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## **Time-frequency representations: wavelet packets and optimal decomposition**

by

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**ABSTRACT.** — We describe the construction of coherent states systems that do not generically come from a square integrable group representation. This property allows the construction of time-frequency representation theorems associated with arbitrary partitions of the Fourier space. As examples, we describe coherent states structures that interpolate between wavelets and Gabor functions, and others that have a wavelet behaviour at high frequencies, and a Gabor behaviour at low frequencies. A continuous analogue of the Coifman-Meyer-Wickerhauser minimal entropy criterion is proposed to select the optimal decomposition for a given analysed function.

*Key words :* Time-frequency analysis, wavelet packets, information entropy, signal analysis.

**RÉSUMÉ.** — Nous décrivons la construction de systèmes d'états cohérents qui ne sont pas directement engendrés par une représentation de carré intégrable d'un groupe localement compact. Cette propriété permet la construction de théorèmes de représentation temps-fréquence associés à des partitions arbitraires de l'espace de Fourier. Comme illustrations, nous décrivons des systèmes d'états cohérents qui interpolent entre les ondelettes et les fonctions de Gabor, ainsi que d'autres se comportant comme des ondelettes à hautes fréquences et comme des fonctions de Gabor à basses fréquences. Nous proposons en outre un analogue continu du critère

d'entropie minimale de Coifman, Meyer et Wickerhauser pour sélectionner la décomposition la mieux adaptée à un signal donné à analyser.

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## I. INTRODUCTION

Recent developments in harmonic analysis, mathematical physics, numerical analysis or signal processing have pointed out the importance of time-frequency decompositions like wavelet or Gabor representations (*see e.g.* [Com-Gr-Tc] for a survey of some applications). Although wavelets and Gabor functions have been known for a long time under different names, it is only since the independent works of A. Grossmann and Y. Meyer and collaborators that they have been recognized and used systematically as basic objects for investigating and solving explicit problems. We will work here in the framework of continuous time-frequency analysis, which basically allows to decompose functions into elementary “wavelets”, assumed to be “well localised” in both direct and Fourier spaces, and indexed by continuous parameters, like usual coherent states [Kl-Sk], [Pe].

In [Gr-Mo-Pa 1] and [Gr-Mo-Pa 2] (*see also* [Fe-Grö], [Sc]), it was shown that continuous wavelet decompositions admit a beautiful interpretation in terms of square integrable group representations, as well as sliding window Fourier analysis (that we will call here Gabor analysis), the only difference between these two methods being a different starting group (the affine group for wavelet analysis, and the Weyl-Heisenberg group for Gabor analysis). The square integrable group representation approach was generalized later to representations which are square integrable with respect to an homogeneous space [Al-An-Ga] rather than to the whole group, and applied to various different groups, like for instance the Poincaré group [Al-An-Ga], [Bo].

At the same time, Meyer and his collaborators (*see e.g.* [Me 1], [Da 2]) developed the theory of orthonormal bases of wavelets, showing that it is possible to build orthonormal bases of  $L^2(\mathbb{R})$  (and many other functional spaces) with functions generated from a unique one (the mother wavelet) by dilations and translations. Collaborations between mathematicians, image processers and electrical engineers led to the discovery of a deep algorithmic structure, called the multiresolution analysis, in which the orthonormal bases of wavelets naturally appeared. This led to a deep

understanding of the connexion with the sub-band coding, a coding scheme familiar to electrical engineers (*see* [Coh]).

Recently, motivated by speech compression problems, Coifman, Meyer and Wickerhauser [Co-Me], [Co-Wi] proposed a generalisation of the wavelet construction, showing that any multiresolution analysis generates a “library of functions”, called the wavelet packets, from which one can extract a countable infinity of orthonormal bases of  $L^2(\mathbb{R})$ ; they moreover proposed a minimal entropy criterion to determine the best adapted basis to a given function to analyse (*see e. g.* [Wi 1]).

It was then natural to look for a continuous (or “coherent states”) analogue of such wavelet packets. An attempt was made in [d’A-Bea], in which the authors proposed a time-frequency analysis modeling the human hearing; nevertheless, they did not derive an exact resolution of the identity, the quality of the signal reconstruction from the time-frequency coefficients being evaluated in terms of purely audiophonic criteria. Moreover, they proposed only one analysis scheme. The same problem was also studied by M. Vetterli [Ve], purely in terms of signal processing, by means of a “three indices transform”, also proposed in [To 1].

Following the original approach of [Gr-Mo-Pa 1], and although no group action obviously appears in Coifman-Meyer’s construction, a natural starting point is the so-called affine Weyl-Heisenberg group  $G_{\text{aWH}}$ , generated by time and frequency translations, and dilations. It was shown in [To 1] that the canonical representation of  $G_{\text{aWH}}$  on  $L^2(\mathbb{R})$  is not square-integrable, but can be made square-integrable with respect to some particular homogeneous spaces over  $G_{\text{aWH}}$ . The main result was nevertheless that all the corresponding resolutions of the identity were trivial modifications of the Gabor and wavelet ones, essentially by frequency translations of the analysed and analyzing functions.

We here address the same problem, but relax the group representation assumption, that is to say that the functions we consider (continuous wavelet packets) are no longer generated by a group action, but by some “deformed” group action. This does not lead directly to a resolution of the identity, but rather to a resolution of a bounded positive operator, with bounded inverse (a continuous frame in the terminology of [Al-An-Ga]), so that the reconstruction is possible. We then derive a continuous infinity of representation theorems (theorem 2), indexed by a scale function, subject to a very weak admissibility condition (proposition 1). They basically have the same structure that the wavelets or Gabor ones, in the sense that they express an arbitrary function  $s \in L^2(\mathbb{R})$  as a sum of elementary functions  $g_\lambda$  ( $\lambda$  belonging to some measurable space), the coefficients of the decomposition being the scalar products  $\langle s, g_\lambda \rangle$ . The coefficients moreover satisfy a reproducing kernel equation (proposition 3). As examples, we exhibit coherent states structures that interpolate between

wavelets and Gabor functions (interpolating wavelet packets), and others that have a wavelet behaviour at high frequencies, and a Gabor behaviour at low frequencies (composite wavelet packets), similar to those of [d'A-Bea]. Moreover, one canonically associates to a given analysed function and a given family of coherent states a probability measure, and then an entropy. As in the Coifman-Meyer-Wickerhauser construction, this gives a way of defining the optimal representation of a function as the one that minimizes the entropy.

