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Configuration space properties of the scattering operator and time delay for potentials decaying like $|x|^{-\alpha}$, $\alpha > 1$

by

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ABSTRACT. — The time-delay in a simple quantum mechanical scattering process is defined to be the limit as $r \to \infty$ of the difference of the sojourn times of a scattering state and of the associated free state in a ball of radius r, with the particle under the influence of a potential $W(x)$. For $W(\underset{\sim}{x}) \sim |\underset{\sim}{x}|^{-\alpha} (\alpha > 1)$ at infinity, we use a smooth localization function with the tail growing like $r^{1-\delta} (\delta > 0$, depending on W) instead of the characteristic function of a ball of radius r and it is shown that the time-delay exists and satisfies the Eisenbud-Wigner relation.

RÉSUMÉ. — Le temps de retard (time delay) pour un système de diffusion simple en mécanique quantique est défini comme la limite lorsque $r \to \infty$ de la différence des temps de séjour dans la boule $B_r \equiv \{ x \in \mathbb{R}^n \mid |x| \leqq r \}$

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d'un état de diffusion et de l'état initial libre qui lui est associé. Nous considérons des potentiels $W(\underline{x})$ à courte portée à décroissance lente $(W(\underline{x}) \sim |\underline{x}|^{-\alpha}$ avec $\alpha > 1$ lorsque $|\underline{x}| \to \infty)$ et nous déterminons l'existence du temps de retard et la validité de la formule d'Eisenbud-Wigner si la fonction caractéristique de B_r est remplacée par une fonction de localisation lisse θ_r, avec $\operatorname{grad} \theta_r(\underline{x}) = 0$ si $|x|/r \notin [1, 1 + r^{-\delta}]$ pour un $\delta > 0$ dépendant de W.

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1. INTRODUCTION

Here we continue the work on time-delay and Eisenbud-Wigner formula following those of [1] and [2]. In these papers, the time-delay was defined as the limit as $r \to \infty$ of the difference of sojourn times of a scattering state of a quantal particle and of the associated asymptotic free state in a ball of radius r, with the particle under the influence of a potential $W(\underline{x})$ behaving like $|\underline{x}|^{-2-\varepsilon}$ at infinity. For short-range potentials with slower decay like $|\underline{x}|^{-1-\varepsilon}$ at infinity, Wang [3] and Nakamura [4] gave proofs of the existence of time-delay, with sojourn times being calculated with the help of a C^∞ localisation function χ_r, where $\chi_r(\underline{x}) = \chi(|\underline{x}|/r)$ with $\chi(\underline{x}) = 1$ if $|\underline{x}| \leqq 1$, $= 0$ if $|\underline{x}| \geqq 2$. From a physical point of view, this is not satisfactory since as $r \to \infty$, the tail of the localisation function grows at the same rate (linearly) as its main support (*i.e.* all $\underline{x}$ for which $\chi_r(\underline{x}) = 1$. We use instead a localization function φ_r (defined in Section 2) whose tail grows at the rate $r^{1-\delta}$ (δ postive and depends on the potential) while the main support grows linearly. For large r, this gives better localization than that of [3] and [4] though not as satisfactory as the one given by the characteristic function of the ball of radius r. In [3] and [4], the potential is assumed to be C^∞ and pseudo-differential operator techniques were used while we use exclusively commutator methods. We would also like to mention the work of Jensen [5] in which the equality of the Eisenbud-Wigner and Lavine's expressions for time-delay was established for essentially the same class of potentiels though no attempt was made to derive either of them from the sojourn times. For a survey of earlier works on time delay, the reader is referred to [6].

It is well known [7] that for short range potentials (which may have some local singularities, but decaying like $|\underline{x}|^{-1-\varepsilon}$, $\varepsilon > 0$, as $|\underline{x}| \to \infty$), the wave operators exist and are complete. However, for the existence of the time delay for such type of potentials, one seems to need a modified free evolution similar to that used to show the existence of wave operators for smooth long range potentials (*see* Chapter 13 of [7]).

This paper is organized as follows: the Section 2 deals with the notations, definitions, known results of scattering theory and the statement of the main result. The next section is devoted to the study of some of the properties of the modified free evolution and the abstract theorem of Martin (*see* Chapter 7 of [7] and [8]) in the present context. The main theorem is proved in Section 4. In the Appendix we verify the hypotheses of the abstract theorem and an auxilary result needed for the main theorem.

2. NOTATIONS AND THE MAIN RESULT

As in [1] and [2], we denote by

$$\underline{Q}=(Q_1, \ldots, Q_n) \quad \text{and} \quad \underline{P}=(P_1, \ldots, P_n),$$

the position and momentum operator respectively in the complex Hilbert space $\mathscr{H}=L^2(\mathbb{R}^n)$. The free Hamiltonian is $H_0=\underline{P}^2=\sum_{j=1}^{n} P_j^2$ and the total Hamiltonian has the form $H=H_0+W(Q)$, where W satisfy the following conditions:

$$\left.\begin{array}{l}\text{(i) W is a real valued } C^5\text{-function on } \mathbb{R}^n \\ \text{(ii) for any multi-index } m=(m_1, \ldots, m_n), \text{ with} \\ \qquad 0 \leqq |m| = \sum_{j=1}^{n} m_j \leqq 5 \text{ and some } \alpha\,(1<\alpha\leqq 2), \\ |D^m W(\underline{x})| = |\partial_1^{m_1}\partial_2^{m_2}\ldots\partial_n^{m_n} W(\underline{x})| \leqq K(1+|\underline{x}|)^{-|m|-\alpha}.\end{array}\right\} \quad (2.1)$$

It should be noted that W need not be spherically symmetric. Under the hypothesis of (2.1), the total Hamiltonian H is self-adjoint on $D(H_0)$ and bounded below, where $D(H_0)$ is the maximal domain of H_0. If we denote the two unitary groups generated by H_0 and H as U_t and V_t respectively, then it is know [7] that the wave operators $\Omega_\pm = s.\lim_{t\to\pm\infty} V_t^* U_t$ exist and are complete so that the scattering operator $S=\Omega_+^*\Omega_-$ is unitary. Further S commutes with H_0. In the spectral representation $L^2[[0, \infty), L^2(S^{(n-1)}); d\lambda]$ of H_0, S is decomposable as $\{S(\lambda)\}_{\lambda\in(0,\infty)}$, where $S(\lambda)$ is unitary in $L^2(S^{(n-1)})$ for all $\lambda\in(0, \infty)$, $S^{(n-1)}$ being the unit sphere of dimension $(n-1)$ embedded in $\mathbb{R}^n$. It is also known that under the condition (2.1) the Hamiltonian H does not have any positive eigenvalues [9]. In fact, H does not have any positive singular spectrum.

We shall denote by A, the selfadjoint infinitesimal generator of the dilation group in $L^2(\mathbb{R}^n)$ and observe that $A=\frac{1}{2}(\underline{P}.\underline{Q}+\underline{Q}.\underline{P})$ on $C_0^\infty(\mathbb{R}^n)$. In the sequel we use the following notations:

$$|\underline{Q}|=\left(\sum_{j=1}^{n} Q_j^2\right)^{1/2}, \qquad \langle Q\rangle=(1+|\underline{Q}|^2)^{1/2}. \quad \langle A\rangle=(I+A^2)^{1/2},$$

$$|\underline{P}|=\left(\sum_{j=1}^{n} P_j^2\right)^{1/2}, \qquad \tilde{W}(\underline{x})=W(\underline{x})+\frac{1}{2}\underline{x}.\nabla W(\underline{x}),$$

$$A_0=\frac{1}{2}\left(\frac{1}{|\underline{P}|^2}\right)\underline{P}.\underline{Q}+\underline{Q}.\underline{P}\left(\frac{1}{|\underline{P}|^3}\right).$$

For any $\mu\geqq 0$, let

$\mathscr{D}_\mu\equiv\{f\in\mathscr{H} : \langle Q\rangle^\mu f\in\mathscr{H}$ and the Fourier transform $\tilde{f}$ of f has compact support in $\mathbb{R}^n\setminus\{0\}\}$. It is clear that $\mathscr{D}_\mu$ is dense in $\mathscr{H}$ for every $\mu\geqq 0$, $\mathscr{D}_{\mu_1}\subseteqq\mathscr{D}_{\mu_2}$ for $\mu_1\geqq\mu_2$ and $\mathscr{D}_\mu\subset D(\langle\mathscr{A}\rangle^\mu)$, $D(\langle\mathscr{A}\rangle^\mu)$ being the domain of $\langle\mathscr{A}\rangle^\mu$, and that A_0 is well defined on $\mathscr{D}_1$. The symbol K in the following is used for generic constants.

We need a result on the norm differentiability of $S(\lambda)$ due to Jensen [10], which we state without proof.

PROPOSITION 2.1. – *Let* W *satisfies* (2.1). *Then* $S(\lambda)$ *is five times continuously norm differentiable in* $(0, \infty)$.

This follows from Theorem 3.6 and equation (3.2) of [10].

COROLLARY 2.2. – *Let* $f\in D(\langle A\rangle^\mu)$ *for* $0\leqq\mu\leqq 5$ *be such that* $f=\psi(H_0)f$ *for some* $\psi\in C_0^\infty(0, \infty)$. *Then* $Sf\in D(\langle A\rangle^\mu)$.

Proof. – The proof is trivial for $\mu=0$. For $\mu=5$, we note that $A^5 S f=\sum_{j=0}^{5}\binom{5}{j} ad_A^j(S)A^{5-j}$, where $ad_A^0(S)=S$ and

$$ad_A^j(S)=\underbrace{[A, [A, \ldots, [A, S]]]\ldots]}_{j-\text{times}}.$$

Then the result follows from the fact that $(Af)_\lambda=2i\lambda\frac{df_\lambda}{d\lambda}+if_\lambda$ in the spectral representation of H_0 (*see* [10]) and Proposition 2.1.

To state the main result we need to have a few definitions.

