ANNALES DE L'I. H. P., SECTION A

M. COMBESCURE

Recurrent versus diffusive dynamics for a kicked quantum oscillator

Annales de l'I. H. P., section A, tome 57, nº 1 (1992), p. 67-87
<http://www.numdam.org/item?id=AIHPA_1992__57_1_67_0>

Ann. Inst. Henri Poincaré,
Vol. 57, n° 1, 1992, p. 67-87. *Physique théorique*

Recurrent versus diffusive dynamics for a kicked quantum oscillator

by

M. COMBESCURE
Laboratoire de Physique Théorique et Hautes Énergies (*)
Bâtiment 211, Université de Paris-Sud, 91405 Orsay cedex, France

ABSTRACT. – We study the long time behavior of a quantum oscillator driven by sequences of kicks at integer times. These kicks have constant size, but signs alternating in an aperiodic way, distributed either randomly or along suitable "substitution sequences" such as the Thue-Morse or the Rudin-Shapiro sequences. We show a "typical" diffusive energy growth in time connected with the espace to infinity of the corresponding classical trajectories. However this growth can depend in a subtle way on the frequency of the unperturbed oscillator; in particular, suitable resonant frequencies induce a partly recurrent behavior in the quantum as well as classical evolution. In the Thue-Morse case, this recurrent behavior is explicitely manifested in the calculus of the quantum autocorrelation measure, which splits into pure point and singular continuous parts. Furthermore the singular continuous part is directly related to the singular continuous correlation measure of the Thue-Morse sequence itself. Therefore we think this simple model provides a good scenario of a "stochastic long time quantum behavior". This study is an extension of my previous work on a kicked quantum 2-level system, to a quantum system with infinitely many energy levels.

RÉSUMÉ. – On étudie le comportement asymptotique en temps d'un oscillateur quantique convenablement pulsé en temps aux instants entiers. La taille de ces pulsations est constante, mais leur signe alterne d'une

(*) Laboratoire Associé au Centre National de la Recherche Scientifique.

manière apériodique, étant distribué aléatoirement ou suivant des « suites de substitution » telles que les suites de Thue-Morse ou de Rudin-Shapiro. On montre que « typiquement » l'énergie croît en temps selon la fuite à l'infini des trajectoires classiques. Cependant cette croissance peut dépendre d'une manière subtile de la fréquence de l'oscillateur non perturbé; en particulier certaines fréquences résonantes induisent un comportement partiellement récurrent dans l'évolution quantique aussi bien que classique. Dans le cas de la suite de Thue-Morse, ce comportement récurrent est manifesté explicitement dans l'expression de la mesure d'autocorrélation quantique qui se décompose en une partie purement ponctuelle et une partie continue-singulière. De plus la partie continue-singulière est directement reliée à la mesure de corrélation continue-singulière de la suite de Thue-Morse elle-même. C'est pourquoi nous pensons qu'un tel modèle simple fournit un bon scénario d'un « comportement stochastique asymtotique en temps » pour l'évolution quantique. Cette étude est une extension de mon précédent travail sur un système quantique « à 2 niveaux » pulsé en temps, à un système quantique « à une infinité de niveaux ».

1. INTRODUCTION

The study of the long time behavior of both classical and quantum systems subjected to time-dependent Hamiltonians has received recently an increasing attention. The first reason is that a wide variety of microscopic systems governed by such Hamiltonians seems to exhibit some kind of chaotic long time behavior, namely an irregular diffusive behavior in phase space, or momentum space, and strong decorrelations in time ([1]-[5]). The second reason, which partly overlaps with the first, is the desire to investigate which kind of mechanism in both classical and quantum dynamics of these systems could be responsible for this chaotic long-time behavior. In fact while a precise mathematical definition is attached to the "classical chaos" [6], no similar notions in quantum mechanics are available to describe such irregular long-time behaviors. More seriously, a number of numerical and theoretical studies on periodically driven Hamiltonians have exhibited the so-called "quantum suppression of classical chaos", namely the quantum long-time behavior appears to be strongly recurrent although the corresponding classical dynamics is perfectly chaotic for the same parameter values ([7]-[9]). Although some particular time-periodic quantum models, studied both rigorously and numerically exhibit an irregular diffusive behavior in momentum space ([10], [30]), it is generally believed that periodically driven quantum systems are much more stable than their classical counterparts. However it is expected that some randomness in the time-dependent driving force destroys the quantum interference effects which enforce stability in the time-periodic case. This was actually exhibited by I. Guarneri [11] for a "randomly kicked quantum rotator". But pure randomness in the time dependent driving is an extreme case which can seem too wild. Halfway between the purely periodic and the purely random cases are several more or less disordered time-dependences such as

(i) the quasiperiodic case

(ii) the case of "deterministic disorder" induced by suitable substitution sequences or automata [12]

(iii) the case where the time-dependence arises from some particular stochastic process. One may then ask how the type of disorder contained in the driving term manifests itself both in the classical and quantum time evolution of the system.

The case (i) of a quasiperiodic driving was considered in several papers ([13]-[[17]). In particular Luck, Orland and Smilansky [18] (*see* also [19]) have considered a quantum two-level system perturbed quasiperiodically via the Fibonacci sequence; their approach provides analytical and numerical evidence that the quantum evolution is not quasiperiodic but exhibits some intermediate kind of behavior between quasiperiodic and random. The case (ii) of more general substitution sequences which are no longer quasiperiodic, as for example the Thue-Morse sequence, has been treated by myself in reference [20] again in the particular case of a quantum two level system. Here, the self-similarity of the substitution sequence governing the driving in time is shown to induce a quantum evolution which can be both recurrent and diffusive. Furthermore, the diffusive part of the quantum evolution is explicitely calculated, for a non-trivial (but thin) set of parameters and is shown to be directly related to the (singular continuous) correlation measure of the Thue-Morse sequence.

The general case (iii) where the time-dependent driving arises from a given stochastic process has been considered in reference [21] for the particular case of classical and quantum driven oscillators. This approach goes back to the work of Hagedorn, Loss and Slawny [22] where only the time-periodic and the purely random cases were treated. In reference [21], the following remarkable features are demonstrated:

– the classical and quantum dynamics can be treated on an equal footing

– varying the degree of randomness of the driving stochastic process provides very different dynamical responses; in particular, the asymptotic energy growth is precisely related to the ergodic properties of the driving process.

– however a diffusive energy growth is typical provided the autocorrelation function of the process decays fast enough.

In this paper, we consider classical and quantum oscillators driven by a sequence of kicks distributed along a deterministic substitution sequence like the well known Fibonacci, Thue-Morse, or Rudin-Shapiro sequences. In these simple systems, we show how the deterministic disorder of these sequences manifests itself in both the classical and the quantum long-time behavior, in particular in the quantum autocorrelation measure. Thus it is an extension of our previous study of reference [20], to a system with an infinity of energy levels.

The paper is organized as follows:

In section 2 we compute explicitly the classical and quantum time evolution of the system described by Hamiltonian (2.1, 2). This part is strongly related to a similar study performed by Bunimovich *et al.* in reference [21] where the (less singular) driving $F(t)$ originates from suitable stochastic processes. In section 3, we show the typical diffusive growth of the energy of the oscillators, for various sequences (ε_n) (either random, or deterministic substitution sequences), and we explicitly calculate the quantum autocorrelation function $C(n)$. For suitable resonant frequencies, we show that $C(n)$ splits into a purely recurrent and a purely diffusive part. In section 4, we summarize the main results of this paper in the form of concluding remarks.