The paper is organised as follows: In section II, we briefly recall the basics of the group theoretical approach, focusing on the wavelet and Gabor cases, and the results of [To 1] on wavelets on homogeneous spaces. In section III, we present the generic construction of continuous wavelet packets and introduce the associated entropy. We illustrate the construction by examples in section IV. Section V is devoted to conclusions.

## II. THE GROUP THEORETICAL APPROACH

In [Gr-Mo-Pa 1] and [Gr-Mo-Pa 2], wavelets were identified as coming from the canonical square integrable representation of the so-called affine group (or “ $ax+b$ ” group). Let us briefly recall here the basics of the construction, together with the (similar) construction of Gabor functions.

### II-1. Wavelet analysis

Consider first the affine group  $\mathbf{G}_{\text{aff}}$ , generated by translations and dilations. Throughout this paper, we will consider positive and negative dilations, so that the affine group is topologically isomorphic to  $\mathbb{R} \times \mathbb{R}^*$  (in [Gr-Mo-Pa 2] and [Gr-Mo], the authors restricted to positive dilations, so that the group  $\mathbf{G}_{\text{aff}}^0$  they considered was isomorphic to  $\mathbb{R} \times \mathbb{R}_+^*$ ; the differences are minor, and we will focus on them when necessary). Let

$$d\mu_{\text{aff}}(b, a) = \frac{da db}{a^2} \quad (\text{II-1})$$

be the left-invariant measure on  $\mathbf{G}_{\text{aff}}$ . Then the continuous irreducible unitary representation  $\pi$  of  $\mathbf{G}_{\text{aff}}$  on  $L^2(\mathbb{R})$ , defined by:

$$[\pi(b, a) \cdot f](t) = \frac{1}{\sqrt{|a|}} f\left(\frac{t-b}{a}\right) \quad (b, a) \in \mathbf{G}_{\text{aff}}, \quad f \in L^2(\mathbb{R}) \quad (\text{II-2})$$

is square-integrable, i. e. there exists a  $g \in L^2(\mathbb{R})$ , called *admissible vector*, such that:

$$0 < \int_{\mathbf{G}_{\text{aff}}} |\langle \pi(b, a) \cdot g, g \rangle|^2 d\mu_{\text{aff}}(b, a) < \infty. \tag{II-3}$$

By general results (see e. g. [Ca], [Du-Mo], [Gr-Mo-Pa 1]), this in turn implies that one has the following resolution of the identity:

$$\int_{\mathbf{G}_{\text{aff}}} |\pi(b, a) \cdot g \rangle \langle \pi(b, a) \cdot g| d\mu_{\text{aff}}(b, a) = c_g \mathbf{1} \tag{II-4}$$

(where the integral converges strongly); otherwise stated, setting:

$$g_{(b, a)} = \pi(b, a) \cdot g \quad (b, a) \in \mathbf{G}_{\text{aff}} \tag{II-5}$$

every  $s \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  can be decomposed as follows:

$$s = \frac{1}{c_g} \int_{\mathbf{G}_{\text{aff}}} T_s(b, a) g_{(b, a)} d\mu_{\text{aff}}(b, a) \tag{II-6}$$

where the equality holds in the  $L^2(\mathbb{R})$  sense, the admissibility constant  $c_g$  being given by:

$$c_g = \int_{\mathbb{R}} |\hat{g}(\xi)|^2 \frac{d\xi}{|\xi|} \tag{II-7}$$

and the  $T_s(b, a)$  coefficients, called the wavelet coefficients read:

$$T_s(b, a) = \langle s, g_{(b, a)} \rangle = \int_{\mathbb{R}} s(t) g_{(b, a)}(t)^* dt. \tag{II-8}$$

They satisfy an integral equation, called the reproducing kernel equation:

$$T_s = \mathcal{K} \cdot T_s \tag{II-9}$$

$\mathcal{K}$  being a kernel operator, with kernel:

$$\mathcal{K}(b, a; b', a') = \frac{1}{c_g} \langle g_{(b', a')}, g_{(b, a)} \rangle. \tag{II-10}$$

*Remark.* — The same analysis can be carried out starting from the group  $\mathbf{G}_{\text{aff}}^0$ ; the only difference is that in that case, the  $a$  parameter runs over the positive half-line, as the integration variable in equation (II-7). The irreducibility constraint on  $\pi$  also implies that the analysis is restricted to the second real Hardy space  $H^2(\mathbb{R})$ .

### II-2. Gabor analysis

We use exactly the same scheme than in the wavelet case, but start from the Weyl-Heisenberg group  $\mathbf{G}_{\text{WH}}$ , i. e. the group generated by time

and frequency translations. It is worth noticing that the usual Weyl-Heisenberg group is a 3-parameters group, but that the third dimension is actually a central extension, which is irrelevant in the coherent states approach. We will then only consider the action of the time and frequency translation parameters.

Let  $g \in L^2(\mathbb{R})$ , and introduce the following family of functions:

$$g_{(b, \omega)}(t) = e^{i\omega t} g(t-b) \quad (b, \omega) \in \mathbb{R}^2 \quad (\text{II-11})$$

that we will call the Gabor functions (actually, the functions Gabor originally used [Ga] were Gaussians, but we keep the name of Gabor functions for simplicity). Such functions are nothing but the orbit of the canonical representation of the Weyl-Heisenberg group  $\mathbf{G}_{\text{WH}}$  through  $g$ . By general results, one then has the following resolution of the identity: any  $s \in L^2(\mathbb{R})$  can be decomposed as follows:

$$s = \int_{\mathbf{G}_{\text{WH}}} G_s(b, \omega) g_{(b, \omega)} d\mu_{\text{WH}}(b, \omega) \quad (\text{II-2})$$

where the equality holds in the  $L^2(\mathbb{R})$  sense, the measure  $d\mu_{\text{WH}}(b, \omega)$  is the left-invariant Haar measure on  $\mathbf{G}_{\text{WH}}$ :

$$d\mu_{\text{WH}}(b, \omega) = db d\omega \quad (\text{II-13})$$

and the  $G_s(b, \omega)$  coefficients, called the Gabor coefficients, are given by:

$$G_s(b, \omega) = \langle s, g_{(b, \omega)} \rangle = \int_{\mathbb{R}} s(t) g_{(b, \omega)}(t)^* dt \quad (\text{II-14})$$

and also satisfy a reproducing kernel equation similar to (II-9).