Let $\varphi_r\,(r\geqq 1)$ be a C^1-function on $[0, \infty)$ such that

$$\left.\begin{array}{l}\text{(i) } 0 \leqq \varphi_r(u) \leqq 1 \text{ for all } u \in [0, \infty),\\ \text{(ii) } \varphi_r(u) = \begin{cases} 1 & \text{for } 0 \leqq u \leqq 1 \\ 0 & \text{for } u \geqq 1 + cr^{-\delta} \\ & \text{for some fixed } c > 0 \text{ and } 0 < \delta < 1, \end{cases}\\ \text{and}\\ \text{(iii) } |\varphi_r'(u)| \leqq \mathrm{K}\, r^{\delta}\\ \quad \text{for some } \mathrm{K} > 0 \text{ for all } u \in [0, \infty).\end{array}\right\} \tag{2.2}$$

Define P_r to be the multiplication operator by the function $\varphi_r\left(\frac{|x|}{r}\right)$ in $\mathscr{H}$ and observe that $\|\mathrm{P}_r\| = 1$. We define the sojourn time for the particle in the fuzzy ball $\left\{\underline{x} : \varphi_r\left(\frac{|x|}{r}\right) \neq 0\right\}$ with and without the potential as $\int_{-\infty}^{\infty} dt \|\mathrm{P}_r \mathrm{V}_t \Omega_- f\|^2$ and $\int_{-\infty}^{\infty} dt \|\mathrm{P}_r \mathrm{U}_t f\|^2$ respectively ($f \in \mathscr{H}$). Then the time delay $\tau_\tau(f)$ for the fuzzy ball in the state f is defined as:

$$\tau_r(f) = \int_{-\infty}^{\infty} (\|\mathrm{P}_r V_t \Omega_- f\|^2 - \|\mathrm{P}_r \mathrm{U}_t f\|^2)\, dt. \tag{2.3}$$

The (global) time-delay $\tau(f)$ in the state f is defined as $\tau(f) = \lim_{r-\infty} \tau_r(f)$, if the limit exists. As we shall show in the sequel, for a suitable dense set of vectors f, $\tau(f)$ exists and can be computed in terms of the S-matrix $\mathrm{S}(\lambda)$. The precise result is our main theorem.

MAIN THEOREM. — Let P_r be as above with $\delta < \dfrac{\alpha - 1}{2(\alpha + 2)}$ and $f \in \mathscr{D}_{1+\eta}$ for some $\eta > 0$. Then

(i) for each $r > 1$, $\tau_r(f)$ exists,

(ii) $\tau(f) = \lim_{r \to \infty} \tau_r(f)$ exists, and

(iii) $\tau(f) = \displaystyle\int_0^{\infty} d\lambda \left(f_\lambda, \left\{-i\mathrm{S}^*(\lambda)\frac{d\mathrm{S}(\lambda)}{d\lambda}\right\} f_\lambda\right).$

3. MODIFIED FREE EVOLUTION

Here we define and study some of the properties of the modified free evolution. We set

$$\left.\begin{array}{l}\mathrm{X}_t^{\pm}(\underline{\mathrm{P}}) = \pm \displaystyle\int_t^{\pm\infty} \mathrm{W}(2s\underline{\mathrm{P}})\, ds,\\ \mathrm{Y}_t^{\pm} = \exp(-i\mathrm{X}_t^{\pm}),\ \mathrm{T}_t^{\pm} = \mathrm{U}_t\, Y_t^{\pm},\end{array}\right\} \tag{3.1}$$

where the signs $\pm$ are to be considered according as $t \gtrless 0$. Then it is easy to see that by virtue of (2.1), $X_t^\pm$ is selfadjoint on the maximal domain and thus $Y_t^\pm$ and $T_t^\pm$ are unitary operators (though not groups) for all t.

LEMMA 3.1:
(i) $Y_t^\pm \to I$ *strongly as* $t \to \pm\infty$, *and*
(ii) $\tilde{\Omega}_\pm = s.\lim_{t\to\pm\infty} V_t^* T_t^\pm$ *exists and* $\tilde{\Omega}_\pm = \Omega_\pm$.

Proof. — Part (i) follows from the inequality

$$|\exp(-iX_t^\pm(\underline{k})) - 1| \leqq K \left| \int_t^{\pm\infty} ds (1 + 2s|k|)^{-\alpha} \right| \text{ for } \underline{k} \neq 0$$

and an application of the dominated convergence theorem.

(ii) Let $t > 0$ and $f \in \mathcal{H}$. Then

$$\begin{aligned} \| V_t^* T_t^+ f - \Omega_+ f \| &\leqq \| V_t^* T_t^+ f - V_t^* U_t f \| + \| V_t^* U_t f - \Omega_+ f \| \\ &= \| Y_t^+ f - f \| + \| V_t^* U_t f - \Omega_+ f \| \to 0 \end{aligned}$$

as $t \to \infty$. The proof for $t \to -\infty$ is identical. ■

LEMMA 3.2. — *Let* $\psi \in C_0^\infty(0, \infty)$. *Then for all* $t \in \mathbb{R}$, *we have the following:*

(i) *for any multi-indes m, with*

$$1 \leqq |m| \leqq 4, \qquad |D^m X_t^\pm(\underline{k}) \psi(|\underline{k}|^2)| \leqq K_1 (1 + |t|)^{1-\alpha}$$

for some constant K_1 *depending only on m and* ψ *and not on t and* $\underline{k}$,

(ii) $\| \langle Q \rangle^\mu Y_t^\pm \psi(H_0) \langle Q \rangle^{-\mu} \| \leqq K_2$, *where* K_2 *depends only on* μ *and* ψ *for* $0 \leqq \mu \leqq 4$,

(iii) *for any integer* j $(1 \leqq j \leqq n)$ *and* $t \neq 0$,

$$\| Q_j (Y_t^\pm - I) \psi(H_0) \langle Q \rangle^{-1} \| \leqq K_3 |t|^{1-\alpha},$$

where K_3 *is a constant depending only on* ψ,

(iv) *for* $1 \leqq j \leqq n$, $0 \leqq \mu \leqq 3$, $\| \langle Q \rangle^{-\mu} \partial_j X_t^\pm \psi(H_0) \langle Q \rangle^\mu \| \leqq K_4$, *where* K_4 *depends on* ψ *and* μ *but not on t.*

Proof. — It follows from the definition (3.1) and (2.1) that for any multi-index m with $|m| \leqq 5$ and $\underline{k} \neq 0$,

$$\begin{aligned} |D^m X_t^\pm(\underline{k})| &\leqq K \left| \int_t^{\pm\infty} ds |2s|^{|m|} (1 + 2|s\underline{k}|)^{-\alpha - |m|} \right| \\ &\leqq \left\{ \begin{array}{ll} K(2\alpha - 2)^{-1} |\underline{k}|^{-(1+|m|)} & \text{for all } t \in \mathbb{R} \\ K(\alpha - 1)^{-1} 2^{-\alpha} |\underline{k}|^{-(\alpha+|m|)} |t|^{1-\alpha} & \text{for } t \neq 0. \end{array} \right\} \end{aligned} \tag{3.2}$$

Thus (i) follows from (3.2) and the support property of ψ.

A simple computation shows that for each j $(1 \leqq j \leqq n)$,

$$[Q_j^4, Y_t^\pm](\underline{k}) = \sum_{\substack{1 \leqq m_1 + m_2 \leqq 4 \\ m_1, m_2 \geqq 0,\ \text{integers}}} K(m_1, m_2)[\partial_j^{m_1} X_t^\pm(\underline{k})]^{\mu_1} \times [\partial_j^{m_2} X_t^\pm(\underline{k})]^{\mu_2} Y_t^\pm Q_j^{4-m_1-m_2}, \quad (3.3)$$

where μ_1 and μ_2 are nonnegative integers depending on $m_1 + m_2$.

Note that for any nonnegative integer l, $Q_j^l \psi(H_0)\langle Q\rangle^{-l}$ is bounded by Proposition A.2 (i) and that $Q_j^l \psi(H_0) = \sum_{m=0}^{l} K(m)\psi_{j,m}(\underline{P}) Q_j^m$, with constants $K(m)$ and $\psi_{j,m} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. This, (3.3) and part (i) leads to

$$\| Q_j^4 Y_t^\pm \psi(H_0)\langle Q\rangle^{-4} \| \leqq K_2, \quad (3.4)$$

where K_2 is independant of t. The proof of (ii) is completed from (3.4) on observing that

$$\| |\underline{Q}|^4 Y_t^\pm \psi(H_0)\langle Q\rangle^{-4} \| \leqq n \sum_{j=1}^{n} \| Q_j^4 Y_t^\pm \psi(H_0)\langle Q\rangle^{-4} \|,$$

and by interpolation between $\mu = 0$ and $\mu = 4$ (*see* [11]). The proof of (iii) follows from (i), Proposition A.2 (vi) and the estimate

$$|(Y_t^\pm(\underline{k}) - 1)\psi(|\underline{k}|^2)| \leqq |X_t^\pm(\underline{k})\psi(|\underline{k}|^2)| \leqq K|t|^{1-\alpha}.$$

The proof of (iv) is similar to that of (iii) if we use the part (i) and the Propositions A.2 (i), (vi). ■

We now prove a result similar in spirit to that of Martin [8]. Before that we need a simple lemma.

LEMMA 3.3. – *Let* H_0, H, $\Omega_\pm$, S *and* P_r *be as in Section* 2. *Then for each* $r \geqq 1$

(i) $\| P_r U_t f \| \in L^2(\mathbb{R}, dt)$ *for all* $f \in \mathcal{D}_0$,

(ii) $\| P_r T_t^\pm f \| \in L^2(\mathbb{R}_\pm, dt)$ *for all* $f \in \mathcal{D}_0$, *where* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *are* $[0, \infty)$ *and* $(-\infty, 0]$ *respectively*,

(iii) $\| P_r V_t \Omega_- f \| \in L^2(\mathbb{R}; dt)$ *for all* $f \in \mathcal{D}_0$, *and*

(iv) $\tau_r(f)$, *as defined in* (2.3), *exists for all* $f \in \mathcal{D}_0$.

