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On the homomorphisms of sum logics

by

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ABSTRACT. — We show that every σ-homomorphism of logics $h: L_1 \to L_2$, where $(L_1, \mathcal{S}(L_1))$ and $(L_2, \mathcal{S}_2)$ are sum logics [where $\mathcal{S}(L_1)$ is the set of all σ-additive states on L_1 and $\mathcal{S}_2$ is a quite full set of σ-additive states on L_2] is necessarily a lattice σ-homomorphism. Some applications of this statement to Hilbert space logics and projection logics of von Neumann algebras are given.

RÉSUMÉ. — Nous prouvons que tout σ-homomorphisme de logiques $h: L_1 \to L_2$ où $(L_1, \mathcal{S}(L_1))$ et $(L_2, \mathcal{S}_2)$ sont des logiques sommes [$\mathcal{S}(L_1)$ est l'ensemble de tous les états σ-additifs sur L_1 et $\mathcal{S}_2$ est un ensemble fort d'états σ additifs sur L_2] est nécessairement un σ-homomorphisme de treillis. Nous donnons des applications de ce résultat aux logiques d'espaces de Hilbert et aux logiques d'algèbres de Von-Neumann.

1. INTRODUCTION

By a (quantum) logic we will mean an orthomodular σ-lattice $L(0, 1, ', \wedge, \vee)$ (*see* [13] for detailed definition). Two elements a, b of L

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are *orthogonal* (written $a \perp b$) if $a \leqq b'$, and $a, b \in \mathrm{L}$ are *compatible* if there are mutually orthogonal elements a_1, b_1, c in L such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

Let L_1 and L_2 be two logics. A map $h: \mathrm{L}_1 \to \mathrm{L}_2$ is said to be a *homomorphis* (of logics) if (i) $h(1)=1$, (ii) $h(a')=h(a)'$ for any $a \in \mathrm{L}_1$, (iii) $h(a \vee b) = h(a) \vee h(b)$ for any $a, b \in \mathrm{L}_1$, $a \perp b$. A homomorphism $h: \mathrm{L}_1 \to \mathrm{L}_2$ is said to be a σ-*homomorphism* if (iii′) $h\left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} h(a_i)$ for any sequence $(a_i)_{i \in \mathrm{N}}$ of mutually orthogonal elements of L_1. A homomorphism $h: \mathrm{L}_1 \to \mathrm{L}_2$ is a *complete* homomorphism if (iii″) $h(\bigvee_{i \in \mathrm{I}} a_i) = \bigvee_{i \in \mathrm{I}} h(a_i)$ for any set $(a_i)_{i \in \mathrm{I}}$ of pairwise orthogonal elements of L_1.

A logic homomorphism $h: \mathrm{L}_1 \to \mathrm{L}_2$ is a *lattice homomorphism* if $h(a \vee b) = h(a) \vee h(b)$ [dually, $h(a \wedge b) = h(a) \wedge h(b)$] for any $a, b \in \mathrm{L}_1$. A σ-homomorphism (complete homomorphism) of logics which is a lattice homomorphism, is a lattice σ-homomorphism (complete lattice homomorphism).

It is easy to see that if $h: \mathrm{L}_1 \to \mathrm{L}_2$ is a logic homomorphism, then $h(0)=0$, $a, b \in \mathrm{L}_1$ and $a \leqq b$ imply $h(a) \leqq h(b)$ in L_2 and $a, b \in \mathrm{L}_1$ and $a \perp b$ imply $h(a) \perp h(b)$ in L_2. Moreover, if $a \leftrightarrow b$, then $h(a) \leftrightarrow h(b)$ and $h(a \vee b) = h(a) \vee h(b)$, $h(a \wedge b) = h(a) \wedge h(b)$. In particular, if L_1 is a Boolean (σ-) algebra, then every (σ-) homomorphism from L_1 into a logic L_2 is a lattice (σ-) homomorphism.

In general, a logic σ-homomorphism need not be a lattice σ-homomorphism. A simple reasoning shows that for a two-dimensional Hilbert space H, there is a two-valued homomorphism from L (H) into itself, which is not a lattice homomorphism. It is easy to find a logic homomorphism from a horizontal sum of three four-element Boolean algebras into the Boolean algebra with three atoms that is even a bijection but it is not a lattice homomorphism.