### II-3. The affine Weyl-Heisenberg group

The affine Weyl-Heisenberg group  $\mathbf{G}_{\text{aWH}}$ , discussed in [To 1], is nothing but the semi direct product of the Weyl Heisenberg group by the the real line (without the origin)  $\mathbb{R}^*$ , the new parameters being interpreted as dilations. Let  $\pi$  be the canonical representation of  $\mathbf{G}_{\text{aWH}}$  on  $L^2(\mathbb{R})$ , defined by:

$$[\pi(b, \omega, a).f](t) = \frac{1}{\sqrt{|a|}} e^{i\omega t} f\left(\frac{t-b}{a}\right) \quad (\text{II-5})$$

$\pi$  is clearly a projective representation of  $\mathbf{G}_{\text{aWH}}$ , but the projective term does not play any role in the construction.

It is not difficult [To 1] to see that  $\pi$  is not square integrable, but can be made square integrable when restricted to an homogeneous space  $\mathbf{X}$ , quotient of  $\mathbf{G}_{\text{aWH}}$  by a one-parameter subgroup, generated by elements of

the form:

$$\left( 0, \frac{1}{\lambda} \left( \frac{1}{a} - 1 \right), a \right) \quad a \in \mathbb{R}^*$$

$G_{aWH}$  canonically inherits a structure of principal bundle, and one has to imbed the homogeneous space into the whole, group, by means of a cross-section

$$\beta : X \rightarrow G_{aWH}, \tag{II-16}$$

the restriction of  $\pi$  to  $X$  being defined by:

$$\pi_X(x) = \pi[\beta(x)]. \tag{II-17}$$

As a result, we showed that the only possible piecewise differentiable sections compatible with the group action are generically given by:

$$\beta(b, \omega) = \frac{1}{\sigma\omega + \tau}, \tag{II-18}$$

leading to wavelet-type ( $\sigma \neq 0$ ) or Gabor-type ( $\sigma = 0$ ) representations. Otherwise stated, the representations of the affine Weyl-Heisenberg group, square-integrable when restricted to the considered homogeneous spaces, do not allow to interpolate between wavelet and Gabor analysis.

### II-4. Remarks and generalisations

#### (a) Decoupling the analysing and reconstructing functions

We only described here the case where the same functions are used for the analysis and the reconstruction. Such an assumption is not necessary at all, and one can start from two functions  $g$  and  $h$ , and use the wavelets  $g_{(b,a)}$  for the computation of the coefficients, and the wavelets  $h_{(b,a)}$  for the reconstruction (and the same for Gabor analysis). The only differences lie in a different admissibility constant:

$$c_{g,h} = \int \frac{\hat{g}(\xi)^* \hat{h}(\xi)}{|\xi|} d\xi \tag{II-19}$$

that must be finite and nonzero, and a different reproducing kernel. Note that the same can be done in the Gabor case. This decoupling is crucial for signal processing, where one can use a “ $\delta$ -function” for the reconstruction, leading to reconstruction formulas indexed by a one-dimensional parameter. The decoupling of the analysing and reconstructing wavelets also played an important role in the analysis of singularities (see e. g. [Ho-Tc]).

(b) *n*-dimensional wavelets

In the *n*-dimensional case, the canonical representation of the *n*-dimensional affine group on  $L^2(\mathbb{R}^n)$  is highly reducible, but one can introduce rotation degrees of freedom to recover the irreducibility of the representation. This was done in [Mu], where the *n*-dimensional affine group  $\mathbf{G}_{\text{aff}}^0$  (*n* translation parameters and one positive dilation parameter) was extended by the *n*-dimensional rotation group  $\text{SO}(n)$ . This allows to introduce the notion of angular selectivity into wavelet analysis, since it was then possible to choose an analyzing wavelet whose Fourier transform is compactly supported in a cone in  $\mathbb{R}^n$ .

## III. WAVELET PACKETS

We now remove the assumption that all analyzing functions are directly generated by a group action. Start from a basic function  $g \in L^2(\mathbb{R})$ , and consider a piecewise differentiable section  $\beta$  of the principal bundle  $\mathbf{G}_{\text{aWH}}$ , such that the subset of the real line in which  $\beta$  vanishes has Lebesgue measure zero. One first needs to introduce the notion of admissibility of  $\beta$ , which will be used in III-2.

## III-1. Admissible section

For all  $\xi \in \mathbb{R}$ , denote by  $u_\xi$  the piecewise differentiable function:

$$u_\xi(\omega) = (\xi - \omega) \beta(\omega) \quad (\text{III-1})$$

and let  $J_\xi$  be the Jacobian of  $u_\xi$ . Introduce the function  $\chi$ , called the *filter* of the representation, defined by its Fourier transform (the *transfer function* of the filter):

$$\hat{\chi}(\xi) = \int_{\mathbb{R}} |\hat{g}(u_\xi(\omega))|^2 |J_\xi| d\omega. \quad (\text{III-2})$$

We will denote by  $\mathcal{C}_\beta$  the corresponding convolution operator:

$$\mathcal{C}_\beta \cdot f = f \star \chi. \quad (\text{III-3})$$

The section  $\beta$  is said to be *admissible* (with respect to the function  $g$ ) if there exist two positive numbers  $K_1$  and  $K_2$  such that for almost all  $\xi \in \mathbb{R}$ :

$$0 < K_1 \leq \hat{\chi}(\xi) \leq K_2 < \infty \quad (\text{III-4})$$

*i. e.*  $\mathcal{C}_\beta$  if is a positive bounded operator, with bounded inverse.

Let  $\{\Xi_k, k \in \Lambda\}$  be a dense partition of the real line into disjoint connected open intervals such that  $u_\xi$  is one-to-one on each  $\Xi_k$ , and not on any  $\Xi_k \cup \Xi_l, l \neq k$ , with  $\overline{\Xi_k} \cup \overline{\Xi_l}$  connected. Clearly, if  $\Lambda$  is a finite set, the filter of the representation satisfies the bound:

$$\hat{\chi}(\xi) \leq 2\pi \|g\|^2 \text{Card}(\Lambda)$$

for all  $\xi$ , so that for a reasonable  $g$  [i.e.  $g$  is such that  $\hat{g}(u_\xi(\omega)) \neq 0$  on a set with non zero measure for almost all  $\xi$ ], the section is admissible. This condition is however not necessary, and one has the finer necessary and sufficient condition, which essentially states that  $\beta$  is not "too much oscillating":

PROPOSITION 1. —  $\beta$  is admissible if and only if the following conditions are satisfied:

(i) For almost all  $\xi \in \mathbb{R}$ , one has:

$$\int_{u_\xi(\mathbb{R})} |\hat{g}(u)|^2 du \neq 0.$$

(ii) For almost all  $\xi \in \mathbb{R}$ , for any  $u \in u_\xi(\mathbb{R})$ , and any neighborhood  $O(u)$  of  $u$  such that  $\int_{O(u)} |\hat{g}(v)|^2 dv \neq 0, u_\xi(\Xi_k) \supset O(u)$  only for a finite number of intervals  $\Xi_k$ .