Proof. – The part (i) follows from the local smoothness [12] of P_r with respect to H_0. Since

$$\| P_r T_t^\pm f \| \leqq \| P_r \langle Q\rangle^\mu \| \, \| \langle Q\rangle^{-\mu} Y_t^\pm \psi(H_0)\langle Q\rangle^\mu \| \| \langle Q\rangle^{-\mu} U_t f \|,$$

where $\psi \in C_0^\infty(0, \infty)$ such that $\psi(H_0) f = f$, the required result (ii) is a consequence of Lemma 3.2 (ii) and the smoothness of $\langle Q\rangle^{-\mu}$ for $\mu > \frac{1}{2}$. Similarly (iii) is arrived at by the local smoothness of $\langle Q\rangle^{-\mu}$ with respect to the total Hamiltonian for $\mu > \frac{1}{2}$. Thus it is clear from (2.3) that $\tau_r(f)$ exists for every f in $\mathscr{D}_0$. ■

THEOREM 3.4. — *Assume the hypothesis of Lemma* 3.3. *Let furthermore* $f \in \mathscr{D}_0$ *be such that*

$$\| V_t \Omega_- f - T_t^- f \| \in L^1(\mathbb{R}_-, dt) \tag{3.5}$$

and

$$\| V_t \Omega_- f - T_t^+ S f \| \in L^1(\mathbb{R}_+, dt). \tag{3.6}$$

Then

$$\tau(f) = \lim_{r \to \infty} \tau_r(f) = \lim_{r \to \infty} \sum_{j=1}^{3} r_r^{(j)}(f)$$
$$\equiv \lim_{r \to \infty} \left[\left\{ \int_0^\infty dt (\| P_r T_t^+ S f \|^2 - \| P_r T_t^+ f \|^2) \right\} + \left\{ \int_0^\infty dt (\| P_r T_t^+ f \|^2 - \| P_r U_t f \|^2) \right\} + \left\{ \int_{-\infty}^0 dt (\| P_r T_t^- f \|^2 - \| P_r U_t f \|^2) \right\} \right].$$

Proof. — Since $f \in \mathscr{D}_0$, $\tau_r(f)$ is well defined by Lemma 3.3 (iv). We write

$$\tau_r(f) - \sum_{j=1}^{3} \tau_r^{(j)}(f) = \int_0^\infty dt\, J_r^+(S f, t) + \int_{-\infty}^0 dt\, J_r^-(f, t), \tag{3.7}$$

where $J_r^\pm(g, t) = \| P_r V_t \Omega_\pm g \|^2 - \| P_r T_t^\pm g \|^2$ for $t \gtrless 0$. Since

$$| J_r^-(f, t) | \leqq 2 \| f \| \| V_t \Omega_- f - T_t^- f \|$$

and

$$| J_r^+(S f, t) | \leqq 2 \| f \| \| V_t \Omega_+ S f - T_t^+ S f \| = 2 \| f \| \| V_t \Omega_- f - T_t^+ S f \|,$$

the results follows by an application of the dominated convergence theorem to (3.7) and the fact P_r converges strongly to I as $r \to \infty$. ■

4. PROOF OF MAIN THEOREM

In this section we show that if f and $S f \in \mathscr{D}_{1+\eta} \subset \mathscr{D}_0$ for some $\eta > 0$, and (3.5) and (3.6) are satisfied, then $\tau_r^{(2)}(f)$ and $\tau_r^{(3)}(f)$ converge to zero while $\tau_r^{(1)}(f)$ converges to the Eisenbud-Wigner form as $r \to \infty$, thereby proving the main theorem stated in Section 2. The verifications of (3.5), (3.6) and the fact that $S f \in \mathscr{D}_{1+\beta}$, for some $\beta > 0$ [depending on α of (2.1)] when $f \in \mathscr{D}_{4+\eta}$ are done in the Appendix.

Let φ_r be as in Section 2 and we write

$$\theta_r = \varphi_r^2 \text{ and } \hat{\theta}_\nu = \theta_{(2\nu)^{-1}}, \qquad \text{where} \quad \nu = \frac{1}{2r}. \tag{4.1}$$

We also note from (2.2) that

$$\begin{aligned} &\text{(i) } \hat{\theta}_\nu \in C^1 [0, \infty),\ 0 \leqq \hat{\theta}_\nu(u) \leqq 1, \\ &\text{(ii) } \hat{\theta}_\nu(u) = \begin{cases} 1 \text{ if } 0 \leqq u \leqq 1 \\ 0 \text{ if } u \geqq 1 + c\,(2\nu)^\delta, \end{cases} \\ &\text{(iii) } |\hat{\theta}'_\nu(u)| \leqq 2\,K\,(2\nu)^{-\delta}. \end{aligned} \tag{4.2}$$

It is well known (Chapter 3 of [7]) that for any bounded function φ of $\underline{Q}$ one has:

$$U_t^* \varphi(\underline{Q})\, U_t = Z_{1/4t}^* \varphi(2\,t\,\underline{P})\, Z_{1/4t}, \tag{4.3}$$

where $Z_t = \exp(i\,\underline{Q}^2\, t)$.

Then an easy calculation as in [1], using the change of variables $s = \dfrac{r}{2|t|}$ and $\nu = \dfrac{1}{2r}$ $\left(\text{so that } 0 < s < \infty \text{ and } 0 < \nu < \dfrac{1}{2}\right)$ and the definition of P_r in Section 2 shows that

$$\int_{\mathbb{R}_\pm} dt\, \| P_r T_t^\pm f \|^2 = \int_0^\infty \frac{ds}{4\nu s^2} \left(Y_{\pm(4\nu s)^{-1}}^\pm f,\ Z_{\pm\nu s}^* \hat{\theta}_\nu \left(\frac{|\underline{P}|}{s} \right) Z_{\pm\nu s}\, Y_{\pm(4\nu s)^{-1}}^\pm f \right), \tag{4.4}$$

$$\int_{\mathbb{R}_\pm} dt\, \| P_r U_t f \|^2 = \int_0^\infty \frac{ds}{4\nu s^2} \left(f,\ Z_{\pm\nu s}^* \hat{\theta}_\nu \left(\frac{|\underline{P}|}{s} \right) Z_{\pm\nu s} f \right), \tag{4.5}$$

and

$$\int_0^\infty \frac{ds}{4\nu s^2} \left(f,\ \hat{\theta}_\nu \left(\frac{|\underline{P}|}{s} \right) f \right) = \left[\frac{1}{4\nu} \int_0^\infty \hat{\theta}_\nu(u)\, du \right] \int_{\mathbb{R}^n} dk\, |\underline{k}|^{-1} |\tilde{f}(\underline{k})|^2. \tag{4.6}$$

We know that the first two of the above integrals exist for $f \in \mathscr{D}_0$ by Lemma 3.3, while the last one in (4.6) also exists for all $f \in \mathscr{D}_0 \subset D(H_0^{-1/4})$. The following theorem is an improvement of the result in the Appendix of [1] adapted to the present situation. For brevity in the presentation we adopt the convention that $Y_{\pm\infty}^\pm = I$, which is consistent with the result in Lemma 3.1.

THEOREM 4.1. – Suppose that $f \in \mathscr{D}_{1+\beta}$ with $0<\beta<1$ and that in (4.2) $0<\delta<\min(\alpha-1, \beta/2)$. Then

$$\lim_{\nu\to 0^+} \int_0^\infty \frac{ds}{4\nu s^2}\left[\left(Y^{\pm}_{\pm\rho} f, Z^*_{\pm\nu s}\hat{\theta}_\nu\left(\frac{|P|}{s}\right)Z_{\pm\nu s}Y^{\pm}_{\pm\rho} f\right)-\left(f, \hat{\theta}_\nu\left(\frac{|P|}{s}\right)f\right)\right] \pm\frac{1}{2}(f, A_0 f)=0, \quad (4.7)$$

where ρ is either ∞ or $(4\nu s)^{-1}$, and A_0 is given in Section 2.

Remark 4.2. – Note that since the support of $\widetilde{Q_j f}$ is contained in the support of $\tilde{f}$ for all $f\in\mathscr{D}_\mu\,(\mu\geqq 1)$, it follows that such an $f\in D(A_0)$ and thus $(f, A_0 f)$ is well defined.

The proof of the theorem proceeds via a few lemmas.

LEMMA 4.3. – *Let* $0\leqq\beta\leqq 1$ *and let* $f\in\mathscr{D}_{1+\beta}$ *and* $g\in\mathscr{D}_1$. *Then for any real* $\tau\neq 0$,

$$\text{(i):}\quad \|(Z_\tau-I)f\|\leqq K|\tau|^{(1+\beta)/2}\,\||Q|^{1+\beta}f\|, \quad (4.8)$$

$$\text{(ii)}\quad \left|(\tau^{-1}(Z_\tau-I)f, g)+i\sum_{j=1}^n (Q_j f, Q_j g)\right| \leqq K'|\tau|^{\beta/2}\sum_{j=1}^n \||Q_j|^{1+\beta}f\|\,\||Q_j|g\|, \quad (4.9)$$

where the constants K and K′ depend only on β.

The proofs of (i) and (ii) are elementary using the estimate

$$|\tau^{-1}(e^{i\tau x^2}-1)|\leqq 2^{1-\eta}|\tau|^\eta|x|^{2\eta}$$

and

$$|\tau^{-1}(e^{i\tau x^2}-1)-ix^2|\leqq 2^{1-\eta}|\tau|^\eta|x|^{2+2\eta}$$

respectively for any $\eta\in[0, 1]$.

LEMMA 4.4. – *Let* $f\in\mathscr{D}_1$. *Then for every* $\nu>0$,

$$\frac{-i}{4}\int_0^\infty \frac{ds}{s}\sum_{j=1}^n\left[\left(Q_j f, Q_j\hat{\theta}_\nu\left(\frac{|P|}{s}\right)f\right)-\left(Q_j\hat{\theta}_\nu\left(\frac{|P|}{s}\right)f, Q_j f\right)\right] = -\frac{1}{2}(f, A_0 f).$$

The proof of this lemma is exactly as in the Appendix of [1] since $\hat{\theta}_\nu(0)=1$ and $\hat{\theta}_\nu(\infty)=0$ for any $\nu>0$.

LEMMA 4.5. – *Let* $f \in \mathscr{D}_0$ *and* $N_0 = \sup\{|\underline{k}|;\ \underline{k}$ *in the support of* $\tilde{f}\}$. *Denote by* $\hat{\theta}_\nu^\perp = 1 - \hat{\theta}_\nu$. *Then* $\hat{\theta}_\nu^\perp\left(\frac{|\underline{P}|}{s}\right) f = 0$ *for* $s > N_0$ *and* $0 < \nu < \frac{1}{2}$.

This is an easy consequence of the fact that $\tilde{f}(\underline{k}) = 0$ for $|\underline{k}| > N_0$ while $\hat{\theta}_\nu^\perp\left(\frac{|\underline{k}|}{s}\right) = 0$ for $|\underline{k}| \leqq s$.