2. THE CLASSICAL AND QUANTUM DYNAMICS

Given ω and $\lambda>0$, and F being a piecewise continuous real function, we consider a system governed by the following Hamiltonian

$$H(t)=H_\omega+\lambda x F(t)\equiv\frac{p^2+\omega^2 x^2}{2}+\lambda x F(t). \qquad (2.1)$$

Below, we shall extend this approach to the case of distributional F of the form

$$F(t)=\sum_{-\infty}^{+\infty}\varepsilon_n\delta(t-n) \qquad (2.2)$$

where $(\varepsilon_n)_{\mathbb{Z}}$ is a deterministic (or random) sequence taking values in $\{-1, +1\}$.

In the piecewise continuous case, it is easy to see that any classical trajectory for Hamiltonian (2.1) satisfies

$$\begin{pmatrix} x(t) \\ p(t)/\omega \end{pmatrix} = \mathrm{R}_\omega(t-s) \begin{pmatrix} x(s) + z_1(s,t)/\omega \\ p(s)/\omega + z_2(s,t)/\omega \end{pmatrix} \tag{2.3}$$

where

$$\mathrm{R}_\omega(t) = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \tag{2.4}$$

and

$$\begin{aligned} z_1(s,t) &= \lambda \int_s^t dt'\, \mathrm{F}(t') \sin \omega t' \\ z_2(s,t) &= -\lambda \int_s^t dt'\, \mathrm{F}(t') \cos \omega t' \end{aligned} \tag{2.5}$$

Therefore the trajectories in phase space are continous with piecewise continuous derivatives. We now turn to the quantum evolution problem in the Hilbert space $\mathscr{H} = \mathrm{L}^2(\mathbb{R})$. We start with simple lemmas:

LEMMA 1:

$$e^{+it\mathrm{H}_\omega} \begin{pmatrix} x \\ p/\omega \end{pmatrix} e^{-it\mathrm{H}_\omega} = \mathrm{R}_\omega(t) \begin{pmatrix} x \\ p/\omega \end{pmatrix}$$

as operators acting on the form domain of H_ω.

Proof. – This is a standard result for quantum harmonic oscillators.

LEMMA 2. – *Let* F *be a piecewise continuous real function, and let* $\varphi \in \mathscr{D}(\mathrm{H}_\omega)$. *Then the time-dependent Hamiltonian* (2.1) *is essentially self-adjoint on* $\mathscr{D}(\mathrm{H}_\omega)$ *for any* t, *and admits a unitary evolution operator* $\mathrm{U}(t, s)$ *satisfying*

(i) $\mathrm{U}(t, s)\varphi \in \mathscr{D}(\mathrm{H}_\omega)$ (*any* t *and* s)

(ii) $\mathrm{U}(t, s)\varphi$ *admits derivatives from the right and from the left for any* t, *which satisfy*

$$i\hbar \frac{d}{dt} \mathrm{U}(t,s)\varphi = \mathrm{H}(t)\mathrm{U}(t,s)\varphi$$

(iii) $\mathrm{U}(t,s)\varphi = e^{-it\mathrm{H}_\omega} e^{ixz_2(s,t) - iz_1(s,t)p/\omega} e^{is\mathrm{H}_\omega}\varphi$.

Proof. – (i) and (ii) are easy consequences of a general result of Kato [23]. To prove (iii), it is enough to note that the time-dependent Hamiltonian

$$t \to \tilde{\mathrm{H}}(t) = e^{it\mathrm{H}_\omega} \lambda x \mathrm{F}(t) e^{-it\mathrm{H}_\omega}$$

equals $\lambda F(t)\left(x \cos \omega t+\frac{p}{\omega} \sin \omega t\right)$ due to lemma 1; therefore it is essentially self-adjoint on $\mathscr{D}(H_\omega)$ and admits a unitary evolution operator

$$\tilde{U}(t, s)=\exp \left\{-i\lambda \int_s^t dt' F(t')\left(x \cos \omega t'+\frac{p}{\omega} \sin \omega t'\right)\right\}$$

satisfying statements analogous to (i) and (ii). It follows that $U(t, s)=e^{-itH_\omega}\tilde{U}(t, s)e^{isH_\omega}$ satisfies (i) and (ii), which completes the proof.

THEOREM 1. — *Let* $F(t)$ *be a piecewise continuous real function, and* z_1 *and* z_2 *be given by* (2.5). *Define*

$$\begin{pmatrix} \alpha(s, t) \\ \beta(s, t) \end{pmatrix} = R_\omega(t) \begin{pmatrix} z_1(s, t) \\ z_2(s, t) \end{pmatrix} \tag{2.6}$$

$$\gamma(s, t)=\frac{1}{2\omega} z_1(s, t)\, z_2(s, t) \cos 2\omega t + \frac{1}{4\omega}(z_2^2(s, t)-z_1^2(s, t)) \sin 2\omega t. \tag{2.7}$$

Then for any $\varphi \in \mathscr{D}(H_\omega)$ we have:

$$U(t, s)\varphi = e^{-i\gamma(s, t)} e^{i\beta(s, t)x} e^{-i\alpha(s, t)p/\omega} e^{-i(t-s)H_\omega}\varphi. \qquad 2.8)$$

The proof is elementary, and left to the reader.

We now want to show that the classical (2.3) and quantum (2.8) dynamical results extend to distributional F of the form (2.2). For such a "kicking force", the classical momenta are expected to be no longer continuous, but to have jumps at integer times. The jump of the momentum at time n equals $-\lambda\varepsilon_n$. Therefore, if x_n and p_n denote respectively the position and momentum just after the n-th kick, they obey the recurrence formula:

$$\begin{aligned} x_{n+1} &= x_n \cos\omega + \frac{p_n}{\omega} \sin\omega \\ \frac{p_{n+1}}{\omega} &= -x_n \sin\omega + \frac{p_n}{\omega} \cos\omega - \frac{\lambda\varepsilon_n}{\omega} \end{aligned} \tag{2.9}$$

whose solution is

$$\begin{pmatrix} x_n \\ \frac{p_n}{\omega} \end{pmatrix} = R_\omega(n) \begin{pmatrix} x_0 \\ \frac{p_0}{\omega} \end{pmatrix} - \frac{\lambda}{\omega} \sum_{p=1}^{n} R_\omega(n-p) \begin{pmatrix} 0 \\ \varepsilon_p \end{pmatrix} \tag{2.10}$$

But (2.10) can be rewritten as

$$R_\omega(n) \begin{pmatrix} x_0 + z_1/\omega \\ (p_0+z_2)/\omega \end{pmatrix}$$

where $z_1=\lambda \sum_1^n \varepsilon_p \sin \omega p$ and $z_2=-\lambda \sum_1^n \varepsilon_p \cos \omega p$ are nothing but equation (2.5) where $s\searrow 0$ and $t\searrow n$ (from above). Furthermore, if $n\leqq t<n+1$, the classical evolution from time n to time t is nothing but the rotation $R_\omega(t-n)$; therefore:

$$\begin{pmatrix} x(t) \\ p(t)/\omega \end{pmatrix} = R_\omega(t-n)\begin{pmatrix} x_n \\ p_n/\omega \end{pmatrix} = R_\omega(t)\begin{pmatrix} x_0+z_1/\omega \\ (p_0+z_2)/\omega \end{pmatrix} \tag{2.11}$$

because $R_\omega(t-n)\,R_\omega(n)=R_\omega(t)$. This proves that the classical trajectory in this case is still given by equations (2.3)-(2.5).