Proof. — For any interval  $\Omega$  in  $\mathbb{R}$ , set:

$$I_g(\Omega) = \int_{\Omega} |\hat{g}(\xi)|^2 d\xi. \tag{III-5}$$

Notice that  $I_g(\Omega) \geq 0$  for all  $\Omega$ . Condition (i) is clearly necessary and sufficient to ensure the left hand side of the bound (III-4), yielding the following estimate:

$$\hat{\chi}(\xi) \geq \text{Max}_{k \in \Lambda} I_g(\Xi_k). \tag{III-6}$$

We now focus on (ii). Let us assume that (ii) does not hold for some  $\xi \in \mathbb{R}$ , and for some  $O(u)$ . Then  $\hat{\chi}(\xi)$  is a countable sum of positive terms of the form  $I_g(\Omega)$ , and more precisely is the sum of an infinity of times  $I_g(O(u))$ , and a positive number, so that the right hand inequality of (III-4) is not satisfied.

Conversely, let  $u_\xi(\Xi_k) = \bigcup_n a_k^n$  be a partition of  $u_\xi(\Xi_k)$  into disjoint connected intervals, such that  $a_k^n \cap a_l^m = \emptyset$  if  $a_k^n \neq a_l^m$ . Let then  $m(k, n)$  denote

the number of intervals  $\Xi_l$  such that  $\Xi_l \supset a_k^n$ , and set:

$$\left. \begin{aligned} M &= \text{Max}_{k, n} m(k, n) \\ x &= \text{Sup}_{\omega \in \mathbb{R}} u_{\xi}(\omega) \\ y &= \text{Inf}_{\omega \in \mathbb{R}} u_{\xi}(\omega). \end{aligned} \right\} \quad \text{(III-7)}$$

Then M is finite by assumption, and:

$$\widehat{\chi}(\xi) \leq M I_g(|y, x|) \leq 2 \pi M \|g\|^2 < \infty \quad \text{(III-8)}$$

and the proposition is proved.

### III-2. The wavelet packets

Start from the functions  $g, \beta$  introduced previously, and build the following family of functions:

$$g_{(b, \omega)}(t) = \frac{1}{\sqrt{|\beta(\omega)|}} e^{i\omega t} g\left(\frac{t-b}{\beta(\omega)}\right), \quad (b, \omega) \in \mathbb{R}^2 \quad \text{(III-9)}$$

whose Fourier transform are given by:

$$\widehat{g_{(b, \omega)}}(\xi) = \sqrt{|\beta(\omega)|} e^{-i(\xi-\omega)b} \widehat{g}[u_{\xi}(\omega)]. \quad \text{(III-10)}$$

Such functions are simply generated from  $g$  by time-frequency translations, and dilation by a function of the frequency translation. Such functions were already described in [To 1]. Introduce now the *wavelet packets*  $\underline{g}_{(b, \omega)}$ , defined by their Fourier transform:

$$\widehat{\underline{g}_{(b, \omega)}}(\xi) = \widehat{g_{(b, \omega)}}(\xi) \sqrt{\frac{|J_{\xi}|}{|\beta(\omega)|}} \quad \text{(III-11)}$$

It is easy to check that for any  $(b, \omega) \in \mathbb{R}^2$ , the wavelet packet  $\underline{g}_{(b, \omega)}$  is by construction a  $L^2$ -function. One then has the following resolution of the identity:

**THEOREM 2.** — *Let  $g$  and  $\beta$  be respectively a  $L^2(\mathbb{R})$  and a piecewise differentiable function, admissible with respect to  $g$ . Then any  $s \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  can be decomposed as follows:*

$$s = \mathcal{C}_{\beta}^{-1/2} \cdot \int_{\mathbb{R} \times \mathbb{R}} T_s(b, \omega) \underline{g}_{(b, \omega)} db d\omega \quad \text{(III-12)}$$

where the generalized time-frequency coefficients are defined by:

$$T_s(b, \omega) = \langle \mathcal{C}_{\beta}^{-1/2} \cdot s, \underline{g}_{(b, \omega)} \rangle \quad \text{(III-13)}$$

(the equality holding in the  $L^2$ -sense), and  $\mathcal{C}_\beta$  is the convolution operator defined by the multiplier.

$$\hat{\chi}(\xi) = \int_{\mathbb{R}} |\hat{g}(u_\xi(\omega))|^2 |J_\xi| d\omega.$$

*Proof.* — Let  $W(b, \omega)$  denote the coefficient defined by:

$$W(b, \omega) = \langle s, \underline{g}_{(b, \omega)} \rangle \tag{III-14}$$

( $W(b, \omega)$  is finite by Cauchy-Schwartz inequality) and set:

$$S = \int_{\mathbb{R}^2} W(b, \omega) \underline{g}_{(b, \omega)} db d\omega. \tag{III-15}$$

A straightforward calculation gives:

$$\hat{S}(\xi) = \hat{s}(\xi) \hat{\chi}(\xi) \tag{III-16}$$

yielding the theorem.

*Remarks:*

— At that point, let us stress that one is now *a priori* forced to use the same  $g$  function for the analysis and the reconstruction: introducing a reconstructing function  $h$  different than the analyzing function would yield a filter with a complex-valued transfer function, and the admissibility of the  $\beta$  section would be more difficult to ensure.

— Since the only assumption made on the analyzing function is a  $L^2$ -assumption, one can choose it with arbitrary regularity and localisation properties (one can for instance choose Gaussian functions, to minimize the Heisenberg inequality). However, the Jacobian factor  $|J_\xi|^{-1/2}$  involved in the definition of the wavelet packets (III-10), while it has only a local action in the Fourier space if the  $g$  function has been chosen rapidly decreasing in this space, damages their localisation in the direct space.

### III-3. Reproducing kernel

The resolution of the identity implies in particular the energy conservation (*i.e.* a time-frequency Plancherel formula), and then expresses that the transforms  $s \rightarrow T_s$  maps  $L^2(\mathbb{R})$  into  $L^2(\mathbb{R}^2)$ . However, like in the case of wavelet or Gabor analysis, the image of  $L^2(\mathbb{R})$  is not the whole  $L^2(\mathbb{R}^2)$ , but some reproducing kernel subspace. More precisely, one has the following result, the proof of which is immediate from the resolution of the identity:

**PROPOSITION 3.** — *The wavelet coefficients satisfy the following (reproducing kernel) integral equation:*

$$T_s = \mathcal{K} \cdot T_s \tag{III-17}$$

where  $\mathcal{K}$  is a kernel operator, with kernel:

$$\mathcal{K}(b', \omega'; b, \omega) = \langle \mathcal{C}_\beta^{-1/2} \cdot \underline{g}(b', \omega'), \mathcal{C}_\beta^{-1/2} \cdot \underline{g}(b, \omega) \rangle$$

$\mathcal{K}$  is moreover an orthonormal projection operator.