Proof of Theorem 4.1. – We shall prove the result for the positive sign only, the proof for the negative sign being identical. Setting $f_\rho = Y_\rho^\pm f$ and using Lemma 4.4, we can rewrite the left hand side of (4.7) as $\lim_{\nu \to 0^+} \int_0^\infty ds\, \mathscr{I}(\nu, s)$, where

$$\mathscr{I}(\nu, s) = \frac{1}{4\nu s^2}\left[\left(f_\rho,\ Z_{\nu s}^* \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right) Z_{\nu s} f_\rho\right) - \left(f_\rho,\ \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right) f_\rho\right)\right.$$
$$\left. + i\nu s \sum_{j=1}^{n} \left\{\left(Q_j f,\ Q_j \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right) f\right) - \left(Q_j \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right) f,\ Q_j f\right)\right\}\right]$$
$$\equiv \frac{1}{4\nu s^2}[\mathscr{I}^{(1)}(\nu, s) + \mathscr{I}^{(2)}(\nu, s)] \tag{4.10}$$
$$\equiv -\frac{1}{4\nu s^2}[\mathscr{I}^{(3)}(\nu, s) + \mathscr{I}^{(4)}(\nu, s)]. \tag{4.11}$$

In the above,

$$\mathscr{I}^{(1)}(\nu, s) = \left((Z_{\nu s} - I) f_\rho,\ \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right)(Z_{\nu s} - I) f_\rho\right),$$
$$\mathscr{I}^{(2)}(\nu, s) = 2\,\mathrm{Re}\left[\left((Z_{\nu s} - I) f_\rho,\ \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right) f_\rho\right)\right.$$
$$\left. + i\nu s \sum_{j=1}^{n}\left(Q_j f,\ Q_j \hat{\theta}_\nu\left(\frac{|\underline{P}|}{s}\right) f\right)\right], \tag{4.12}$$

and $\mathscr{I}^{(3)}$ and $\mathscr{I}^{(4)}$ are same as $\mathscr{I}^{(1)}$ and $\mathscr{I}^{(2)}$ respectively with $\hat{\theta}_\nu$ replaced by $\hat{\theta}_\nu^\perp$. We divide the range of integration in s into $(0, N_0]$ and $[N_0, \infty)$, where N_0 is given in Lemma 4.5. Note that by Lemma 4.5, $\mathscr{I}^{(4)}(\nu, s) = 0$ for all $s > N_0$. On the other hand, we have

$$|\mathscr{I}^{(3)}(\nu, s)| \leqq \frac{1}{s^2}\left\|\left(\frac{s}{|\underline{P}|}\right)^2 \hat{\theta}_\nu^\perp\left(\frac{|\underline{P}|}{s}\right)\right\| \,\|\,|\underline{P}|(Z_{\nu s} - I) f_\rho\|^2, \tag{4.13}$$

and by Lemma 4.3 (i),

$$\| |\underline{P}| (Z_{\nu s}-I) f_\rho \|^2 = \sum_{j=1}^{n} \| P_j (Z_{\nu s}-I) f_\rho \|^2$$
$$\leqq K(\nu s)^{1+\beta} \sum_{j=1}^{n} \| |Q|^{1+\beta} Y_\rho^+ P_j f \|^2 + 8\nu^2 s^2 \| \langle Q \rangle Y_\rho^+ f \|. \quad (4.14)$$

By Lemma 3.2 (ii), $\| |Q|^{1+\beta} Y_\rho^+ P_j f \|$ and $\| \langle Q \rangle Y_\rho^+ f \|$ are bounded uniformly in ρ for all $f \in \mathscr{D}_{1+\beta}$. Since $\left| \frac{1}{u^2} \hat{\theta}_\nu^\perp (u) \right| \leqq 1$, (4.11), (4.13) and (4.14) yields

$$\left| \int_{N_0}^{\infty} ds \, \mathscr{I}(\nu, s) \right| \leqq K' \left[\nu^\beta \int_{N_0}^{\infty} \frac{ds}{s^{3-\beta}} + \nu \int_{N_0}^{\infty} \frac{ds}{s^2} \right] \to 0 \quad \text{as } \nu \to 0^+.$$

Next we deal with the integral over $(0, N_0]$. By Lemma 4.3 (i),

$$\left| \int_0^{N_0} \frac{ds}{4\nu s^2} \mathscr{I}^{(1)}(\nu, s) \right| \leqq \int_0^{N_0} \frac{ds}{4\nu s^2} \| (Z_{\nu s}-I) f_\rho \|^2$$
$$\leqq K'' \int_0^{N_0} \frac{ds}{4\nu s^2} (\nu s)^{1+\beta} \| |Q|^{1+\beta} f_\rho \| \to 0 \quad \text{as } \nu \to 0^+,$$

since $0<\beta<1$ and $\| \langle Q \rangle^{1+\beta} Y_\rho^+ f \|$ is bounded uniformly in ρ for $f \in \mathscr{D}_{1+\beta}$ by Lemma 3.2 (ii).

Finally we write $\mathscr{I}^{(2)} = 2 \operatorname{Re} [\mathscr{I}_1^{(2)} - \mathscr{I}_2^{(2)}]$, where

$$\mathscr{I}_1^{(2)}(\nu, s) = ((Z_{\nu s}-I) f_\rho, \hat{\theta}_\nu f_\rho) + i\nu s \sum_{j=1}^{n} (Q_j f_\rho, Q_j \hat{\theta}_\nu f_\rho)$$

and

$$\mathscr{I}_2^{(2)}(\nu, s) = i\nu s \sum_{j=1}^{n} \{ (Q_j f_\rho, Q_j \hat{\theta}_\nu f_\rho) - (Q_j f, Q_j \hat{\theta}_\nu f) \}. \quad (4.15)$$

Here we have suppressed the argument $|\underline{P}|/s$ of the function $\hat{\theta}_\nu$ for brevity. By Lemma 4.3 (ii) we get that

$$\left| \int_0^{N_0} \frac{ds}{4\nu s^2} \mathscr{I}_1^{(2)}(\nu, s) \right|$$
$$\leqq K_1 \int_0^{N_0} \frac{ds}{4s} (\nu s)^{\beta/2} \sum_{j=1}^{n} \| |Q_j|^{1+\beta} f_\rho \| \| Q_j \hat{\theta}_\nu f_\rho \|. \quad (4.16)$$

Now by Lemma 3.2 (ii) and the support properties of f,

$$\| Q_j \hat{\theta}_\nu f_\rho \| \leqq \| Q_j f_\rho \| + \left\| \frac{\partial}{\partial P_j} \hat{\theta}_\nu \left(\frac{|\underline{P}|}{s} \right) f_\rho \right\|$$
$$\leqq \| Q_j f_\rho \| + \left\| \frac{P_j}{|\underline{P}| s} \hat{\theta}_\nu' \left(\frac{|\underline{P}|}{s} \right) f_\rho \right\| \leqq K_2 (1+\nu^{-\delta}), \quad (4.17)$$

for $f \in \mathscr{D}_{1+\beta}$. Combining (4.16) and (4.17), we have

$$\left| \int_0^{N_0} \frac{ds}{4\nu s^2} \mathscr{I}_1^{(2)}(\nu, s) \right| \leqq K_2 \int_0^{N_0} ds\, s^{(\beta/2)-1} (\nu^{\beta/2} + \nu^{(\beta/2)-\delta})$$

$\to 0$ as $\nu \to 0^+$ since $\delta < \beta/2$. In (4.15), note that $\mathscr{I}_2^{(2)}(\nu, s)$ is identically zero where $\rho = \infty$ and hence for this case the proof is complete. In the case when $\rho = (4\nu s)^{-1}$, we make the following estimate using (4.17) and Lemma 3.2 (iii):

$$\begin{aligned} |\mathscr{I}_2^{(2)}(\nu, s)| \leqq \nu s \sum_{j=1}^{n} \{ & \| Q_j (Y_\rho^+ - I) f \| \| Q_j \hat{\theta}_\nu f_\rho \| \\ & + \| Q_j f \| \| Q_j (Y_\rho^+ - I) \psi(H_0) \langle Q \rangle^{-1} \| \| \langle Q \rangle \hat{\theta}_\nu f \| \} \\ & \leqq K_3 (\nu s) \rho^{1-\alpha} (1 + \nu^{-\delta}) = K_3' (\nu^\alpha + \nu^{\alpha-\delta}) s^\alpha. \end{aligned}$$

Thus

$$\left| \int_0^{N_0} \frac{ds}{4\nu s^2} \mathscr{I}_2^{(2)}(\nu, s) \right| \leqq \frac{1}{4} K_3' (\nu^{\alpha-1} + \nu^{\alpha-1-\delta}) \int_0^{N_0} s^{\alpha-2}\, ds$$

$\to 0$ as $\nu \to 0^+$ since $\alpha > 1$ and $\delta < \alpha - 1$.

We end this section by giving the proof of the main theorem stated in Section 2.

Proof of the main theorem. – As mentioned in the beginning of this section we shall assume here that for $f \in \mathscr{D}_{1+\eta}$, with $\eta > 0$, $S f \in \mathscr{D}_{1+\beta}$ for $0 < \beta < \dfrac{\alpha-1}{\alpha+2}$ and that (3.5) and (3.6) are verified for such f, the proofs being given in the Appendix (Theorems A.10, A.3 and A.15 respectivsely). Thus given δ such that $0 < \delta < \dfrac{\alpha-1}{2(\alpha+2)}$, we can find a $\beta \in \left(0, \dfrac{\alpha-1}{\alpha+2}\right)$ such that $\delta < \dfrac{\beta}{2}$ and apply Theorem 3.4. And using the notations of the same theorem, we shall show that $\tau_r^{(j)}(f) \to 0$ as $r \to \infty$ for $j = 2$ and 3.

In fact by (4.4)-(4.6) we have

$$\begin{aligned} \tau_r^{(2)}(f) = \int_0^\infty \frac{ds}{4\nu s^2} [& (Y_{(4\nu s)^{-1}}^+ f, Z_{\nu s}^* \hat{\theta}_\nu Z_{\nu s} Y_{(4\nu s)^{-1}} f) \\ & - (f, \hat{\theta}_\nu f)] - \int_0^\infty \frac{ds}{4\nu s^2} [(f, Z_{\nu s}^* \hat{\theta}_\nu Z_{\nu s} f) - (f, \hat{\theta}_\nu f)], \end{aligned}$$

where we have again suppressed the argument $\dfrac{|P|}{s}$ of the function $\hat{\theta}_\nu$.

Since we can find a $\beta \in \left(0, \dfrac{\alpha-1}{\alpha+2}\right)$ such that $\delta < \dfrac{\beta}{2} < \alpha - 1$ and since by

Theorem 4.1 each of the above integrals converges to $-\frac{1}{2}(f, A_0 f)$ as $\nu \to 0^+$, we conclude that $\tau_r^{(2)}(f) \to 0$ as $r \to \infty$. Similarly one arrives at the result $\lim_{r \to \infty} \tau_r^{(3)}(f) = 0$.

