V} \mathbf{V}^{-1} \mathbf{U} - \mathcal{T}). \quad (67)$$

If, for $p \in \mathcal{L}^*$ and $\text{Im}(k) > 0$, we denote by $z_i(p, k)$ the determination of $(k^2 - (n + p)^2)^{1/2}$ with positive imaginary part, the operator \mathcal{T} (resp. \mathcal{V}) is the (holomorphic) restriction to the smooth manifold \mathcal{G}_∞ of a bounded operator from $L_a^2(\Omega)$ (resp. $H^{-1/2}(\Sigma)$) into $H_{-a}^1(\Omega)$.

Using proposition 11, we can write (65) as:

$$\begin{aligned} (\mathbf{H} - k^2)^{-1} u &= \frac{1}{(2\pi)^2} \int_{\mathcal{C}^*} e^{ip \cdot x'} \frac{\tilde{D}(p, k, z(p, k))}{f(p, k, z(p, k))} \\ &\times \left(\sum_{n \in \mathcal{L}} u(x' + n, x_3) e^{ip \cdot (x' + n)} \right) dp. \end{aligned} \quad (68)$$

Now, to study the properties of the total resolvent, we introduce “global” (isotropic) weighted spaces [1]:

$$L_\alpha^2(\Omega^e) = \{u \in L_{\text{loc}}^2(\Omega^e) : e^{\alpha(\langle |x'| \rangle + \langle x_3 \rangle)} u \in L^2(\Omega^e)\}, \quad (69)$$

and:

$$H_\alpha^1(\Omega^e) = \{u \in H_{\text{loc}}^1(\Omega^e) : e^{\alpha(\langle |x'| \rangle + \langle x_3 \rangle)} u \in H^1(\Omega^e)\}, \quad (70)$$

and we put on these spaces the natural associated hilbertian structure.

Then we have:

PROPOSITION 4. – For $\text{Im}(k) > 0$, and $u \in C_0^\infty(\Omega^e)$, let us write the resolvent of \mathbf{H} as:

$$(\mathbf{H} - k^2)^{-1} u = \frac{1}{(2\pi)^2} \int_{\mathcal{C}^*} \frac{M(p, k, z(p, k))}{f(p, k, z(p, k))} u dp, \quad (71)$$

where, in $z = z(p, k) = (z_1 \dots z_N)$, the $z_i = z_i(p, k)$ are the determinations with positive imaginary part, of $[(k^2 - (n_i + p)^2)^{1/2}]$, for $p \in \mathcal{C}^*$.

Then, for $(p, k, z) \in \mathcal{W} \times \mathcal{U} \times \mathcal{Z}$, the operator $M(p, k, z)$ can be extended into a bounded operator from $L_a^2(\Omega^e)$ into $H_{-a}^1(\Omega^e)$, if $a > \sup_{p \in \mathcal{W}} |\text{Im}(p)| - \inf_{z \in \mathcal{Z}} (\text{Im}(z_i))$.

Proof. – First, the mapping:

$$\phi \rightarrow \sum_{n \in \mathcal{L}} \phi(x' + n, x_3) e^{ip \cdot (x' + n)}, \tag{72}$$

extends into a bounded operator from $L_a^2(\Omega^e)$ into $L_a^2(\Omega)$, as soon as a satisfies the above condition.

Then $x' \rightarrow e^{-ip \cdot x'} u(x', x_3)$, extends, by periodicity, into a function in $H_{\text{loc}}^1(\Omega^e)$ which satisfies the bound: $\|e^{-ip \cdot x'} u\|_{L_a^2(\Omega^e)} \leq \|u\|_{L_a^2(\Omega)}$, if $a > \sup_{p \in \mathcal{W}} |\text{Im}(p)|$, and the same estimate holds for the derivative of $e^{-ip \cdot x'} u$.

As, after theorem 4, $(\mathbf{H}_p - k^2)^{-1}$ is continuous from $L_a^2(\Omega)$ into $H_{-a}^1(\Omega)$, the conclusion follows, by composition of these three mappings. \square

Now, a study parallel to the lines of [1] shows the following general result for the global analytic continuation for the total resolvent:

THEOREM 3. – *Let \mathcal{U} be an open bounded set in \mathbf{C} .*

For f and $g \in C_0^\infty(\Omega^e)$, the mapping: $k \rightarrow ((\mathbf{H} - k^2)^{-1} f, g)_{H^1(\Omega^e)}$ defined in $\mathcal{U} \cap \{\text{Im}(k) > 0\}$ can be analytically continued to the universal covering of the complex analytic set $\mathcal{U} \setminus LA \cup LA_\infty$, where LA is a finite set of points in \mathcal{U} , and LA_∞ is a closed set of Lebesgue measure zero in \mathcal{U} .