Such a property is very important for practical applications, as discussed for instance in [Gr-KM-Mo]. In particular, it allows to derive interpolation formulas, and then to recover the continuous transform  $T_s$  from sampled versions.

### III-4. Optimality and implementation

In [Co-Me], the authors proposed a simple modification of the multiresolution analysis algorithm, leading to a “library” or orthonormal bases of  $L^2(\mathbb{R})$ . They then solved the problem of discrimination between these bases by introducing a Hilbert space decomposition version of Shannon-Weaver’s entropy function [Sh-We], well known in the information theory community. Given a function to analyze, the best adapted (or optimal) basis was then defined as the basis in the library which minimizes the entropy. We propose here a continuous analogue of their procedure.

Let then  $s \in L^2(\mathbb{R})$ , and let  $T_s$  be its wavelet packet transform. To the representation is canonically associated a density  $\rho_{(s, \beta)}$ , defined by:

$$\rho_{(s, \beta)}(b, \omega) = \frac{|T_s(b, \omega)|^2}{\|s\|^2} \tag{III-19}$$

$\rho_{(s, \beta)}$  is clearly positive definite, and the energy conservation implies that  $\rho_{(s, \beta)}$  is of integral unity, so that

$$d\nu_{(s, \beta)}(b, \omega) = \rho_{(s, \beta)}(b, \omega) db d\omega \tag{III-20}$$

is a probability measure. By general results [Li], one associates to  $\rho_{(s, \beta)}$  an *information entropy* function:

$$\varepsilon[s, \beta] = - \int_{\mathbb{R}^2} \rho_{(s, \beta)}(b, \omega) \ln[\rho_{(s, \beta)}(b, \omega)] db d\omega \tag{III-21}$$

which measures the “dispersion of  $\rho_{(s, \beta)}$ ” in the time-frequency plane. We refer to [Li] for a description of the basic properties of the entropy function. Given a  $s \in L^2(\mathbb{R})$ , the admissible section  $\beta$  is said to be *adapted* to  $s$  if the entropy  $\varepsilon[s, \beta]$  is minimal.

It must be stressed that in the Coifman-Meyer-Wickerhauser construction, the algorithmic structure of the multiresolution analysis allows a fast implementation of the minimal entropy criterion. In the continuous case, we do not know for the moment if such a fast implementation is possible.

The following simple scheme can be used for the numerical implementation:

*Transform:*

$$\begin{aligned} s \in L^2(\mathbb{R}) &\rightarrow s_1 = \mathcal{C}_\beta^{-1/2} \cdot s \in L^2(\mathbb{R}) \\ s_1 &\rightarrow T_s = \langle s_1, \underline{g}(b, \omega) \rangle \in L^2(\mathbb{R}^2). \end{aligned} \tag{III-22}$$

*Reconstruction:*

$$\begin{aligned} T_s &\rightarrow s_2 = \int T_s(b, \omega) \underline{g}(b, \omega) db d\omega \in L^2(\mathbb{R}) \\ s_2 &\rightarrow s = \mathcal{C}_\beta^{-1/2} \cdot s_2. \end{aligned} \tag{III-23}$$

Nevertheless, it is worth noticing that due to the real character of the transfer function of the filter, the  $\mathcal{C}_\beta^{-1/2}$  operator is hermitian, so that one can look at the representation theorem as a decomposition and a reconstruction of  $L^2(\mathbb{R})$  directly with the functions:

$$\mathcal{C}_\beta^{-1/2} \cdot \underline{g}(b, \omega)$$

However, the scheme described in (III-22) and (III-23) seems better suited for numerical computations, since it only involves twice the action of  $\mathcal{C}_\beta^{-1/2}$ .

Let us stress that in order to introduce the information entropy  $\varepsilon[s, \beta]$ , we have chosen a completely symmetric analysis-reconstruction scheme (a non-symmetric scheme would in general lead to a complex-valued density  $\rho_{(s, \beta)}$ , and the entropy would not be well-defined). However, if one does not care about the information entropy, one can for instance use the following non-symmetric scheme, which could be simpler for some signal analysis problems:

*Transform:*

$$s \in L^2(\mathbb{R}) \rightarrow S_s(b, \omega) = \langle s, \underline{g}(b, \omega) \rangle. \tag{III-24}$$

*Reconstruction:*

$$S_s \rightarrow s = \mathcal{C}_\beta^{-1} \cdot \int S_s(b, \omega) \underline{g}(b, \omega) db d\omega \tag{III-25}$$

where:

$$\widehat{\underline{g}(b, \omega)}(\xi) = \widehat{g}(b, \omega)(\xi) \frac{|J_\xi|}{|\beta(\omega)|} \tag{III-26}$$

with the appropriate properties on  $g$  to ensure the convergence of the integrals (III-25) and (III-26).

#### IV. EXAMPLES

We will describe here two basic examples of the wavelet packet decompositions introduced in the previous section. The first example corresponds to a decomposition of  $L^2(\mathbb{R})$  into functions which behave like Gabor functions in the low frequency domain, and wavelets in the high frequency domain. The second example, which is rather a series of examples than a single one, gives representation theorems which interpolate between Calderon's identity (wavelets) and the Gabor representation, going through Cordoba-Fefferman's wave-packets. Each of these examples corresponds to a particular choice of the section  $\beta(\omega)$ .

##### IV-1. Wavelets and Gabor functions

The first step is to check is that for some special choices of  $\beta$ , one recovers the usual Gabor and wavelet analysis. Clearly, the trivial choice

$$\beta_{\text{WH}} = \text{Const.} \neq 0 \quad (\text{IV-1})$$

is admissible, and leads to the standard Gabor analysis (the dilation is constant, and can be absorbed in a redefinition of  $g$ ). Indeed, the  $\chi$  filter is trivial, and equal to  $4\pi \|g\|^2$  for all  $t \in \mathbb{R}$ .