We not that $(S f, \hat{\theta}_\nu S f) = (f, \hat{\theta}_\nu f)$ and using (4.4)-(4.6) we write

$$\tau_r^{(1)}(f) = \int_0^\infty \frac{ds}{4\nu s^2} [(Y^+_{(4\nu s)^{-1}} S f, Z^*_{\nu s} \hat{\theta}_\nu Z_{\nu s} Y^+_{(4\nu s)^{-1}} S f) - (S f, \hat{\theta}_\nu S f)] - \int_0^\infty \frac{ds}{4\nu s^2} [(f, Z^*_{\nu s} \hat{\theta}_\nu Z_{\nu s} f) - (f, \hat{\theta}_\nu f)]$$

which by Theorem 4.1 converges to $-\frac{1}{2}(S f, A_0 S f) + \frac{1}{2}(f, A_0 f)$ as $\nu \to 0^+$. Thus by Theorem 3.4, $\tau(f)$ exists for all $f \in \mathscr{D}_{4+\eta}$ and $\tau(f) = -\frac{1}{2}(f, S^* [A_0, S] f)$. Finally we obtain the Eisenbud-Wigner formula

$$\tau(f) = \int_0^\infty d\lambda \left(f_\lambda, \left\{ -i S^*(\lambda) \frac{dS(\lambda)}{d\lambda} \right\} f_\lambda \right)$$

in the spectral representation of H_0 once we take note of the Proposition 2.1 and recall that $(A_0 f)_\lambda = 2i \frac{df_\lambda}{d\lambda}$. ■

APPENDIX

Here we prove the three assumptions made in the proof of the main theorem. First we collect the known results in the form of a few propositions and then we prove the easiest of the three viz. (3.5). Next we establish the decay properties of $V_{t-s} U_s$ upto the order $\alpha + 1$ using commutator method which leads to the proof of the result that if $f \in \mathscr{D}_{3+\eta}$, then $S f \in \mathscr{D}_{1+\beta}$ for $0 < \beta < \frac{\alpha - 1}{\alpha + 2}$. The proof of (3.6) is long, though not complicated. For this part, we omit most of the details since the methods are identical to those for getting the lower order estimates.

PROPOSITION A.1. — *Let* $\psi \in C_0^\infty(0, \infty)$ *and* $\mu \in \mathbb{R}_+$. *Then there exist constants* K *and* K_1, *independent of t, such that*

(i) $\| \langle Q \rangle^{-\mu} U_t \psi(H_0) \langle Q \rangle^{-\mu} \| \leqq K (1 + |t|)^{-\mu}$,

(ii) $\| \langle A \rangle^{-\mu} U_t \psi(H_0) \langle A \rangle^{-\mu} \| \leqq K_1 (1 + |t|)^{-\mu}$.

For proof, the reader is referred to [13] or Lemma 2.4 (ii) of [14].

PROPOSITION A.2. — *Let either* $\psi \in C_0^\infty(0, \infty)$ *or* $\psi(\lambda)=(\lambda+\omega)^{-1}$ *for some* $\omega>0$ *such that* $-\omega \in \rho(H) \cap \rho(H_0)$, $\rho(H_0)$ *and* $\rho(H)$ *being the resolvent sets of* H_0 *and* H *respectively. Then for any* $\mu \in \mathbb{R}$

(i) $\langle Q \rangle^\mu \psi(H_0) \langle Q \rangle^{-\mu}$, $\langle Q \rangle^\mu P_j \psi(H_0) \langle Q \rangle^{-\mu}$, $\langle Q \rangle^{\mu-1} A \psi(H_0) \langle Q \rangle^{-\mu}$ *are all bounded operators,*

(ii) *the three expressions in* (i) *are bounded, when* H_0 *is replaced by* H,

(iii) $\langle Q \rangle^\mu \exp(-it(H_0+\omega)^{-1}) \langle Q \rangle^{-\mu} \| \leq K(1+|t|)^{|\mu|}$ *and*

$$\| \langle Q \rangle^{-\mu} \exp(-it(H+\omega)^{-1}) \langle Q \rangle^{\mu} \| \leq K(1+|t|)^{|\mu|},$$

(iv) $\langle Q \rangle^\mu \{ \psi(H) - \psi(H_0) \} \langle Q \rangle^{\alpha-\mu}$ *is a bounded operator.*

(v) *Let* W *satisfy* (2.1), *and let* $\xi, \psi \in C_0^\infty(0, \infty)$. *Then for each* $\mu \in [0, \alpha]$ *and* $\varepsilon>0$, *there is a constant* K *such that for all* $s, t \in \mathbb{R}$,

(*a*) $\| \langle Q \rangle^{-\mu} \xi(H) V_t \langle Q \rangle^{-\mu} \| \leq K(1+|t|)^{-\mu+\varepsilon}$

(*b*) $\| \langle Q \rangle^{-\mu} \xi(H) V_{t-s} U_s \psi(H_0) \langle Q \rangle^{-\mu} \| \leq K(1+|t|)^{-\mu+\varepsilon}$.

(vi) *Let* $f \in \mathscr{D}_1$ *and let* $\psi, \psi_1 \in C_0^\infty(0, \infty)$ *such that* $\psi(H_0) f = f$ *and* $\psi_1 \psi = \psi$. *Then* $Q_j f = \psi_1(H_0) Q_j f$ *and* $A f = \psi_1(H_0) A f$.

This proposition is proven in Sections 3 and 4 of [2] except for (vi) which is easy to verify.

THEOREM A.3. — *Let* $f \in \mathscr{D}_{3+\eta}$ *for some* $\eta>0$. *Then*

$$\int_{-\infty}^{0} \| V_t \Omega_- f - T_t^- f \| \, dt < \infty.$$

Proof. — Using the identity (*see* Chapter 13 of [7]) :

$$U_t^* (W(\underline{Q}) - W(2t\underline{P})) U_t = \int_0^1 d\rho \, U_{t/\rho}^* \left[\sum_{j=1}^{n} \partial_j W(\rho \underline{Q}) U_{t/\rho} Q_j + it \Delta W(\rho \underline{Q}) U_{t/\rho} \right],$$

we have that

$$\| V_t \Omega_- f - U_t Y_t^- f \| \leq \int_{-\infty}^{t} ds \int_0^1 d\rho \left[\sum_{j=1}^{n} \| \partial_j W(\rho \underline{Q}) U_{s/\rho} \psi_1(H_0) Q_j Y_s^- \psi(H_0) f \| \right.$$
$$\left. + |s| \, \| \Delta W(\rho \underline{Q}) U_{s/\rho} \psi(H_0) Y_s^- f \| \right], \quad (a.1)$$

where we have used the Proposition A.2 (vi) with ψ and ψ_1 such that $f = \psi(H_0) f$ and $\psi_1 \psi = \psi$. Now the result follows from (a.1) and (2.1) by using the Lemma 3.2 (ii) and Proposition A.1. ■

Now we give a few preliminary results.

LEMMA A.4. — *Let* $\xi \in C_0^\infty(\mathbb{R})$. *Then*

(i) *for any real numbers* μ_1 *and* μ_2 *with the proterty that* $\mu_1+\mu_2 \leq \alpha$,

$$\langle Q \rangle \mu_1 \, ad_A^m \{ \xi(H) - \xi(H_0) \} \langle Q \rangle \mu_2 \text{ is bounded for } m=1, 2, 3,$$

(ii) $\langle Q \rangle^\mu ad_A^m (\xi(H)) \langle Q \rangle^{-\mu}$ *is bounded for all* $\mu \in \mathbb{R}$, *and each* $m=1, 2, 3$.

(iii) *Let* μ_1 *and* μ_2 *be as in* (i). *Then for any* j $(1 \leqq j \leqq n)$, $\langle Q \rangle^{\mu_1} P_j \{ \xi(H) - \xi(H_0) \} \langle Q \rangle^{\mu_2}$ *is a bounded operator.*

Proof. – Without loss of generality assume that $\xi \in C_0^\infty(-\omega, \infty)$, where $\omega > 0$ be such that $(H_0 + \omega) > I$ and $(H + \omega) > I$. Let ζ be defined by $\zeta(\lambda) = \xi(\lambda^{-1} - \omega)$; observe that $\zeta \in C_0^\infty(\mathbb{R})$ and that support of ζ is contained in $(0, \infty)$. With $L_0 = (H_0 + \omega)^{-1}$ and $L = (H + \omega)^{-1}$, we have

$$\xi(H) = \zeta(L) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \tilde{\zeta}(t) e^{iLt} \, dt$$

and

$$\xi(H_0) = \zeta(L_0) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \tilde{\zeta}(t) e^{iLt} \, dt.$$

Now

$$[A, \xi(H) - \xi(H_0)] = (-i)(2\pi)^{-1/2} \int_{-\infty}^{\infty} dt \, \tilde{\zeta}(t) \int_0^t ds \, [A, e^{iL(t-s)} LWL_0 e^{iL_0 s}].$$

The proof for $m=1$ in (i) is completed by expanding the commutator above and observing that by Proposition A.2 (i)-(iii)

$$\| \langle Q \rangle^{\mu_1} [A, e^{iL(t-s)} LWL_0 e^{iL_0 s}] \langle Q \rangle^{\mu_2} \|$$

is bounded by a polynomial in t and s. The proofs for $m=2$ and $m=3$ are identical, and that of (ii) is a consequence of (i) and Proposition A.2 (i).

The proof of (iii) is same as that of Lemma 5 of [2]. ■

LEMMA A.5. – *Let* $\Phi: \mathbb{R}^n \to \mathbb{R}$ *satisfy* (2.1) *with* $\Phi(\underline{x}) \leqq K(1 + |\underline{x}|)^{-\nu}$, $\nu \geqq \alpha$, *and let* $\psi, \xi, \xi_1 \in C_0^\infty(0, \infty)$ *such that* $\xi_1 \xi = \xi$. *Then for each* $\mu \in [0, \alpha]$ *and each* $\varepsilon > 0$, *there is a constant* C *such that for all* $s, t \in \mathbb{R}$

(i) $$\| \langle Q \rangle^{-\mu} \xi(H) V_{t-s} \Phi(\underline{Q}) U_s \psi(H_0) \langle Q \rangle^{-\mu - \varkappa(\nu)} \| \leqq C(1 + |t|)^{-\mu + \varepsilon} (1 + |s|)^{-\varkappa(\nu)},$$

where

$$\kappa(\nu) = \begin{cases} 1 & \text{for } \alpha \leqq \nu < \alpha + 1 \\ 2 & \text{for } \nu \geqq \alpha + 1, \end{cases}$$

and

(ii) $$\| \langle Q \rangle^{-\mu} \xi(H) V_{t-s} [A, \xi_1(H)] W U_s \psi(H_0) \langle Q \rangle^{-\mu-1} \| \leqq C(1 + |t|)^{-\mu + \varepsilon} (1 + |s|)^{-1}.$$

Proof. – Note that (i) is trivially true for $\mu = 0$ and is seen to be true for $\mu = \alpha - 1$ as follows. By Propositions A.1 and A.2 (v) *a*, the expression on the L.H.S. of (i) is bounded by

$$C(1 + |t-s|)^{-\min(\mu, \nu - \varkappa(\nu)) + \varepsilon} (1 + |s|)^{-\varkappa(\nu)}.$$

If $|t-s| \geqq \frac{|t|}{2}$, then the above estimate is majorized by

$$C(1+|t|)^{-\mu+\varepsilon}(1+|s|)^{-\varkappa(\nu)}.$$

On the other hand if $|t-s|<|t|/2$ then the same expression is bounded by $C(1+|s|)^{-\nu} \leqq C(1+|t|)^{-\mu}(1+|s|)^{-\varkappa(\nu)}$ since in this case $|s|>\frac{|t|}{2}$.