We now turn to the quantum evolution problem in the case (2.2). Using $\mathscr{C}^\infty$ approximants of the δ-pulses, equation (2.8) holds true for the quantum evolution $U_\varepsilon(t, s)$ with regularized F_ε instead of F. But it has been shown rigorously (*see* ref. [24]) that the limit $\varepsilon\searrow 0$ can be taken in (2.8) in the strong convergence sense. Since for fixed s and t, $z_i^{(\varepsilon)}(s, t)$ converges to $z_i(s, t)$ as $\varepsilon\searrow 0$, and since translations in position and in momentum space are continuous from $\mathscr{D}(H_\omega)$ to $\mathscr{H}=L^2(\mathbb{R})$, $U_\varepsilon(t, s)$ φ converges in $\mathscr{H}$ to $U(t, s)\varphi$ ($\varphi\in\mathscr{D}(H_\omega)$) as $\varepsilon\searrow 0$.

In particular,

$$\lim_{\substack{s\nearrow n\\ t\searrow n}} \begin{pmatrix} z_1(s, t) \\ z_2(s, t) \end{pmatrix} = \lambda\varepsilon_n \begin{pmatrix} \sin \omega n \\ -\cos \omega n \end{pmatrix}$$

and therefore

$$\lim_{\substack{s\nearrow n\\ t\searrow n}} \gamma(s, t)=0, \qquad \text{and} \qquad \lim_{\substack{s\nearrow n\\ t\searrow n}} \begin{pmatrix} \alpha(s, t) \\ \beta(s, t) \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda\varepsilon_n \end{pmatrix}$$

which proves that $\lim_{\substack{s\nearrow n\\ t\searrow n}} U(t, s)\varphi = e^{-i\lambda\varepsilon_n x}\varphi$ as expected. Thus we have:

THEOREM 2. — *Let* F(*t*) *be given by* (2.2), *for an arbitrary real sequence* $(\varepsilon_n)_{\mathbb{Z}}$, *and let* z_i, α, β, γ *be given by* (2.5)-(2.7). *Then for any* $\varphi\in\mathscr{D}(H_\omega)$, *equation* (2.8) *holds true for the quantum evolution.*

The Hamiltonian (2.1) being time-dependent, there is no conserved energy. However we can explicitely study how the energy of the oscillator

$$\frac{1}{2}(p_t^2+\omega^2 q_t^2)=E(t) \tag{2.12}$$

varies in time, either along the classical trajectory (2.5), or along the quantum time evolution, p_t (resp. q_t) being in the last case the Heisenberg observables $U(t, 0)^* p\, U(t, 0)$ [resp. $U(t, 0)^* U(t, 0)$] averaged over a given initial state ψ. This has been done in reference [21] in the general case where $F(t)$ is given by a stationary process with given invariant measure.

COROLLARY 3. – *Let* $E_Q(t, \psi) = \langle U(t, 0)\psi, H_\omega U(t, 0)\psi \rangle$ *be the quantum energy of the oscillator at time t (for the initial state* ψ*), Then if* $F(t)$ *is as in Theorems* 1 *or* 2 *and if* ψ *is any normalized eigenstate of* H_ω, *we have:*

$$E_Q(t, \psi) - E_Q(0, \psi) = |z(0, t)|^2/2 \equiv \Delta E(t, \omega) \tag{2.13}$$

where

$$z(s, t) = z_1(s, t) + iz_2(s, t) \tag{2.14}$$

with $z_i(s, t)$ *given by* (2.5).

Proof. – Using (2.8) and the fact that ψ is an eigenstate of H_ω [so that $x\psi$ (resp. $p\psi$) is orthogonal to ψ] it is easy to see that:

$$\|pU(t, 0)\psi\|^2 = \|pe^{i\beta(0, t)x}\psi\|^2 = |\beta(0, t)|^2 + \|p\psi\|^2$$

and similarly

$$\|xU(t, 0)\psi\|^2 = \|xe^{-i\alpha(0, t)p/\omega}\psi\|^2 = \omega^{-2}|\alpha(0, t)|^2 + \|x\psi\|^2.$$

Thus

$$E_Q(t, \psi) = E_Q(0, \psi) + \frac{1}{2}(|\alpha(0, t)|^2 + |\beta(0, t)|^2 = E_Q(0, \psi) + \frac{1}{2}|z(0, t)|^2$$

due to (2.6) and unitarity of the matrix $R_\omega(t)$.

Remark 1. – Formula (2.13) for the quantum energy can be compared to a similar calculus for the classical oscillator energy, which is, due to (2.3):

$$\frac{1}{2}|\omega x(0) + z_1(0, t)|^2 + \frac{1}{2}|p(0) + z_2(0, t)|^2 \equiv E_c(t)$$

and which in general differs from

$$E_c(0) + \frac{1}{2}|z(0, t)|^2$$

because of the cross-terms. In the quantum case, the cross-terms disappear when we choose ψ as an eigenstate of H_ω, but they do not in general. However, the long time behaviors of both $E_c(t)$ and $E_Q(t, \psi)$ are essentially the same.

We now turn to the quantum "return probability at time t" for initial state ψ, defined as:

$$d_\psi(t) = \langle \psi, U(t, 0)\psi \rangle \tag{2.15}$$

Then:

THEOREM 3. – *For any eigenstate* ψ_m *of* H_ω *belonging to the eigenvalue* $\left(m+\frac{1}{2}\right)\omega$, *we have:*

$$d_{\psi_m}(t)=\exp\left(-\frac{|z(0, t)|^2}{4\omega}\right)L_m^0(|z(0, t)|^2/2\omega)\,e^{-it\omega(m+(1/2))}$$

where L_m^0 are Laguerre polynomials [25] and $z(0, t)$ is given by (2.14).

Proof:

$$\langle\psi_m, U(t, 0)\psi_m\rangle=e^{-it\omega(m+(1/2))-i\gamma(0, t)}\langle\psi_m, e^{ix\beta(0, t)}e^{-ip\alpha(0, t)/\omega}\psi_m\rangle$$

Now the result follows from lemma 3 below, and from the fact that $\omega^{-1}\alpha(0, t)\beta(0, t)=2\gamma(0, t)$, and $\alpha^2+\beta^2=|z(0, t)|^2$.