Consider now the case:

$$\beta_{aff}(\omega) = \frac{1}{\sigma\omega + \tau} \quad \omega \in \mathbb{R}, \quad \omega \neq -\frac{\tau}{\sigma}. \quad (\text{IV-2})$$

Then:

$$\frac{\beta'_{aff}(\omega)}{\beta_{aff}(\omega)} = -\sigma \beta_{aff}(\omega) \quad (\text{IV-3})$$

so that, by a redefinition of the analyzing function:

$$\hat{h}(\xi) = \hat{g}(\xi) |\sigma\xi + 1|^{1/2} \quad (\text{IV-4})$$

one has that:

$$\underline{g}_{(b, \omega)} = h_{(b, \omega)}. \quad (\text{IV-5})$$

Note that by construction,  $h$  fulfills a "shifted" admissibility condition:

$$\hat{h}(-1/\sigma) = 0. \quad (\text{IV-6})$$

Setting  $a = \beta_{aff}(\omega)$ , one then has that:

$$\omega = \frac{1}{\sigma} \left[ \frac{1}{a} - \tau \right] \quad (\text{IV-7})$$

and  $h_{(b, \omega)}$  is nothing but:

$$h_{(b, \omega)}(t) = e^{ib/\sigma a} e^{-it/\sigma} e^{i(t-b)/\sigma a} h\left(\frac{t-b}{a}\right). \tag{IV-8}$$

The computation of the wavelet packets coefficients  $T_s(b, \omega)$  of  $s \in L^2(\mathbb{R})$  then involves the two following steps:

- Computation of the wavelet coefficients of:

$$\tilde{s}(t) = e^{it/\sigma} s(t)$$

with respect to the wavelet:

$$\tilde{h}(t) = e^{it/\sigma} h(t)$$

- Multiplication by the complex number:  $e^{-ib/\sigma a}$ .

Note that  $\tilde{h}$  is admissible by construction, since the Fourier transform of  $h$  vanishes at  $-1/\sigma$ , and thus the Fourier transform of  $\tilde{h}$  vanishes at 0.

The section  $\beta_{aff}$  then leads to a simple modification of wavelet analysis.

### IV-2. Composite wavelet packets

We will call *composite wavelet packets* the functions we will describe now, which have a Gabor behaviour at low frequencies, and a wavelet behaviour at high frequencies. To construct such functions, start from a  $L^2(\mathbb{R})$  function  $g$ , fix two real strictly positive parameters  $\lambda$  and  $\omega_0$ , and consider the following section  $\beta_{comp}$ :

$$\left. \begin{aligned} \beta_{comp}(\omega) &= 1 && \text{for } |\omega| \leq \omega_0 \\ \beta_{comp}(\omega) &= \frac{\lambda}{\omega - \omega_0 + \lambda} && \text{for } \omega \geq \omega_0 \\ \beta_{comp}(\omega) &= \frac{-\lambda}{\omega + \omega_0 + \lambda} && \text{for } \omega \leq -\omega_0 \end{aligned} \right\} \tag{IV-9}$$

Such a choice simply forces the Gabor behaviour at frequencies less than  $\omega_0$  in absolute value, and the wavelet behaviour at high frequencies. Obviously, such a  $\beta$  is admissible. A straightforward calculation leads to the following expression for the transfer function of the  $\chi$  filter:

$$\hat{\chi}_{comp}(\xi) = I_g([\xi - \omega_0, -\lambda]) + I_g([\xi - \omega_0, \xi + \omega_0]) + I_g([\lambda, \xi + \omega_0]). \tag{IV-10}$$

One trivially checks the admissibility of  $\beta$  on such expressions, *i.e.* that there exist two bounds  $K_1$  and  $K_2$  (for example depending linearly on  $\|g\|^2$ ) such that:

$$0 < K_1 \leq \hat{\chi}_{comp}(\xi) \leq K_2 < \infty$$

for all  $\xi \in \mathbb{R}$ , so that the deconvolution is numerically stable.

### IV-3. Interpolating wavelet packets

We now address the problem of finding representation theorems interpolating between wavelet analysis and Gabor analysis. As we will see, this can be done by the following family of sections:

$$\beta_\alpha(\omega) = |\sigma\omega + \tau|^\alpha \quad (\text{IV-11})$$

where  $\alpha$  is a real number. As discussed in IV-1,  $\alpha=0$  corresponds to Gabor functions, and  $\alpha=-1$  corresponds to wavelets. We will restrict in a first stage to the case  $-1 < \alpha < 0$ , and describe the behaviour of the corresponding functions. We will then briefly describe the behaviour of the interpolating wavelet packets for  $\alpha < -1$  and  $\alpha > 0$ .

Let us start with the study of  $u_\xi$ . Set:

$$\omega_c(\xi) = \frac{\alpha\xi - \tau/\sigma}{\alpha + 1} \quad \xi \in \mathbb{R} \quad (\text{IV-12})$$

and:

$$u_c(\xi) = u_\xi(\omega_c) = -\frac{\alpha}{\sigma(\alpha + 1)^2} |\sigma\xi + \tau|^\alpha (\sigma\xi + \tau) < 0 \quad (\text{IV-13})$$

For  $\xi < -\tau/\sigma$ ,  $\omega_c(\xi) > -\tau/\sigma$ , and  $u_\xi$  is a change of variable on the following intervals:  $] -\infty, -\tau/\sigma[$ ,  $] -\tau/\sigma, \omega_c(\xi)[$ ,  $] \omega_c(\xi), +\infty[$ . When  $\omega \rightarrow -\infty$ ,  $u_\xi(\omega) \rightarrow +\infty$ ; when  $\omega \rightarrow -\tau/\sigma$ ,  $u_\xi(\omega) \rightarrow -\infty$ ; when  $\omega \rightarrow +\infty$ ,  $u_\xi(\omega) \rightarrow -\infty$ .

For  $\xi > -\tau/\sigma$ ,  $\omega_c(\xi) < -\tau/\sigma$ , and  $u_\xi$  is a change of variable on the following intervals:  $] -\infty, \omega_c(\xi)[$ ,  $] \omega_c(\xi), -\tau/\sigma[$ ,  $] -\tau/\sigma, +\infty[$ . When  $\omega \rightarrow -\infty$ ,  $u_\xi(\omega) \rightarrow +\infty$ ; when  $\omega \rightarrow -\tau/\sigma$ ,  $u_\xi(\omega) \rightarrow +\infty$ ; when  $\omega \rightarrow +\infty$ ,  $u_\xi(\omega) \rightarrow -\infty$ .

For  $\xi = -\tau/\sigma$ ,  $u_\xi$  is a change of variable on the real line. When  $\omega \rightarrow -\infty$ ,  $u_\xi(\omega) \rightarrow +\infty$ ; when  $\omega \rightarrow +\infty$ ,  $u_\xi(\omega) \rightarrow -\infty$ .