Choose $\psi_1 \in C_0^\infty(0, \infty)$ such that $\psi_1 \psi=\psi$ and set $\xi_2(\lambda)=\lambda^{-1}\xi(\lambda)$ implying $\xi \in C_0^\infty(0, \infty)$. Then

$$\begin{aligned}
&[A, V_{t-s}\xi_1(H)\Phi(Q)U_s\psi_1(H_0)] \\
&\quad = 2tHV_{t-s}\xi_1(H)\Phi U_s\psi_1(H_0)+2iV_{t-s}\xi_1(H)\Phi U_s\psi_1'(H_0)H_0 \\
&\quad + V_{t-s}\xi_1(H)[A, \Phi]U_s\psi_1(H_0)-2sV_{t-s}\xi_1(H)\{W\Phi+[H_0, \Phi]\}U_s\psi_1(H_0) \\
&\quad + V_{t-s}[A, \xi_1(H)]\Phi U_s\psi_1(H_0)-2 \\
&\quad \times \int_s^t d\tau\, V_{t-\tau}\tilde{W}V_{\tau-s}\xi_1(H)\Phi U_s\psi_1(H_0). \qquad (a.2)
\end{aligned}$$

We premultiply and postmultiply $(a.2)$ by $\langle Q\rangle^{-\alpha}\xi_2(H)$ and by $\psi(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)}$ respectively, note that $\psi_1'\psi=0$, use Proposition A.2 (i) and (ii), and rearrange terms to get the estimate that for some constant $K_1>0$,

$$\begin{aligned}
&K_1|t|\,\|\langle Q\rangle^{-\alpha}\xi(H)V_{t-s}\Phi U_s\psi(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)}\| \\
&\quad \leqq \|\langle Q\rangle^{-\alpha+1}V_{t-s}\xi_1(H)\Phi U_s\psi(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)}\| \\
&\quad + \|\langle Q\rangle^{-\alpha}\xi_2(H)\xi_1(H)V_{t-s}\Phi U_s\psi_1(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)+1}\|. \\
&\quad + \|\langle Q\rangle^{-\alpha}\xi_2(H)\xi_1(H)V_{t-s}[A, \Phi]U_s\psi(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)}\| \\
&\quad + |s|\,\|\langle Q\rangle^{-\alpha}\xi_2(H)\xi_1(H)V_{t-s} \\
&\quad \times \{W\Phi-\Delta\Phi-2i\nabla\Phi.P\}U_s\psi(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)}\| \\
&\quad + \|\langle Q\rangle^{-\alpha}\xi_2(H)V_{t-s}[A, \xi_1(H)]\Phi U_s\psi(H_0)\langle Q\rangle^{-\alpha-\varkappa(\nu)}\| \\
&\quad + \int_s^t d\tau\,\|\langle Q\rangle^{-\alpha}\xi_2(H)V_{t-\tau}\tilde{W}V_{\tau-s}\xi_1(H)\Phi U_s\psi(H_0) \\
&\qquad \langle Q\rangle^{-\alpha-\varkappa(\nu)}\| \qquad (a.3)
\end{aligned}$$

Now, the first, second, and third terms in $(a.3)$ are majorized by constant. $|t|^{-\alpha+1+\varepsilon}|s|^{-\varkappa(\nu)}$ by the estimate given in the first paragraph of the proof. In the fourth term of $(a.3)$, first we note that by Proposition A.2 (i) we can essentially ignore the P_i in it and that $W\Phi$, $\Delta\Phi$ and $\nabla\Phi$ are bounded by $(1+|\underline{x}|)^{-(\nu+\alpha)}$, $(1+|\underline{x}|)^{-(\nu+2)}$ and $(1+|\underline{x}|)^{-(\nu+1)}$ respectively. For $|t-s| \geqq |t|/2$, we split these exponents into $\nu+\alpha-1-\kappa(\nu)$ $(((\nu+1-\kappa(\nu)), (\nu-\kappa(\nu))$ respectively) and $1+\kappa(\nu)$ and use Proposition A.1 to the right of the expression while on the left we use Proposition A.2 (v). Since all the exponents $\nu+\alpha-1-\kappa(\nu)$, $\nu+1-\kappa(\nu)$ and $\nu-\kappa(\nu)$ are greater than or equal to $\alpha-1$, this shows that the fourth term in $(a.3)$ is bounded by const. $|t|^{-\alpha+1}|s|^{-1-\varkappa(\nu)}$. For $|t-s|<|t|/2$, we use the Proposition A.1 to the right of the expression to get a majorization by Const. $|s||s|^{-(\alpha+\varkappa(\nu))} \leqq$ Const. $|t|^{-\alpha+1}|s|^{-\varkappa(\nu)}$ since $|s|>|t|/2$ in this case. Thus the fourth term of $(a.3)$ has the requisite bound. The fifth term can be handled similarly on using the Lemma A.4 (ii).

In the sixth term for $|t-s|>|t|/2$, the decay in $|t|$ is obtained from the τ-integral on using Proposition A.2 (v) a while that in $|s|$ is from the free group decay. For $|t-s|<|t|/2$, the free group decay from Proposition A.1 gives decays in $|s|$ and $|t|$ (since $|s|>|t|/2$ in this case) while the integrability in τ is assured by the Proposition A.2 (v) a as before. The proof of (i) of this lemma is finally completed by dividing by $|t|$ in $(a.3)$, observing that the L.H.S. of (i) is a uniformly bounded operator and by interpolation. Note that

$$\xi(H)[A, \xi_1(H)] = \xi(H)\, ad_A^1(\xi_1(H)-\xi_1(H_0)) - 2i\,\xi(H)[H\xi_1'(H) - H_0\xi_1'(H_0)]$$

since $\xi_1'\xi=0$. Now the proof of (ii) follows easily from Lemma A.4 (i), Proposition A.2 (iv) and an argument identical to that of (i). ■

Remark A.6. – Since A is not a bounded operator, the calculation (and all similar ones in the sequel) of commutators of A with a bounded operator is to be understood in the sense of a quadratic from on $\mathscr{D}_{\nu_1} \times \mathscr{D}_{\nu_2}$ for suitable $\nu_1, \nu_2 \geqq 1$. However, it is often the case, as in $(a.2)$ for example, that the commutator has a bounded extension [e. g. that given by the R.H.S. of $(a.2)$].

The next theorem is an improvement over the result in Proposition A.2 (v).

THEOREM A.7. – *Let* W *satisfy* (2.1) *and* $\xi, \psi \in C_0^\infty(0, \infty)$. *Then for each* $\mu \in [0, \alpha+1]$ *and* $\varepsilon>0$, *there exists a constant* K *such that for all* $s, t \in \mathbb{R}$

(i) $\| \langle Q \rangle^{-\mu} \xi(H) V_t \langle Q \rangle^{-\mu} \| \leqq K(1+|t|)^{-\mu+\varepsilon}$,

(ii) $\| \langle Q \rangle^{-\mu} \xi(H) V_{t-s} U_s \psi(H_0) \langle Q \rangle^{-\mu} \| \leqq K(1+|t|)^{-\mu+\varepsilon}$.

Proof. – Since (i) follows from (ii) on setting $s=0$ and $\psi=1$, we shall obtain both results simultaneously if we do this substitution at each step. Without loss of generality we assume $0<\varepsilon<\alpha-1$ and choose ξ_1, ψ_1, ξ_2 and write down the estimate as in the proof of Lemma A.5:

$$\begin{aligned} K'|t| \, \| \langle Q \rangle^{-\alpha-1} \xi(H) V_{t-s} U_s \psi(H_0) \langle Q \rangle^{-\alpha-1} \| \\ \leqq \| \langle Q \rangle^{-\alpha-1} \xi_2(H)[A, V_{t-s}\xi_1(H) U_s \psi_1(H_0)] \psi(H_0) \langle Q \rangle^{-\alpha-1} \| \\ + |s| \, \| \langle Q \rangle^{-\alpha-1} \xi_2(H) V_{t-s} \xi_1(H) W U_s \psi(H_0) \langle Q \rangle^{-\alpha-1} \| \\ + \| \langle Q \rangle^{-\alpha-1} \xi_2(H) V_{t-s} [A, \xi_1(H)] U_s \psi(H_0) \langle Q \rangle^{-\alpha-1} \| \\ + \int_s^t \| \langle Q \rangle^{-\alpha-1} \xi_2(H) V_{t-\tau} \tilde{W} V_{\tau-s} \xi_1(H) U_s \psi(H_0) \langle Q \rangle^{-\alpha-1} \| \, d\tau, \quad (a.4) \end{aligned}$$

for some constant $K'>0$.

The first term on the R.H.S. of $(a.4)$ is bounded by constant. $(1+|t|)^{-\alpha+\varepsilon}$ by Proposition A.2 (i), (ii) and (v) b, while the second term admits an identical estimate by Lemma A.5 (i) with $\Phi=W$, $\nu=\alpha$ and $K(\nu)=1$ (note that for case $s=0$, $\psi=1$ this term is absent). An identical bound for the third term of $(a.4)$ results from a calculation similar to that in the proof of Lemma A.5 (i).