We can give the following interpretation of this result.

The first part LA of the singular set is a set of “resonances” obtained by a “condensation” of eigenvalues of the reduced operators, which consists of branch points for $(\mathbf{H} - k^2)^{-1}$, rather than poles as in the compact obstacle problem. Physically, this corresponds to Bragg incidence angles observed in neutron scattering by crystals, in the context of the physics of solids, or in electromagnetic scattering by gratings, in the microwave domain.

The second part, as Gerard notes, is less transparent and comes from the technical problem of varying domains of unbounded operators while one performs the p -integral leading to this subtle singular set LA_∞ .

What is unclear is the following: does the set LA_∞ actually exist, or is it an artefact of the method? An optimistic point of view should be that LA_∞ is absent, at least for small periodic perturbations of a plane. Nevertheless, the following problem remains to connect the structure of LA to the geometric properties of Γ .

7. CONCLUSION

The previous qualitative result is the only one we can obtain, in the general case, and, since we made no precise hypothesis on the surface, it is not surprising that we recover the same results as Gerard, due to the similarities between his Schrödinger framework and ours.

However, as the analytic extension of the resolvent is the first step toward the estimate of the decay of the local energy, it is clear that the analytical structure given by the previous result is not sufficient to obtain precise information about the decay.

In fact, following the ideas of Ralston [11], we should need two types of information, which deserve further study:

1. A lower bound for a number $\varepsilon > 0$, such that:

$$LA \cup LA_\infty \subset \{\operatorname{Im}(k) < -\varepsilon\}, \quad (73)$$

2. an asymptotic behaviour at most of exponential type for $R(k) = (H - k^2)^{-1}$ in the strip $\{|\operatorname{Im}(k)| < \varepsilon\}$, for large $|\operatorname{Re}(k)|$:

$$\|R(k) f\|_{0 < x_3 < R} \leq C e^{T|\operatorname{Im}(k)|} \|f\|, \quad (74)$$

for a smooth compactly supported f . In this case, exponential decay would hold, using the results of Vainberg [13].

ACKNOWLEDGEMENTS

We are indebted to Claude Bardos for suggesting this problem, J. Rauch for helpful discussions, C. Gérard for several explanations of his results in [1], and we thank Clayton Chinn for numerous linguistic improvements.

REFERENCES

- [1] C. GÉRARD, Resonance theory in atom-surface scattering, *Commun. Math. Phys.*, Vol. **126**, 1989, pp. 263-290.
- [2] C. BARDOS, J. C. GUILLOT and J. RALSTON, La relation de Poisson pour l'équation des ondes dans un ouvert non borné : application à la théorie de la diffusion, *Comm. in Partial Differential Equations*, Vol. **7**, 1982, pp. 905-958.
- [3] B. DUCOMET, Decay of solutions of the wave equation outside rough surfaces, Preprint PTN-92, 1992.
- [4] P. D. LAX and R. S. PHILLIPS, Scattering theory, *Academic Press*, 1989.
- [5] H. D. ALBER, A quasi-periodic boundary value problem for the Laplacian and the continuation of its resolvent, *Proc. of the Royal Soc. of Edinburgh*, Vol. **82A**, 1979, pp. 252-272.

- [6] R. DAUTRAY and J.-L. LIONS, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, Masson, 1985.
- [7] J. L. LIONS and E. MAGENES, *Problèmes aux limites non homogènes et applications*, T. 1, Dunod, 1968.
- [8] J. GIROIRE, *Integral equation methods for exterior problems for the Helmholtz equation*, Internal report 40, Centre de Mathématiques Appliquées, École Polytechnique, 1978.
- [9] G. I. ESKIN, *Boundary value problems for elliptic pseudo-differential equations*, AMS Translations, 1981.
- [10] M. REED and B. SIMON, *Methods of modern mathematical physics*, Vol. 4, Academic Press, 1980.
- [11] J. RALSTON, *Note on the decay of acoustic waves*, *Duke Math. J.*, Vol. 46, 1979, pp. 799-804.
- [12] R. MELROSE, *Singularities and energy decay in acoustical scattering*, *Duke Math. J.*, Vol. 46, 1979, pp. 43-59.
- [13] B. R. VAINBERG, *On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t \rightarrow \infty$ of solutions of non-stationary problems*, *Russian Math. Surveys*, Vol. 30, 1975, pp. 1-58.

(Manuscript received June 9, 1993;

Revised November 26, 1993.)