Since the filter of the time-frequency representation only appears in a deconvolution, it is not necessary to care about its behaviour on an single point, and it can be set to:

$$\left. \begin{aligned} \hat{\chi}_\alpha(\xi) &= 2I_g(]-\infty, u_c(\xi)[) + 2\pi \|g\|^2 & \text{for } \xi < -\tau/\sigma \\ \hat{\chi}_\alpha(\xi) &= 6\pi \|g\|^2 & \text{for } \xi = -\tau/\sigma \\ \hat{\chi}_\alpha(\xi) &= 2I_g(]u_c(\xi), \infty[) + 2\pi \|g\|^2 & \text{for } \xi > -\tau/\sigma \end{aligned} \right\} \quad (\text{IV-14})$$

so that  $\beta_\alpha$  is an admissible section. It is then possible to use the analysis-reconstruction procedure described in III-3 from such sections. Note that since the filter is only used in deconvolutions, its value at a single point is irrelevant, and it can be evaluated at  $-\tau/\sigma$  by continuity (indeed,  $u_c(\xi) \rightarrow -\infty$  when  $\xi \rightarrow (-\tau/\sigma)_+$ , and  $u_c(\xi) \rightarrow +\infty$  when  $\xi \rightarrow (-\tau/\sigma)_-$ ).

*Remarks:*

– It is not necessary to assume that the  $\alpha$  coefficient is restricted to  $[-1, 0]$ . The  $\beta_\alpha$  section is actually admissible for all real values of  $\alpha$ , the only differences leaving in different expressions for the filter of the representation. We give here the explicit expressions of the filter in the remaining cases:

If  $\alpha > 0$ :

$$\left. \begin{aligned} \hat{\chi}_\alpha(\xi) &= 2 I_g (|u_c(\xi), 0|) + 2 \pi \|g\|^2 & \text{for } \xi \leq -\tau/\sigma \\ \hat{\chi}_\alpha(\xi) &= 2 I_g (|0, u_c(\xi)|) + 2 \pi \|g\|^2 & \text{for } \xi \geq -\tau/\sigma \end{aligned} \right\} \quad \text{(IV-15)}$$

If  $\alpha < -1$ :

$$\left. \begin{aligned} \hat{\chi}_\alpha(\xi) &= 2 I_g (|0, u_c(\xi)|) + 2 \pi \|g\|^2 & \text{for } \xi \leq -\tau/\sigma \\ \hat{\chi}_\alpha(\xi) &= 2 I_g (|u_c(\xi), 0|) + 2 \pi \|g\|^2 & \text{for } \xi \geq -\tau/\sigma \end{aligned} \right\} \quad \text{(IV-16)}$$

– The case  $\alpha = -1$  does not exactly lead to wavelet analysis as described in VI-1, but to a slight modification of it in which one only considers positive dilation parameters, then to the analysis of  $H^2(\mathbb{R})$  as stressed in II-1. To recover the analysis of  $L^2(\mathbb{R})$ , one can use the following family of sections:

$$\beta'_\alpha(\omega) = \text{Sgn}[\sigma\omega + \tau] |\sigma\omega + \tau|^\alpha \quad \text{(IV-17)}$$

leading to a slightly different expression for the filter of the representation.

– A particularly interesting case is the case where  $\alpha = -1/2$  and  $\tau = 0$ . The corresponding functions are built from  $g_{(b, \omega)}$  functions such that the number of their oscillations is inversely proportional to their size. This is the generic property of the so-called wave-packets used by Cordoba and Fefferman to study certain Fourier integral operators [Cor-Fef].

### V. CONCLUSIONS

We described a generic algorithm for constructing resolutions of the identity from functions indexed by continuous frequency and scale parameters. However, it is not clear up to now whether or not these continuous coherent states are the continuous analogues of the orthonormal bases of Coifman and Meyer. An answer would be given by the discretization of the resolution of the identity, *i.e.* by the construction of the associated frames [Da 1].

In the framework described here, it is in particular possible to build a  $\beta$  function adapted to human hearing (the so-called Bark scale), and then to derive the results of [d'A-Bea] in a rigorous setting. This would lead to interesting results in speech processing.

Let us also stress that a dual version of the wavelet packets can be straightforwardly developed as follows: instead of analysing the function itself, one can compute the wavelet packets coefficients of its Fourier transform. Up to simple modifications, this is essentially equivalent to build wavelet packets similar to those of equations (III-9) to (III-11), in which the  $\beta$  function now depends on the time translations  $b$  instead of the frequency translation  $\omega$ .

The continuous wavelet packets construction illustrates the fact that the square integrable group representation theory, although it provides an elegant and interesting structure in the wavelet and Gabor cases, is irrelevant in a more general setting. Indeed, the wavelet packets are no longer generated by a unitary group representation. One is then led to consider deformed representations of  $\mathbf{G}_{\text{WH}}$  or, what is equivalent [Di], deformed reproducing kernel subspaces of  $L^2(\mathbf{G}_{\text{WH}}, d\mu_{\text{WH}})$  (*i. e.* reproducing kernel subspaces with a deformed kernel).

At that point, let us nevertheless quote the interesting work of Bertrand and Bertrand [Be-Be], in which they use Cartan's classification of the 3-parameters extensions of the affine group  $\mathbf{G}_{\text{aff}}$  to study some representations which are square integrable when restricted to some homogeneous spaces. They then obtain a class of generalized Wigner functions (since they are only interested in bilinear representations), and it is very likely that one can find corresponding representation theorems.

## REFERENCES

- [Al-An-Ga] S. T. ALI, J. P. ANTOINE and J. P. GAZEAU, *Square Integrable Group Representations on Homogeneous Spaces*, I, Preprint UCL-IPT-89-18; II, Preprint UCL-IPT-89-19, Louvain-la-Neuve.
- [Be-Be] J. BERTRAND and P. BERTRAND, Classification of Affine Wigner Functions via an Extended Covariance Principle, *Proc. of the 17th International Colloquium on Group Theoretical Methods in Physics*, Sainte ADÈLE, Y. SAINT-AUBIN and L. VINET Ed., World Scientific (1989), pp. 380-383.
- [Bo] G. BOHNKÉ, Treillis d'ondelettes sur le groupe de Lorentz, *Ann. Inst. H. Poincaré, Physique Théorique*, Vol. **54**, 1991, pp. 245-260.
- [Ca] A. L. CAREY, Square Integrable Representations of Non Unimodular Groups, *Bull. Austr. Math. Soc.*, Vol. **15**, 1976, pp. 1-12.
- [Ca] A. P. CALDERON, Intermediate Spaces and Interpolation, the Complex Method, *Studia Math.*, Vol. **24**, 1964, pp. 113-190
- [Coh] A. COHEN, Ondelettes, analyses multirésolution et traitement numérique du signal, PhD Thesis, CEREMADE, Université de Paris-Dauphine, 1990.
- [Co-Me] R. R. COIFMAN and Y. R. MEYER, *Nouvelles bases orthomormées de  $L^2(\mathbb{R})$  ayant la structure du système de Walsh*, Preprint, Yale University, New Haven, 1990.