The estimate of the last term of $(a.4)$ is not immediate and needs a commutator calculation viz.

$$[A, V_{t-\tau}\xi_3(H)\tilde{W}\xi_4(H)V_{\tau-s}\xi_1(H)U_s\psi_1(H_0)]$$

for suitable choice of ξ_3 and ξ_4 in $C_0^\infty(0, \infty)$. A repetition of the arguments and estimates as above in the proof of Lemma 4.5 leads to the necessary bound for both cases:

$$\int_s^t \|\langle Q\rangle^{-\alpha}\xi_2(H)V_{t-\tau}\tilde{W}V_{\tau-s}\xi_1(H)U_s\psi(H_0)\langle Q\rangle^{-\alpha-1}\|\,d\tau \leqq \text{Const.}\,(1+|t|)^{-\alpha+\varepsilon}. \quad (a.5)$$

This completes the proof. ■

The next two lemmas are preparatory material for the result that $Sf\in\mathscr{D}_{1+\beta}$ for $0<\beta<\dfrac{\alpha-1}{\alpha+2}$ when $f\in\mathscr{D}_{3+\eta}$ for some $\eta>0$.

LEMMA A.8. – *Let* $\Omega_\pm$, S *be as defined in Section* 2. *Then* $\Omega_\pm$ *map* $\mathscr{D}_{2+\eta}$ *into* $D(\langle Q\rangle)$ *while* S *maps* $\mathscr{D}_{2+\eta}$ *into* $\mathscr{D}_1$ *for every* $\eta>0$.

Proof. – Let $f\in\mathscr{D}_1$ and $g\in\mathscr{D}_{2+\eta}$ for some $\eta>0$ such that $\xi(H_0)g=g$ for some $\xi\in C_0^\infty(0, \infty)$. Then

$$(Q_j f, \xi(H)V_t^* U_t g)=(f, \xi(H)V_t^* U_t Q_j g)+(f, [Q_j, \xi(H)V_t^* U_t]g)$$

and thus it is enough to show that $|(f, [Q_j, \xi(H)V_t^* U_t]g)|\leqq \text{Const.}\,\|f\|$ for all $f\in\mathscr{D}_1$. Now,

$$\begin{aligned}[Q_j, \xi(H)V_t^* U_t]g &= [Q_j, \xi(H)-\xi(H_0)]V_t^* U_t g \\ &\quad +2P_j\xi'(H_0)V_t^* U_t g-2tV_t^*[P_j, \xi(H)-\xi(H_0)]U_t g \\ &\quad +2\int_0^t \tau\,d\tau\,V_\tau^*\,\partial_j W V_{\tau-t}\xi(H)U_t g. \quad (a.6)\end{aligned}$$

The first part of the result follows by applying the Proposition A.2 (iv) to the first term, Lemma A.4 (iii) and Proposition A.1 to the third term, Theorem A.7 (ii) to the last term in $(a.6)$, and letting $t\to\pm\infty$.

Let f, g, ξ be as before and choose $\psi\in C_0^\infty(0,\infty)$ such that $\psi\xi=\xi$. Then we have

$$
\begin{aligned}
&[Q_j, \psi(H_0)\, U_t^*\, V_t\, \xi(H)\, V_t^*\, \psi(H)\, U_s]\, g \\
&\quad = 2\, P_j\, \psi'(H_0)\, U_t^*\, V_t\, \xi(H)\, V_s^*\, \psi(H)\, U_s g \\
&\qquad + \psi(H_0)\, U_t^*\, [Q_j, \xi(H)]\, V_{t-s}\, \psi(H)\, U_s g \\
&\qquad - 2t\, \psi(H_0)\, U_t^*\, [P_j, \xi(H) - \xi(H_0)]\, V_{t-s}\, \psi(H)\, U_s g \\
&\qquad + \psi(H_0)\, U_t^*\, V_{t-s}\, \xi(H)\, [Q_j, \psi(H)]\, U_s g \\
&\qquad + 2s\, \psi(H_0)\, U_t^*\, V_{t-s}\, \xi(H)\, [\psi(H) - \psi(H_0), P_j]\, U_s g \\
&\qquad - 2\, \psi(H_0)\, U_t^*\, \xi(H)\, V_t \int_s^t \tau\, d\tau\, V_\tau^*\, \partial_j W V_{\tau-s}\, \psi(H)\, U_s g. \qquad (a.7)
\end{aligned}
$$

The norm boundedness (uniform in s and t) for each of the terms of $(a.7)$ follows exactly as that for the terms in $(a.6)$. Since

$$
s-\lim_{t\to+\infty}\, s-\lim_{s\to-\infty}\, \psi(H_0)\, U_t^*\, V_t\, \xi(H)\, V_s^*\, \psi(H)\, U_s g = S g,
$$

we have te second result. ■

LEMMA A.9. – *Let ψ, $\xi\in C_0^\infty(0,\infty)$. Then for t, $s\in\mathbb{R}$, $\beta\in[0,1]$ and each $1\leqq j\leqq n$, there is a constant* K *independent of s and t such that*

$$
\| |Q_j|^\beta\, \psi(H_0)\, U_s^*\, V_{s-t}\, \xi(H)\, \langle Q\rangle^{-\beta(\alpha+1)} \| \leqq K(1+|t|)^\beta.
$$

Proof. – It suffices to prove the above estimate for $\beta=1$ and then apply interpolation. For this we note that

$$
\begin{aligned}
[Q_j, \psi(H_0)\, U_s^*\, V_{s-t}] &= [Q_j, \psi(H_0)]\, U_s^*\, V_{s-t} - 2t\, \psi(H_0)\, P_J\, U_s^*\, V_{s-t} \\
&\quad + 2\int_0^{s-t} \tau\, \psi(H_0)\, U_s^*\, V_{s-t-\tau}\, \partial_j W V_\tau\, d\tau
\end{aligned}
$$

and that by Theorem A.7 (i), $\displaystyle\int_0^\infty \tau\, \| \partial_j W V_\tau\, \xi(H)\, \langle Q\rangle^{-\alpha-1} \|\, d\tau < \infty$. ■

THEOREM A.10. – S *maps* $\mathscr{D}_{3+\eta}(\eta>0)$ *into* $\mathscr{D}_{1+\beta}$ *for every* $\beta\in\left(0, \dfrac{\alpha-1}{\alpha+2}\right)$.

Proof. – Let $f\in\mathscr{D}_{1+\beta}$ and $g\in\mathscr{D}_{3+\eta}$ for β, $\eta>0$ and let ξ, ψ be as in the proof of Lemma A.8. Set $G(t,s)=\psi(H_0)\, U_t^*\, V_t\, \xi(H)\, V_s^*\, \psi(H)\, U_s$. Then it follows from $(a.7)$ that $G(t,s)\, g$ and $G(t,s)\, Q_j g$ belong to $D(Q_j)$, and that by interpolation we have for $\beta\in[0,1]$

$$
\| |Q_j|^\beta\, G(t,s)\, Q_j g \| \leqq K_1, \text{ independent of } t \text{ and } s. \qquad (a.8)
$$

Since

$$
(Q_j |Q_j|^\beta f, G(t,s)\, g) = (f, |Q_j|^\beta\, G(t,s)\, Q_j g) + (|Q_j|^\beta f, [Q_j, G(t,s)]\, g)
$$

and $G(s, t)g = Sg$ as $s \to -\infty$ followed by $t \to \infty$, we shall have the required result by virtue of (a.8) if we can show that $\| |Q_j|^\beta [Q_j, G(t, s)] g \|$ is bounded uniformly in t and s for β in a suitable subinterval of (0, 1].

By Proposition A.2(i) and Lemma A.8, it is easy to see that with $\psi_1 \in C_0^\infty(0, \infty)$ such that $\psi_1 \psi' = \psi'$,

$$\begin{aligned} \| |Q_j|^\beta P_j \psi'(H_0) U_t^* V_t \xi(H) V_s^* \psi(H) U_s g \| & \\ \leqq \| \langle Q \rangle^\beta P_j \psi'(H_0) \langle Q \rangle^{-\beta} \| & \\ \times \| \langle Q \rangle^\beta \psi_1(H_0) U_t^* V_t \xi(H) V_s^* \psi(H) U_s g \|, & \quad (a.8) \end{aligned}$$

which is bounded uniformly in t, s for $\beta \in [0, 1]$. Next,

$$\begin{aligned} \| |Q_j|^\beta \psi(H_0) U_t^* [Q_j, \xi(H)] V_{t-s} \psi(H) U_s g \| & \\ \leqq \| |Q_j|^\beta \psi(H_0) U_t^* \langle Q \rangle^{-\beta} \| & \\ \times \| \langle Q \rangle^\beta [Q_j, (\xi(H) - \xi(H_0))] \langle Q \rangle^{\alpha-\beta-1} \| & \\ \times \| \langle Q \rangle^{1+\beta-\alpha} V_{t-s} \psi(H) U_s g \| & \\ + 2 \| |Q_j|^\beta P_j \psi(H_0) \xi'(H_0) U_t^* V_{t-s} \psi(H) U_s g \|. & \quad (a.9) \end{aligned}$$

Setting $W=0$ (so that α can be taken to be 0) in Lemma A.9 we get $\| |Q_j|^\beta \psi(H_0) U_t^* \langle Q \rangle^{-\beta} \| \leqq K(1+|t|)^\beta$ while the second factor in the first term of the R.H.S. of (a.9) is bounded by Proposition A.2 (iv). For $\beta < \alpha - 1$, we can apply Proposition A.2 (v)b to get a bound of $(1+|t|)^{1+\beta-\alpha}$ for the third factor in the first term of (a.9) while the second term of (a.9) is uniformly bounded as in (a.8). Thus the L.H.S. of (a.9) is also uniformly bounded if $2\beta < \alpha - 1$. The third term of (a.7), when premultiplied by $|Q_j|^\beta$, can be estimated uniformly in norm in a similar fashion by using Lemma A.4 (iii) for $2\beta < \alpha - 1$.

Note that since $\psi'(H_0) g = 0$, we have

$$\begin{aligned} \| |Q_j|^\beta \psi(H_0) U_t^* V_{t-s} \xi(H) [Q_j, \psi(H)] U_s g \| & \\ \leqq \| |Q_j|^\beta \psi(H_0) U_t^* V_{t-s} \xi(H) \langle Q \rangle^{-\beta(\alpha+1)} \| & \\ \times \| \langle Q \rangle^{\beta(\alpha+1)} [Q_j, \xi(H) - \xi(H_0)] \langle Q \rangle^{\alpha-1-\beta(\alpha+1)} \| & \\ \| \langle Q \rangle^{\beta(\alpha+1)-\alpha+1} U_s g \| & \end{aligned}$$

which is bounded uniformly in s and t by Lemma A.9, Propositions A.2 (iv) and A.1 provided that $\beta < \dfrac{\alpha-1}{\alpha+2}$ which also implies $2\beta < \alpha - 1$. The last two terms of (a.7) on premultiplication by $|Q_j|^\beta$ can similarly be shown to be uniformly bounded in norm if $\beta < \dfrac{\alpha-1}{\alpha+2}$ on using the estimate of Theorem A.7 (ii). ■

We are now left with verification of (3.6), for which we need some lemmas.