LEMMA 3:

$$\langle\psi_m, e^{i\beta x}e^{-i\alpha p/\omega}\psi_m\rangle=e^{-(\alpha^2+\beta^2-2i\alpha\beta)/4\omega}L_m^0\left(\frac{\alpha^2+\beta^2}{2\omega}\right).$$

Proof:

$$\psi_m(x)=c_m(\omega)^{-1/2}e^{-\omega x^2/2}H_m(x\sqrt{\omega})$$

H_m being a Hermite polynomial, and $c_m(\omega)$ the normalization constant (so that the L^2 norm of ψ_m is 1). Therefore

$$\begin{aligned}
\langle\psi_m, e^{i\beta x}e^{-i\alpha p/\omega}\psi_m\rangle
&=c_m(\omega)^{-1}\int dx\, e^{i\beta x-\omega x^2/2-\omega(x-(\alpha/\omega))^2/2}\\
&\quad\times H_m(x\sqrt{\omega})\,H_m\left(\left(x-\frac{\alpha}{\omega}\right)\sqrt{\omega}\right)\\
&=c_m(\omega)^{-1}e^{-(\alpha^2+\beta^2-2i\alpha\beta)/4\omega}\\
&\quad\times\int dx\, e^{-\omega(x-((\alpha+i\beta)/2\omega))^2}H_m(x\sqrt{\omega})\,H_m\left(\left(x-\frac{\alpha}{\omega}\right)\sqrt{\omega}\right)\\
&=c_m(1)^{-1}e^{-(\alpha^2+\beta^2-2i\alpha\beta)/4\omega}\\
&\quad\times\int du\, e^{-u^2}H_m\left(u+\frac{\alpha+i\beta}{2\sqrt{\omega}}\right)H_m\left(u+\frac{-\alpha+i\beta}{2\sqrt{\omega}}\right)\\
&=e^{-(\alpha^2+\beta^2-2i\alpha\beta)/4\omega}L_m^0\left(\frac{\alpha^2+\beta^2}{2\omega}\right)
\end{aligned}$$

(*see* [25])

Therefore, the long-time behavior of $|z(0, t)|^2$ is a central tool in the study of the long-time quantum behavior. It was studied precisely in reference [21] in the case where the driving term $F(t)$ arises from a stationary process ξ_t (*i.e.* $F(t)=f(\xi_t)$), with measure $d\mu$ (for example $F(t)=\lambda \cos(\Omega t+\varphi)$ with φ a random variable uniformly distributed in $[0, 2\pi]$). Then $\langle |z(0, t)|^2 \rangle_\mu$ is related to the correlation function of the process $\langle f(\xi_t) f(\xi_0) \rangle_\mu$, and the resulting growth when $|t| \to \infty$ was studied precisely for varying degrees of randomness of the process ξ_t.

In the next section, we study the long time behavior of $|z(0, t)|^2$ in the case (2.2) where $F(t)$ is a sequence of "kicks" in time distributed along a sequence ε_n which is either

– independent identically distributed random variable

– or a deterministic binary sequence of $+1$ and -1 given by a substitution rule, like for example the Thue-Morse or Rudin-Shapiro sequence (*see* [12]).

3. RECURRENT VERSUS DIFFUSIVE DYNAMICS FOR THE KICKED SYSTEM (2.1-2)

Since the discrete kicks only occur at integer times, we shall now concentrate on $z(0, n)$ given by (2.5, 2.14) and on $U(n, 0)\psi$ for a given initial quantum state ψ, where $n \in \mathbb{Z}$. In particular we shall be interested in the following limit

$$\lim_{N\to\infty} N^{-1} \sum_{p=1}^{N} \langle U(p, 0)\psi, U(p+n, 0)\psi \rangle \tag{3.1}$$

which we denote by $C_\psi(n)$ when it can be shown to exist, and which we call the "quantum autocorrelation function" at time n. If it exists, it obviously satisfies

$$\left.\begin{array}{r} C_\psi(0)=1 \\ C_\psi(-n)=\overline{C_\psi(n)} \end{array}\right\} \quad \text{(for any normalized } \psi) \tag{3.2}$$

and it is therefore the Fourier transform of a positive measure on $\mathbb{T}$, called the "quantum autocorrelation measure":

$$C_\psi(n) = \int_0^1 e^{2i\pi\lambda n} d\sigma_\psi(\lambda) \tag{3.3}$$

Note that for the Hamiltonian (2.1-2) in the time-periodic case (*i.e.* $\varepsilon_n=1$, $\forall n$), $d\sigma_\psi(\lambda)$ is nothing but the spectral measure $d\mu_\psi$ (in the state ψ) of the Floquet operator $U(1, 0)\equiv U$. Namely in that case $U(p, 0)\psi = U^p\psi$, so that $C_\psi(n)$ exists and equals

$$C_\psi(n) = \langle \psi, U^n \psi \rangle = \int_0^1 e^{2i\pi\lambda n} d\mu_\psi(\lambda)$$

by the spectral theorem. Therefore when studying the quantum long time behavior, the analog in the case of aperiodic sequences $(\varepsilon_n)_{\mathbb{Z}}$ of the spectral study of the Floquet operator, is the study of the continuous or pure point nature of the "quantum autocorrelation measure". The first result in this direction is the following:

THEOREM 4. – Let $(\varepsilon_n)_{n\in\mathbb{Z}}$ be the Thue-Morse sequence of $+1$ and -1 defined as follows:

$$\begin{gathered}\varepsilon_0=1\\ \varepsilon_{2p}=(-1)^p\varepsilon_p\\ \varepsilon_{2p+1}=-\varepsilon_p\end{gathered}\tag{3.4}$$

and let $\varepsilon_{-n}=\varepsilon_{1-n}$ *for* $n\in\mathbb{N}$. *Assume that* ω *is such that the base-2 expansion of* $\omega/2\pi$ *is*

$$\begin{gathered}\omega/2\pi=\sum_{p\in\mathbb{Z}}\alpha_p 2^{\mathrm{P}}\\ \text{where } \alpha_p\in\{0,\ 1\} \text{ and } \alpha_p=0 \text{ for } p\leqq -n\end{gathered}\tag{3.5}$$

Then the following properties hold:

(i) $\forall r\in\mathbb{N}$, and $\forall\psi_m$ eigenstate of H_ω:

$$\mathrm{U}(r2^n,\ 0)\psi_m=\psi_m$$

(ii) The quantum autocorrelation function $p\in\mathbb{Z}\to \mathrm{C}_{\psi_m}(p)$ defined by (3.1) for any eigenstate ψ_m of H_ω exists and satisfies (3.3) with

$$\frac{d\sigma_{\psi_m}}{d\lambda}(\lambda)=\sum_{q=0}^{2^n-1} a_q\delta(\lambda-q2^{-n})+\nu(2^n\lambda)\sum_{q=-2^n+1}^{2^n-1} a'_q e^{2i\pi q\lambda}\tag{3.6}$$

$$\left.\begin{aligned}a_q\in\mathbb{R}\\ a'_{-q}=\overline{a'_q}\in\mathbb{C}\end{aligned}\right\}\tag{3.7}$$

$\nu(\lambda)$ being the weak star limit as $\mathrm{N}\to\infty$ of the Riesz product

$$2^{-\mathrm{N}}(1-\cos 2\pi\lambda)(1-\cos 4\pi\lambda)\dots(1-\cos 2^{\mathrm{N}-1}\pi\lambda)\tag{3.8}$$

Proof. – Let $z(s,\ t)$ be given by equations (2.5) and (2.14). Then for any $q\in\mathbb{N}$:

$$z(0,\ q)\overset{\text{def}}{=}\lim_{\substack{s\searrow 0\\ t\nearrow q}} z(s,\ t)=i\lambda\sum_1^q \varepsilon_p e^{i\omega p}\tag{3.9}$$

But, as a property of the Thue-Morse sequence [12],

$$\varepsilon_{q+2^{p-1}}=-\varepsilon_q,\qquad q=1,\ \dots\ 2^{p-1},\quad \forall p\in\mathbb{N}\tag{3.10}$$

so that for any $p\in\mathbb{N}$:

$$z(0,\ 2^p)=(1-e^{i\omega 2^{p-1}})\,z(0,\ 2^{p-1})\tag{3.11}$$

and therefore equals the Riesz product

$$i\lambda(1-e^{i\omega})(1-e^{2i\omega})\dots(1-e^{2^{p-1}i\omega}) \tag{3.12}$$