- [Co-Me-Qu-Wi] R. R. COIFMAN, Y. R. MEYER, S. QUAKE and M. V. WICKERHAUSER, *Signal processing and compression with wavelet packets*, Preprint, Yale University, New Haven 1990.
- [Co-Wi] R. R. COIFMAN and M. V. WICKERHAUSER, *Best Adapted Wave Packet Bases*, Preprint, Yale University, New Haven, 1990.
- [Com-Gr-Tc] J. M. COMBES, A. GROSSMANN and Ph. TCHAMITCHIAN Ed., *Wavelets, Time-Frequency Methods and Phase Space*, Springer, 1989.
- [Cor-Fef] A. CORDOBA and FEFFERMAN, Wave Packets and Fourier Integral Operators, *Comm. Part. Diff. Eq.*, 3, Vol. 3, 1978, pp. 979-1005.
- [d'A-Bea] C. D'ALESSANDRO and D. BEAUTEMPS, *Représentation et modification du signal de parole par transformée en ondelettes utilisant des contraintes auditives*, Preprint, LIMSI 90-10, Orsay, October 1990.
- [Da 1] I. DAUBECHIES, The Wavelet Transform, Time-Frequency Localisation and Signal Analysis, *IEEE Trans. on Information Theory*, Vol. 36, 1991, pp. 961-1005.
- [Da 2] I. DAUBECHIES, Orthonormal Bases of Compactly Supported Wavelets, *Comm. Pure Appl. Math.*, Vol. XLI, 1988, pp. 909-996.
- [Di] J. DIXMIER, Les  $C^*$ -algèbres et leurs représentations, *Cahiers Scientifiques*, Vol. 29, Gauthier-Villars, 1957.
- [Du-Mo] M. DUFLO and C. C. MOORE, On the Regular Representation of a Non Unimodular Locally Compact Group, *J. Funct. An.*, Vol. 21, 1976, p. 209.
- [Fe-Grö] H. G. FEICHTINGER and K. H. GRÖCHENIG, Banach Spaces Related to Integrable Group Representations and their Atomic Decompositions, I, *J. Funct. An.*, Vol. 86, 1989, pp. 307-340; II, *Mh. Math.*, Vol. 108, 1989, pp. 129-148.
- [Ga] D. GABOR, Theory of Communication, *J. Inst. Elec. Eng.*, Vol. 903, 1946, pp. 429-457.
- [Gr-KM-Mo] A. GROSSMANN, R. KRONLAND-MARTINET and J. MORLET, *Reading and Understanding Continuous Wavelet Transform*, in [Com-Gr-Tc].
- [Gr-Mo] A. GROSSMANN and J. MORLET, *Decomposition of Hardy Functions Into Wavelets of Constant Shape*, *S.I.A.M. J. Math. An.*, Vol. 15, 1984, pp. 723-736.
- [Gr-Mo-Pa 1] A. GROSSMANN, J. MORLET and T. PAUL, Transforms Associated with Square Integrable Group Representations, I, *J. Math. Phys.*, Vol. 27, 1985, pp. 2473-2479; II, *Ann. Inst. H. Poincaré*, Vol. 45, 1986, pp. 293-309.
- [Gr-Mo-Pa 2] A. GROSSMANN, J. MORLET and T. PAUL, Transforms Associated with Square Integrable Group Representations, II, *Ann. Inst. H. Poincaré*, Vol. 45, 1986, pp. 293-309.
- [He-Wa] C. E. HEIL and D. F. WALNUT, Continuous and Discrete Wavelet Transform, *S.I.A.M. Rev.*, Vol. 31, 1989, pp. 628-666.
- [Ho-Tc] M. HOLSCHNEIDER and Ph. TCHAMITCHIAN, *Pointwise analysis of Riemann's Nowhere Differentiable Function*, Preprint CPT89/P.2274, *Invent. Math.* (to appear).
- [Kl-Sk] J. R. KLAUDER and B. S. SKAGERSTAM, *Coherent States*, World Scientific, 1985, Singapore.
- [Li] E. LIEB, Some Convexity and Subadditivity Properties of Entropy, *Bull. Am. Math. Soc.*, Vol. 81, 1975, pp. 1-13.
- [Me 1] Y. MEYER, *Ondelettes et opérateurs I : Ondelettes*, Hermann, Paris, 1989.
- [Me 2] Y. MEYER, *Ondelettes et opérateurs II : Opérateurs de Calderón-Zygmund*, Hermann, 1990.

- [Mu] R. MURENZI, Ondelettes multidimensionnelles et applications à l'analyse d'images, *Ph-D Thesis*, IPT-UCL Louvain-La-Neuve, 1990
- [Pe] A. PERELOMOV, *Generalized Coherent States and their Applications*, Springer Verlag, Berlin, 1986.
- [Sc] W. SCHEMP, *Harmonic analysis on the Heisenberg nilpotent Lie group, with applications to signal theory*, Longman, 1986.
- [Sh-We] C. E. SHANNON and W. WEAVER, *The Mathematical Theory of Communication*, The University of Illinois Press, 1964.
- [Tc-To] Ph. TCHAMITCHIAN and B. TORRÉSANI, *Ridge and Skeleton Extraction from the Wavelet Transform*, in *Wavelets and their applications*, B. RUSKAI et al. Eds, Jones & Bartlett Publ. Comp., Boston, 1992.
- [To 1] B. TORRÉSANI, Wavelets Associated with Representations of the Affine Weyl-Heisenberg Group, *J. Math. Phys.*, Vol. 32, 1991, pp. 1273-1279.
- [To 2] B. TORRÉSANI, Wavelet Analysis of Asymptotic Signals: Ridge and Skeleton of the Transform, (to appear) in the *Proceedings of the 2nd Conference on Wavelet and Applications*, 1989, Y. MEYER Ed., Masson, Paris 1991.
- [Ve] M. VETTERLI, Private communication, 1990.
- [Wi 1] M. V. WICKERHAUSER, *Applications of Wavelets to Signal Compression*, Yale University Preprint, Mathematics department, 1990.
- [Wi 2] M. V. WICKERHAUSER, *Picture Compression by Best Basis Sub-Band Coding*, Yale University Preprint, Mathematics department, 1990.

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