LEMMA A.11. — *Let* $\xi, \psi \in C_0^\infty(0, \infty)$. *Then for each* $\eta \in \left(\frac{\alpha-1}{2}, \alpha-1\right]$ *there is a* $\beta \in \left(0, \frac{\alpha-1}{2}\right)$ *such that*

$$\| \langle A \rangle^{2+\beta} \xi(H) V_t^* U_t \psi(H_0) \langle Q \rangle^{-2-\eta} \| \leqq K.$$

Proof. — It is enough to show the following:

(i) $\| |A|^\beta V_t^* \xi(H) U_t \psi(H_0) \langle Q \rangle^{-\beta} \| \leqq K_1$ for $0 \leqq \beta \leqq 1$,

(ii) $\| |A|^\beta ad_A^1 [V_t^* \xi(H) U_t \psi(H_0)] \langle Q \rangle^{-1-\eta} \| \leqq K_2$,

(iii) $\| |A|^\beta ad_A^2 [V_t^* \xi(H) U_t \psi(H_0)] \langle Q \rangle^{-2-\eta} \| \leqq K_3$,

where β and η are as in the statement of this lemma. Choose $\xi_1 \in C_0^\infty(0, \infty)$ such that $\xi_1 \xi = \xi$. Observe that

$$\begin{aligned} ad_A^1 [V_t^* \xi(H) U_t \psi(H_0)] &= [A, \xi_1(H)] V_t^* \xi(H) U_t \psi(H_0) + \xi_1(H) V_t^* [A, \xi(H)] U_t \psi(H_0) \\ &\quad + 2i V_t^* \xi(H) U_t H_0 \psi'(H_0) - 2t V_t^* \xi(H) W U_t \psi(H_0) \\ &\quad - 2 \int_0^{-t} d\tau\, V_{t+\tau}^* \xi_1(H) \tilde{W} \xi(H) V_\tau U_t\, \psi(H_0). \qquad (a.10) \end{aligned}$$

The part (i) follows from (*a*.10) by using Proposition A.1, the local smoothness of $\langle Q \rangle^{-\alpha/2}$ with respect to H and interpolation.

The proof of (ii) and (iii) is similar to that of Theorem A.10. It is easy to see that $|A|^\beta [A, \xi(H)] \langle A \rangle^{-\beta}$ is bounded and that $\| |A|^\beta V_t \xi(H) \langle Q \rangle^{-\beta} \| \leqq K(1+|t|)^\beta$, and thus by part (i), Proposition A.1 and Lemma A.2 *v*(*b*), we have part (ii) if $\alpha - \beta < 1 + \eta$ and $2\beta - \alpha < -1$.

The result (iii) is similarly obtained by computing

$$|A|^\beta ad_A^2 (V_t^* \xi(H) U_t \psi(H_0)) \langle Q \rangle^{-2-\eta}$$

and by using Proposition A.2 (ii). For example, one of the terms is

$$\int_0^t d\tau \left\| \int_0^\tau d\tau_1 \, |A|^\beta \xi_2(H) V_{-t-\tau} \xi_1(H) \tilde{W} \xi_1(H) V_{\tau-\eta} \times \tilde{W} V_{\tau_1} \xi(H) U_t \psi(H_0) \langle Q \rangle^{-2-\eta} \right\|$$

which is

$$\leqq \int_0^t d\tau \, |t+\tau|^\beta \left\| \int_0^\tau d\tau_1 \langle Q \rangle^\beta \tilde{W} \xi_1(H) V_{\tau-\tau_1} \times \tilde{W} V_{\tau_1} \xi(H) U_t \psi(H_0) \langle Q \rangle^{-2-\eta} \right\|$$

The first integral on the R.H.S. of the above inequality looks like the expression (*a*.5) and thus the above is bounded if $\alpha - \beta < 1 + \eta$ and $2\beta - \alpha < -1$. ■

LEMMA A.12. — *Let* W *satisfy* (2.1) *and* $\xi, \psi \in C_0^\infty(0, \infty)$. *Then for each* $\mu \in [0, \alpha+2]$ *and* $\varepsilon > 0$, *there exists a constant* K *such that for all* s, $t \in \mathbb{R}$.

$$\| \langle Q \rangle^{-\mu} \xi(H) V_{t-s} U_s \psi(H_0) \langle Q \rangle^{-\mu} \| \leqq K(1+|t|)^{-\mu+\varepsilon}.$$

LEMMA A.13. — *Let* $g \in \mathcal{D}_{4+\eta}$ *for some* $\eta > 0$. *Then for any* $\eta_1 > 0$ *there exists* $0 < \eta_2 < \eta_1$ *such that*

$$\left\| \int_{-\infty}^{\infty} \tau\, d\tau \langle A \rangle^{-2-\eta_1} V_{\sigma-\tau} \xi(H) \partial_j W V_{\tau-s} \psi(H_0) U_s g \right\| \leqq K(1+|\sigma|)^{-2-\eta_2}.$$

The proofs of Lemma A.12 and A.13 are very long, though not complicated. For example in Lemma A.13, we need to compute the double commutator of A with $V_{\sigma-\tau}\xi(H)\partial_j W V_{\tau-s}\xi(H) U_s \psi(H_0)$ and it is here that we need the five times differentiability of W in (2.1). We do not give the proofs here because of their lengths and refer the reader to [15].

THEOREM A.14. — *Let* W *satisfy* (2.1) *and* S *be the scattering operator defined in Section* 2. *Then for* $g \in \mathcal{D}_{4+\eta}$ $(\eta > 0)$ *and* $\psi \in C_0^\infty(0, \infty)$ *with* $\psi(H_0) g = g$,

$$\| \langle Q \rangle^{-\alpha-1} U_\sigma \psi(H_0) [Q_j S] g \| \leqq K(1+|\sigma|)^{-2-\eta_2} \qquad \text{with} \quad \eta_2 > 0.$$

Proof. — In the expression (*a*.7), if we take limit $s \to -\infty$ and then $t \to \infty$, then all the terms in the R.H.S. except the last one converge to zero by Propositions A.1 and A.2 and Theorem A.7 (ii), while the L.H.S. converges to $[Q_j, \psi(H_0) S] g$. Thus it suffices to show that

$$\left\| \int_{-\infty}^{\infty} \tau\, d\tau \langle Q \rangle^{-\alpha-1} U_{\sigma-t} \psi(H_0) V_{t-\tau} \xi(H) \partial_j W V_{\tau-s} \psi(H) U_s g \right\| \leqq K(1+|\sigma|)^{-2-\eta_2}.$$

But this follows easily from Lemmas A.11 and A.13. ■

THEOREM A.15. — *Let* $f \in \mathcal{D}_{4+\eta}$ *for some* $\eta > 0$. *Then*

$$\int_0^\infty \| V_t \Omega_- f - T_t^+ S f \| \, dt < \infty.$$

Proof. — A simple calculation as in Chapter 13 of [7] shows that

$$\begin{aligned} V_t \Omega_- f - T_t^+ S f &= i \int_t^\infty V_{t-s} (W - W(2s\underline{P})) U_s Y_s^+ S f\, ds \\ &= i \int_t^\infty ds\, V_{t-s} U_s \int_0^1 d\rho\, U_{s/\rho}^* \{ \underline{\nabla} W(\rho \underline{Q}) U_{s/\rho} \underline{Q} Y_s^+ \\ &\qquad + is \Delta W(\rho \underline{Q}) U_{d/\rho} Y_s^+ \} S f, \end{aligned}$$

so that

$$\|V_t\Omega_- f - T_t^+ S f\| \leqq \int_t^\infty ds \int_0^1 d\rho \, \| \underline{\nabla} W(\rho \underline{Q}) U_{s/\rho} . \underline{Q} Y_s^+ S f \| + \int_t^\infty s ds \int_0^1 \| \Delta W(\rho \underline{Q}) U_{s/\rho} Y_s^+ S f \| d\rho. \quad (a.11)$$

The second integrand in $(a.11)$ is majorized by

$$\rho^{-\alpha-2} \| \langle Q \rangle^{-\alpha-2} Y_s^+ \psi(H_0) \langle Q \rangle^{\alpha+2} \| \, \| \langle Q \rangle^{-\alpha-2} U_{s/\rho} S f \|, \quad (a.12)$$

where we have chosen $\psi \in C_0^\infty(0, \infty)$ such that $f = \psi(H_0) f$. Since $S f \in D(|A|^{\alpha+2})$ by Corollary 2.2, the second norm in $(a.12)$ is bounded by $(1+s/\rho)^{-\alpha-2}$ on using the Proposition A.1. This together with Lemma 3.2 (ii) shows that the second integral is dominated by constant. $(1+t)^{-\alpha}$, which leads to the integrability in t for this part in $(a.11)$.

Similarly the first integrand in $(a.11)$ is bounded by

$$K \rho^{-\alpha-1} \sum_{j=1}^{n} \{ \| \langle Q \rangle^{-\alpha-1} Y_s^+ \partial_j X_s^+ \psi(H_0) \langle Q \rangle^{\alpha+1} \| \times \| \langle Q \rangle^{-\alpha-1} U_{s/\rho} S f \| + \| \langle Q \rangle^{-\alpha-1} Y_s^+ \psi_2(H_0) \langle Q \rangle^{\alpha+1} \| \times \| \langle Q \rangle^{-\alpha-1} U_{s/\rho} \psi_1(H) Q_j S f \| \}, \quad (a.13)$$

where we have introduced $\psi_1(H_0)$ by Proposition A.2 (vi) and chosen $\psi_2 \in C_0^\infty(0, \infty)$ such that $\psi_2 \psi_1 = \psi_1$. Since the first two factors in $(a.13)$ are uniformly bounded by Lemmas 3.2 (ii), (iv) and since

$$\| \langle Q \rangle^{-\alpha-1} U_{s/\rho} S f \| \leqq K (1 + |s/\rho|)^{-\alpha-1},$$

it only remains to show that for each j $(1 \leqq j \leqq n)$ and for $f \in \mathscr{D}_{4+\eta}$ $\| \langle Q \rangle^{-\alpha-1} U_t \psi_1(H_0) Q_j S f \| \leqq K_1 (1 + |t|)^{-\alpha-1+\varepsilon}$ for some ε small and positive. For such f, $SQ_j f$ belongs to $D(|A|^{3+\eta})$ and hence the above required estimate follows from Proposition A.1 and Theorem A.14. ■

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