Since by definition of the integer n [*see* (3.5)]

$$\omega\, 2^{n-1}=0 \pmod{2\pi} \tag{3.13}$$

we conclude from (3.11) that

$$z(0, 2^n)=0 \tag{3.14}$$

Then, since z_1 (resp. z_2) is the real part (resp. imaginary part) of z, we immediately get from Lemma 2 (iii) that

$$\mathrm{U}(2^n, 0)\,\psi_m=e^{-i2^n(m+(1/2))\omega}\,\psi_m=\psi_m \tag{3.15}$$

for any eigenstate ψ_m of H_ω of eigenvalue $(m+(1/2))\,\omega$. Moreover, for any integers p and $r\in\mathbb{N}$, and any $q=1, \dots 2^p$, the Thue-Morse sequence satisfies (*see* [12]):

$$\varepsilon_{r\,2^p+q}=\varepsilon_r\,\varepsilon_q \tag{3.16}$$

so that, similarly to (3.11)

$$z(r\,2^p, (r+1)\,2^p)=(1-e^{i\omega 2^{p-1}})\,z(r\,2^p, r\,2^p+2^{p-1})$$

Therefore, due to (3.13), we conclude that

$$z(r\,2^n, (r+1)\,2^n)=0 \qquad [\text{for } n \text{ defined by } (3.5)] \tag{3.17}$$

which implies [similarly to (3.15)]

$$\mathrm{U}(r\,2^n, 0)\,\psi_m=\psi_m \qquad (\text{any } r\in\mathbb{N}) \tag{3.18}$$

This completes the proof of part (i) of Theorem 4.

The proof of part (ii) is very similar to that of Theorem 1 in reference [20]: Denote by $\mathrm{V}(t, 0)$ the evolution operator for Hamiltonian

$$\mathrm{H}'(t)=\mathrm{H}_\omega-\lambda\, x\, \mathrm{F}(t)$$

[recall that $\mathrm{U}(t, 0)$ is the evolution operator for $\mathrm{H}(t)=\mathrm{H}_\omega+\lambda\, x\, \mathrm{F}(t)$, so that the "kicks" in $\mathrm{H}'(t)$ and $\mathrm{H}(t)$ have opposite sign]. Then, due to the self-similarity of the Thue-Morse sequence:

$$\mathrm{U}(p\,2^n+q, p\,2^n)=\begin{cases}\mathrm{U}(q, 0) & \text{if } \varepsilon_p=1\\ \mathrm{V}(q, 0) & \text{if } \varepsilon_p=-1\end{cases}$$

for n defined by (3.5) and $q=0, 1, \dots 2^n-1$, and any $p\in\mathbb{N}$, which can be rewritten as

$$\mathrm{U}(p\,2^n+q, p\,2^n)=\frac{\varepsilon_p+1}{2}\,\mathrm{U}(q, 0)+\frac{1-\varepsilon_p}{2}\,\mathrm{V}(q, 0) \tag{3.19}$$

Therefore, defining

$$\begin{aligned}\Psi_m(q)&=\frac{1}{2}(\mathrm{U}(q,0)\psi_m-\mathrm{V}(q,0)\psi_m)\\ \Phi_m(q)&=\frac{1}{2}(\mathrm{U}(q,0)\psi_m+\mathrm{V}(q,0)\psi_m)\end{aligned}\tag{3.20}$$

we get (using (3.18)):

$$\mathrm{U}(p\,2^n+q,0)\psi_m=\varepsilon_p\Psi_m(q)+\Phi_m(q)\tag{3.21}$$

Using Lemma 5 of reference [20], (3.21) allows a complete calculus of the quantum autocorelation measure:

$$\begin{aligned}\mathrm{C}_{\psi_m}(p\,2^n+q)=\lim_{\mathrm{N}\to\infty}\mathrm{N}^{-1}2^{-n}\sum_{p'=0}^{\mathrm{N}-1}\\ \times\Bigg\{\sum_{q'=0}^{2^n-q-1}\langle\mathrm{U}(q'+p'\,2^n,0)\psi_m,\ \mathrm{U}(q+q'+(p+p')\,2^n,0)\psi_m\rangle\\ +\sum_{q'=2^n-q}^{2^n-1}\langle\mathrm{U}(q'+p'\,2^n,0)\psi_m,\\ \mathrm{U}(q+q'-2^n+(p+p'+1)\,2^n,0)\psi_m\rangle\Bigg\}\end{aligned}\tag{3.22}$$

The scalar products in (3.22) can be calculated using (3.21); now, using the fact that ε_p has mean zero, so that the cross-terms $\langle\Psi_m,\Phi_m\rangle$ do not contribute, we obtain:

$$\begin{aligned}\mathrm{C}_{\psi_m}(p\,2^n+q)=2^{-n}\{d(q)\,\mathrm{C}_p(\varepsilon)+e(q)\\ +\overline{d(2^n-q)}\,\mathrm{C}_{p+1}(\varepsilon)+\overline{e(2^n-q)}\}\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}d(q)&=\sum_{q'=0}^{2^n-q-1}\langle\Psi_m(q'),\Psi_m(q+q')\rangle\\ e(q)&=\sum_{q'=0}^{2^n-q-1}\langle\Phi_m(q'),\Phi_m(q+q')\rangle\\ \mathrm{C}_p(\varepsilon)&=\lim_{\mathrm{N}\to\infty}\mathrm{N}^{-1}\sum_{p'=1}^{\mathrm{N}}\varepsilon_{p'}\varepsilon_{p+p'}\end{aligned}\tag{3.24}$$

But it is known that (3.24), which is the correlation function of the Thue-Morse sequence, is the Fourier transform of $\nu(\lambda)$ (in the distributional sense), so that the result follows with

$$a'_q=d(-q)$$

and a_q solution of the system of 2^n equations of 2^n unknown variables:

$$\sum_{0}^{2^n-1} a_q e^{2i\pi q/2^n} = e(q) + \overline{e(2^n - q)}$$

(For further details, we refer to [20], Lemma 5).

Remark 2. — In reference [20], we obtained a similar result for the quantum autocorrelation measure of a two-level system with "kicks" in time modulated along the Thue-Morse sequence. The set of parameters (E, λ) for which it held was a non-trivial but thin set described in [20, section 3]. Here we get a similar result for a quantum system with an "infinite number of levels", with *no conditions on the size* λ *of the kicks*, and for all dyadic $\omega/2\pi$ (which are dense in $\mathbb{R}$). For these values of the oscillator's frequency, the quantum autocorrelation function splits into a purely recurrent and a purely diffusive part. Furthermore, the way it diffuses is via a well-known *singular continuous measure* (the Riesz measure). Therefore it seems to be a good candidate for what could be called a "chaotic long-time quantum behavior".

Before giving further results for non-dyadic frequencies $\omega/2\pi$, and for other substitution sequences (ε_n), we treat the simple case where ε_n are independent identically distributed random variables (i.i.d.r.v.) taking values $+1$ or -1 with equal probability $1/2$.

LEMMA 4. — *Let* (ε_n) *be i.i.d.r.v. distributed along a probability measure* μ, *with* $\int \varepsilon \, d\mu = 0$ *and* $\int \varepsilon^2 \, d\mu = b > 0$.

(i) *Then we have:*

$$\langle E_c(n) \rangle_\mu = E_c(0) + \lambda^2 bn/2$$

$$\langle E_Q(n, \psi) \rangle_\mu = E_Q(0, \psi) + \lambda^2 bn/2 \qquad [\text{any } \psi \in \mathcal{D}(H_\omega)]$$

i. e. *the mean classical and quantum energies of the oscillator increase linearly in time.*

(ii) *Assume further that* $d\mu = \frac{1}{2}[\delta(\varepsilon+1) + \delta(\varepsilon-1)] \, d\varepsilon$ (i. e. *that* ε_n *takes values* $+1$ *or* -1 *with equal probability* $1/2$). *Then the random variables* $z_i(0, n)/\sqrt{n}$ $(i = 1$, or $2)$ *converge in distribution to a standard gaussian variable.*

Proof. — (i) From (2.13) and remark 1, it is enough to calculate $\langle |z(0, n)|^2 \rangle_\mu$ (because all cross-terms disappear as a consequence of the parity of the probability measure μ). But

$$\lambda^{-2} \langle |z(0, n)|^2 \rangle_\mu = \left\langle \sum_{p, p'=1}^{n} \varepsilon_p \varepsilon_{p'} e^{i\omega(p-p')} \right\rangle_\mu = n \int \varepsilon^2 \, d\mu = bn$$

because $\langle \varepsilon_p \varepsilon_{p'} \rangle_\mu = \langle \varepsilon_p \rangle_\mu \langle \varepsilon_{p'} \rangle_\mu = 0$ for $p \neq p'$.

(ii) is an extension to the present situation of a general central limit theorem established in reference [21]: It is enough to show that the characteristic function

$$\langle e^{ixz_i(0,n)/\sqrt{n}} \rangle_\mu \xrightarrow[n \to \infty]{} e^{-x^2/2d} \quad \text{for} \quad i=1,2 \, (\text{some } d>0)$$

We give the proof for $i=1$ (the case $i=2$ being similar).

$$\langle e^{i\lambda x n^{-1/2} \sum_1^n \varepsilon_p \sin \omega p} \rangle_\mu = \prod_{p=1}^n \langle e^{i\lambda x n^{-1/2} \varepsilon_p \sin \omega p} \rangle_\mu$$
$$= \prod_{p=1}^n \frac{1}{2} (e^{-i\lambda x n^{-1/2} \sin \omega p} + e^{i\lambda x n^{-1/2} \sin \omega p})$$
$$= \prod_{p=1}^n \cos(\lambda x n^{-1/2} \sin \omega p)$$

Therefore

$$\operatorname{Log} \langle e^{i\lambda x n^{-1/2} \sum_1^n \varepsilon_p \sin \omega p} \rangle_\mu = \sum_{p=1}^n \operatorname{Log} \cos(\lambda x n^{-1/2} \sin \omega p)$$
$$\underset{n \to \infty}{\sim} \sum_{p=1}^n \operatorname{Log}\left(1 - \frac{\lambda^2 x^2}{2n} \sin^2 \omega p\right)$$
$$\underset{n \to \infty}{\sim} -\frac{\lambda^2 x^2}{2n} \sum_{p=1}^n \sin^2 \omega p \sim -\lambda^2 x^2/4$$

which completes the proof, with $d = 2/\lambda^2$.

We also have an "almost sure result" [with respect to the probability distribution of sequence (ε_n)] for the growth at infinity of $E_Q(n, \psi)$ and $E_c(n)$.

THEOREM 5. — *Let* (ε_n) *be i.i.d.r.v. of values* $+1$ *or* -1 *with equal probability* $1/2$. *Then we have for any quantum state* ψ *in* $\mathscr{D}(H_\omega)$, *and some positive constant* C:

$$0 \leq E_Q(n, \psi) - E_Q(0, \psi) = E_c(n) - E_c(0) \leq C \lambda^2 n \operatorname{Log} \operatorname{Log} n$$

almost surely.

Proof. — It follows easily from the well known "law of iterated logarithm" (*see* [26-28]), which in the present case implies, for any fixed ω:

$$\overline{\lim_n} \frac{\sum_1^n \sin \omega p \, \varepsilon_p}{\left(2 \sum_1^n \sin^2 \omega p \operatorname{Log} \operatorname{Log} \sum_1^n \sin^2 \omega p\right)^{1/2}} = 1$$

almost surely (and similarly for cos ωp instead of sin ωp).

Remark 3. — We can compare Theorem 5 with a similar problem treated in reference [22], namely the quantum dynamics for

$$H(t)=\frac{p^2}{2}+\lambda(t)x^2 \tag{3.25}$$

where $\lambda(t)=\omega_k$ for $k \leqq t < k+1$, (any $k \in \mathbb{Z}$) and where ω_k are i.i.d. random variables with probability distribution μ. It is shown in [22] that, under some "non-triviality condition", for any initial state ψ in Schwartz space, the following holds:

(i) $E_Q(t, \psi)$ grows exponentially in t,

(ii) $|d_\psi(t)|$ decays faster than any inverse power of t as $|t| \to \infty$, almost surely.

Therefore, the quantum dynamics for (3.25) was argued to be similar to that of states from the "transient absolutely continuous subspace" for time-independent Hamiltonians: in that sense, the quantum motion for (3.25) is said to be "almost surely non-stochastic" as $t \to \infty$.

On the contrary, in the situation of Theorem 5, the growth in energy is (almost surely) linear (up to logarithmic corrections), which is usually referred to as a "diffusive growth". Furthermore, the expression of $d_\psi(n)$ in Theorem 3 (namely the return probability to state ψ at time n) shows wild oscillations between 0 and 1 as $n \to \infty$, almost surely, again as a consequence of the "law of iterated logarithm". Thus, in deep contrast with the "almost sure non-stochasticity" of system (3.25), the random problem of Theorem 5 seems to exhibit an "almost sure stochastic quantum long-time behavior", with non trivial recurrences in time, and a diffusive growth of energy.

We now come back to the case of deterministic binary sequences (ε_n), generated for $n \geqq 0$ by a substitution rule, and completed for non-positive integers, by the rule $\varepsilon_{1-n}=\varepsilon_n$. We have seen that a central ingredient for the long-time classical or quantum dynamics is the behavior of

$$\left|\sum_1^n e^{i\omega p}\varepsilon_p\right|^2=\frac{2}{\lambda^2}\Delta E(n, \omega) \tag{3.26}$$

as n goes to infinity. However it can be a very wild limit, connected with the correlation measure of the sequence (ε_n) as we shall see. In particular, we have already seen that for ω a dyadic number, the limit (3.26) vanishes when taken along the subsequence $n_j=j2^p$ (some $p \in \mathbb{N}$ depending on ω). We shall consider the cases of the Thue-Morse sequence (in the general non dyadic case) and of the Rudin-Shapiro sequence (*see* [12]).

THEOREM 6. – *Let* (ε_n) *be, as in Theorem* 4, *the Thue-Morse sequence, and let* $\nu(x)$ *be its correlation measure defined by* (3.8). *Define* $\alpha_1>0$ *by*

$$\operatorname{Log}\alpha_1=\int_0^1 \nu(x)\,dx\,\operatorname{Log}(1-\cos 2\pi x) \qquad (3.27)$$

Then $\alpha_1 \geqq 1$, *and the following properties hold along the subsequence of times* $N_n=2^n\,(n\in\mathbb{N})$:

(i) $\dfrac{\operatorname{Log}\Delta E(N_n,\omega)}{\operatorname{Log} N_n}\to 1+\dfrac{\operatorname{Log}\alpha_1}{\operatorname{Log} 2}$ *as* $n\to\infty$ for ν *a.e.* ω

(ii) $\dfrac{\operatorname{Log}\Delta E(N_n,\omega)}{\operatorname{Log} N_n}\to 0$ *as* $n\to\infty$

almost everywhere in ω (*with respect to the Lebesgue measure*).

(iii) *Given any interval* $I\subset[0,1]$, *let* $\nu(I)$ *denote the Riesz measure of the interval* I i. e.

$$\nu(I)=\int_I \nu(x)\,dx$$

Then we have:

$$\lim_{n\to\infty} N_n^{-1}\int_I \frac{d\omega}{2\pi}\Delta E(N_n,\omega)=\nu(I).$$

Proof. – From (3.36) and (3.12), we have:

$$\Delta E(N_n,\omega)=\frac{\lambda^2}{2}\left|(1-e^{i\omega})(1-e^{2i\omega})\dots(1-e^{2^{n-1}i\omega})\right|^2$$
$$=\frac{\lambda^2}{2}2^n\prod_0^{n-1}(1-\cos 2^p\omega)=\frac{\lambda^2}{2}N_n P_n\left(\frac{\omega}{2\pi}\right) \qquad (3.28)$$

where $P_n(x)$ on $\mathbb{T}$ is defined by

$$P_n(x)=\prod_0^{n-1}(1-\cos 2^p 2\pi x)$$

Therefore

$$\frac{\operatorname{Log}\Delta E(N_n,\omega)}{\operatorname{Log} N_n}\sim 1+\frac{1}{n\operatorname{Log}2}\operatorname{Log} P_n(\omega/2\pi) \quad \text{as } n\to\infty.$$

Now (i) and (ii) are easy consequences of the Birkhoff ergodic theorem. Namely let for a parameter $t\in[0,1]$

$$\nu_t\,dx=w^*\times\lim_{N\to\infty}\prod_0^N(1-t\cos 2^p 2\pi x)\,dx$$

be a one parameter family of ergodic measures on $\mathbb{T}$ [with respect to $Tx=2x \pmod 1$)], which reduces to the Riesz measure ν (resp. the Lebesgue measure) for $t=1$ (resp. $t=0$). Therefore for any $t \in [0, 1]$

$$\frac{1}{n} \operatorname{Log} P_n(x) = \frac{1}{n} \sum_0^{n-1} \operatorname{Log}(1-\cos 2^p 2\pi x)$$

converges as $n \to \infty$ to

$$\operatorname{Log} \alpha_t = \int_0^1 \nu_t(x)\, dx \operatorname{Log}(1-\cos 2\pi x) \qquad \nu_t\text{-almost everywhere}$$

Now, it is enough to show that $\operatorname{Log} \alpha_1 \geqq 0$, and $\operatorname{Log} \alpha_0 = -\operatorname{Log} 2$.

$$\operatorname{Log} \alpha_1 = \int_0^1 dx (1-\cos 2\pi x) \operatorname{Log}(1-\cos 2\pi x) \prod_1^\infty (1-\cos 2^p 2\pi x)$$

We split the integration interval in $\int_0^{1/2} \ldots$ and $\int_{1/2}^1 \ldots$, and in the second, we make the change of variables $x' = x + 1/2$, so that $\prod_1^\infty (1-\cos 2^p 2\pi x)$ is unchanged, and $\cos 2\pi x' = -\cos 2\pi x$ so that:

$$\operatorname{Log} \alpha_1 = \int_0^{1/2} dx \prod_1^\infty (1-\cos 2^p 2\pi x) \{ (1-\cos 2\pi x) \operatorname{Log}(1-\cos 2\pi x) + (1+\cos 2\pi x) \operatorname{Log}(1+2\pi x) \}$$

Now it is easy to see that the integrand is non-negative, and therefore $\alpha_1 \geqq 1$. Now consider α_0:

$$\operatorname{Log} \alpha_0 = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Log}(1-\cos x)\, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{Log} 2 + 2 \operatorname{Log} \sin x/2)\, dx = \operatorname{Log} 2 + 2\mathrm{I}$$

where

$$\mathrm{I} = \pi^{-1} \int_0^\pi \operatorname{Log} \sin u\, du = \operatorname{Log} 2 + \pi^{-1} \int_0^\pi du \left(\operatorname{Log} \sin \frac{u}{2} + \operatorname{Log} \cos \frac{u}{2} \right)$$

$$= \operatorname{Log} 2 + \frac{2}{\pi} \int_0^{\pi/2} (\operatorname{Log} \sin y + \operatorname{Log} \cos y)\, dy$$

But, letting $y' = y + \frac{\pi}{2}$, we see that

$$\int_0^{\pi/2} \operatorname{Log} \cos y\, dy = \int_{\pi/2}^{\pi} dy' \operatorname{Log} \sin y'$$

and therefore

$$I = \mathrm{Log}\,2 + 2I \quad \Rightarrow \quad I = -\mathrm{Log}\,2 \quad \Rightarrow \quad \mathrm{Log}\,\alpha_0 = -\mathrm{Log}\,2.$$

The proof of (iii) is an immediate consequence of (3.28), and of the fact that the measure $\nu(x) = w^* \lim_{N \to \infty} P_N(x)$ has no point mass.

Remark 4. – (i) The constant α_1 can be calculated numerically and shown to be strictly above 1. Therefore Theorem 6 (i) shows that, along the sequence N_n of times, the energy increase is like N_n^γ with $\gamma = 1 + \frac{\mathrm{Log}\,\alpha_1}{\mathrm{Log}\,2} > 1$, at least for *almost every* ω *with respect to the Riesz measure.*

(ii) However, part (ii) of Theorem 6 just shows that, along the same subsequence, the energy increase is below linear, *for almost every* ω *with respect to the Lebesgue measure.* And of course we already know that $\Delta E(N_n, \omega)$ is exactly zero for all dyadic $\omega/2\pi$. Furthermore we recover the exact linear increase [Theorem 6 (iii)] when integrating over an interval of values of ω. This is not surprising since $\Delta E(N_n, \omega)$ has a very weird dependence on ω.

We now turn to the Rudin-Shapiro sequence $(r_n)_{n \in \mathbb{N}}$ defined by

$$\begin{aligned} r_0 &= 1 \\ r_{2n} &= r_n \\ r_{2n+1} &= (-1)^n r_n \end{aligned} \tag{3.29}$$

and we assume $\varepsilon_n = r_n$, and $\varepsilon_{-n} = \varepsilon_{1+n}$ $(n \in \mathbb{N})$ in order to have a biinfinite sequence. Whereas the correlation measure of the Thue-Morse sequence has been identified with the singular continuous Riesz measure, the correlation measure of the Rudin-Shapiro sequence is nothing but the Lebesgue measure (*see* [12]). This property makes the Rudin-Shapiro sequence, in some sense, closer to the purely random case already considered. Again we are interested in the quantum dynamical response for Hamiltonian (2.1-2), in particular in terms of the long-time energy growth. The result is as follows:

THEOREM 7. – *Let* ε_n *be the Rudin-Shapiro sequence* (3.29) *for* $n \in \mathbb{N}$, *and* $\varepsilon_{-n} = \varepsilon_{1+n}$. *Then the energy growth* $\Delta E(t, \omega)$ *defined by* (2.13) *satisfies*

(i) $\sup_\omega \Delta E(n, \omega) \leq 6n$ *any* $n \in \mathbb{N}$,

(ii) *Along the subsequence* $N_n = 2^n$ $(n \in \mathbb{N})$ *of times, we have, for any interval* I *of Lebesgue length* $|I|$:

$$N_n^{-1} \int_I d\omega\, \Delta E(N_n, \omega) \to |I| \quad \textit{as } n \to \infty.$$

Proof. – (i) follows from an estimate obtained by Saffari ([31]): $\operatorname{Sup}_{\omega}\left|\sum_{0}^{n-1} e^{i\omega p}\varepsilon_p\right| \leqq \sqrt{6n}$ for ε_n the n-th term of the Rudin-Shapiro sequence.

(ii) is the analog, for the Rudin-Shapiro sequence, of the corresponding result of Theorem 6, (iii) for the Thue-Morse sequence. It is a simple consequence of the fact that the correlation measure of the Rudin-Shapiro sequence is the Lebesgue measure on $\mathbb{T}$ (*see* [12]).

4. CONCLUDING REMARKS

We have seen that perturbing a free oscillator of frequency ω by aperiodic sequences of kicks in time with alternating signs produces a quantum increase of energy exactly connected with the escape to infinity of the classical trajectories in phase space.

In the purely random case (size of kicks are i.i.d. random variables), this escape to infinity occurs via a "random walk like" process which ensures, almost surely, a diffusive linear growth of energy. Furthermore, the "return probability at time n" for quantum states wildly oscillates when n becomes large (again almost surely) and therefore the quantum long time behavior appears to be "almost surely stochastic".

The story seems to be much more involved in the case of deterministic aperiodic sequences of kicks generated by a substitution rule (or automaton). In this paper, we have considered the case of two substitution sequences known as the Thue-Morse, and the Rudin-Shapiro sequences, but the study seems to be generalizable to many other substitution sequences, because there is already a large mathematical knowledge on them. The long time quantum behavior is shown to depend weirdly on the frequency ω of the unperturbed oscillator. However, when integrating over any interval of frequencies, we recover the linear diffusive energy growth in time, at least over suitable subsequences of integer times. In the Thue-Morse case, exploiting the self-similarity of the sequence, and the structure of Riesz polynomials, we show that for all dyadic frequencies $\omega/2\pi$, the quantum autocorrelation measure is an exact sum of a purely recurrent term, and a purely diffusive term connected with the (singular continuous) correlation measure of the sequence itself. But given an arbitrary non-dyadic frequency, the energy growth can be anything between 0 and t^{γ}, where γ is strictly above 1. We think that such aperiodic substitution sequences provide a good scenario of what could be called a "stochastic quantum long time behavior".

ACKNOWLEDGEMENTS

I acknowledge useful discussions, at various stages of this work, with J. P. Allouche, A. Bovier, H. Jauslin. M. Mendès France, J. Peyrière, M. Queffelec and B. Saffari, who, in particular, have provided to me a number of useful references.

REFERENCES

[1] J. E. Bayfield and P. M. Koch, *Phys. Rev. Lett.*, Vol. **33**, 1974, p. 258.
[2] J. G. Leopold and I. C. Percival, *Phys. Rev. Lett.*, Vol. **41**, 1978, p. 944.
[3] G. Casati, B. V. Chirikov and D. L. Shepelyanski, *Phys. Rev. Lett.*, Vol. **53**, 1984, p. 2525.
[4] R. Jensen, *Phys. Rev. Lett.*, Vol. **49**, 1982, p. 1365.
[5] S. Washburn and R. A. Webb, *Adv. Phys.*, Vol. **35**, 1986, p. 375.
[6] J. Moser, Stable and random motions in dynamical systems, *Annals of Math. Studies*, Princeton Univ. Press, Princeton, 1973.
[7] G. Casati, B. V. Chirikov, D. L. Shepelyanski and I. Guarneri, *Phys. Rep.*, Vol. **154**, 1987, p. 77.
[8] G. Casati and L. Molinari, *Progr. Theo. Phys. Suppl.*, Vol. **98**, 1989, p. 287.
[9] S. Fishman, D. R. Grempel and R. E. Prange, *Phys. Rev. Lett.*, Vol. **49**, 1982, p. 509.
[10] M. Combescure, *J. Stat. Phys.*, Vol. **59**, 1990, p. 679.
[11] I. Guarneri, *Lett. Nuovo Cimento*, Vol. **40**, 1984, p. 171.
[12] M. Queffelec, *Substitution Dynamical Systems. Spectral Analysis*, Springer, Berlin, 1987.
[13] B. Dorizzi, B. Grammaticos and Y. Pomeau, *J. Stat. Phys.*, Vol. **37**, 1984, p. 93.
[14] T. Hogg and B. A. Huberman, *Phys. Rev. Lett.*, Vol. **48**, 1982, p. 711.
[15] P. W. Milonni, J. R. Ackerhalt and M. E. Goggin, *Phys. Rev. A*, Vol. **35**, 1987, p. 1714.
[16] Y. Pomeau, B. Dorizzi and B. Grammaticos, *Phys. Rev. Lett.*, Vol. **56**, 1986, p. 681.
[17] K. Rzazewski and J. Mostowski, *Phys. Rev. A*, Vol. **35**, 1987, p. 4414.
[18] J. M. Luck, H. Orland and U. Smilansky, *J. Stat. Phys.*, Vol. **53**, 1988, p. 551.
[19] B. Sutherland, *Phys. Rev. Lett.*, Vol. **57**, 1986, p. 770.
[20] M. Combescure, *J. Stat. Phys.*, Vol. **62**, 1991, issue 3-4.
[21] L. Bunimovich, H. R. Jauslin, J. J. Lebowitz, A. Pellegrinotti and P. Nielaba, *J. Stat. Phys.*, 1991.
[22] G. A. Hagedorn, M. Loss and J. Slawny, *J. Phys. A*, Vol. **19**, 1986, p. 521.
[23] T. Kato, *J. Fac. Sci. Univ. Tokyo*, Sect. I, Vol. **17**, 1970, p. 241.
[24] J. Piepenbrink, *Pulsed Quantum Mechanical Hamiltonians*, West Georgia College preprint, 1990.
[25] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press Inc., 1965.
[26] Y. S. Chow and H. Teicher, *Probability Theory*, Springer, Berlin-Heidelberg-New York, 1978.
[27] W. Stout, *Almost Sure Convergence*, Academic Press, New York, 1974.
[28] A. N. Shiryayev, *Probability*, Springer, Berlin-Heidelberg-New York, 1984.
[29] R. Blümel, R. Meir and U. Smilansky, *Phys. Lett.*, Vol. **103 A**, 1984, p. 353.
[30] B. Milek and P. Seba, *Phys. Rev. A*, Vol. **42**, 1990, pp. 3213-3220.
[31] B. Saffari, *C. R. Acad. Sci. Paris*, T. **304**, 1987, pp. 127-130.
[32] B. Saffari, *Barker Sequences and Littlewood's "Two-Sided Conjectures" on Polynomials with* ± 1 *Coefficients*, Orsay review article, 1990.

(*Manuscript received march* 12, 1991.)