The aim of this paper is to show that every σ-homomorphism of logics $h: \mathrm{L}_1 \to \mathrm{L}_2$ between two sum logics $(\mathrm{L}_1, \mathscr{S}(\mathrm{L}_1))$ and $(\mathrm{L}_2, \mathscr{S})$, is necessarily a lattice σ-homomorphism. We note that a special case of this statement is proved in [12]. Since a projection logic of a von Neumann albegra of operators acting on a separable complex Hilbert space which does not contain any I_2-factor as a direct summand is a sum logic in the sense of our definition (*see*, e. g., the review paper [9]), we obtain as a consequence that every logic σ-homomorphism from such a projection logic into a projection logic of any von Neumann algebra is necessarily a lattice σ-homomorphism. In particular, if H and K are complex separable Hilbert spaces and dim H ≠ 2, then using Hamhalter's result (*see* [7]), we find that every homomorphism of projection logics $h: L(H) \to L(K)$ is a lattice σ-homomorphism.

2. HOMOMORPHISMS OF SUM LOGICS

Let L be a logic. A *state* on L is a map $s: L \to [0, 1]$ such that $s(1)=1$, and $s(\bigvee_{i \in N} a_i) = \sum_{i \in N} s(a_i)$ for any sequence $(a_i)_{i \in N}$ of mutually orthogonal elements in L. Let $\mathcal{S}(L)$ denote the set of all states on L. A logic L is called *quite full* if for any $a, b \in L$ such that $a \nleq b$ there is $s \in \mathcal{S}(L)$ such that $s(a)=1$ and $s(b) \neq 1$.

An *observable* on a logic L is a σ-homomorphism from the σ-algebra $\mathcal{B}(R)$ of Borel subsets of the real line R into L. If x is an observable and $s \in \mathcal{S}(L)$, then the map $s_x: \mathcal{B}(R) \to [0, 1]$ defined by $s_x(E) = s(x(E))$ is a probability measure on $\mathcal{B}(R)$. The expectation of an observable x in a state s is then defined by $s(x) = \int t s_x(dt)$, if the integral exists. An observable x is *bounded* if there is a compact subset $C \subset R$ such that $s(C)=1$. An observable x is a *proposition observable* if $x(\{0, 1\})=1$. To every $a \in L$, there is a unique proposition observable q_a such that $q_a(\{1\})=a$.

If x is a bounded observable on L, then $s(x)$ exists and is finite for every $s \in \mathcal{S}(L)$. In [5], there is proved that also the opposite statement is true: If $\mathcal{S} \subset \mathcal{S}(L)$ is σ-convex and quite full, then an observable x is bounded if and only if $s(x)$ exists and is finite for all $s \in \mathcal{S}$.

Let x, y be bounded observables on a logic L with a quite full set of states $\mathcal{S}$. We say that a bounded observable z is a *sum* of x and y if $s(x)+s(y)=s(z)$ for all $s \in \mathcal{S}(L)$. A couple $(L, \mathcal{S})$, where L is a logic and $\mathcal{S}$ is a σ-convex quite full set of of states is said to be a *sum logic* if for every pair x, y of bounded observables on L there is a unique sum z. We shall write $z=x+y$ if z is the sum of x and y. For more details about sum logics we refer to [6].

Now we are ready to state and prove our main result.

THEOREM 1. – *Let* $(L_1, \mathcal{S}(L_1))$ *and* $(L_2, \mathcal{S})$, $\mathcal{S} \subset \mathcal{S}(L_2)$ *be sum logics. Then every logic* σ*-homomorphism* $h: L_1 \to L_2$ *is a lattice* σ*-homomorphism.*

Proof. – Let $h: L_1 \to L_2$ be a σ-homomorphism. Let x be an observable on L_1. Define $h(x): \mathcal{B}(R) \to L_2$ by $h(x)(E) = h(x(E))$. Then $h(x)$ is an observable on L_2. For every $s \in \mathcal{S}(L_2)$, the map $s.h: L_1 \to [0, 1]$ defined by $s.h(a) = s(h(a))$ is a state on L_1. If x is a bounded observable on L_1, then for any $s \in \mathcal{S}(L_2)$, $s(h(x)) = s.h(x)$, where $s.h \in \mathcal{S}(L_1)$, and therefore $s.h(x)$ exists and is finite. This implies that $h(x)$ is bounded. For $a \in L_1$, let q_a denote the proposition observable such that $q_a(\{1\})=a$. Then

$h(q_a)(\{1\}) = h(q_a(\{1\})) = h(a)$, similarly $h(q_a(\{0\}) = h(a)'$. This proves that $h(q_a) = q_{h(a)}$.

Now let x, y be bounded observables on L_1 and let $x + y = z$. Since for every $s \in \mathcal{S}$, $s.h \in \mathcal{S}(L_1)$, we find $s.h(z) = s.h(x) + s.h(y)$, *i.e.*, $s(h(z)) = s(h(x)) + s(h(y))$ for all $s \in \mathcal{S}$, which entails $h(z) = h(x) + h(y)$.

By Gudder [6], for any pair a, b of elements of a sum logic L we have $(q_a + q_b)(\{2\}) = a \wedge b$. Let a, $b \in L_1$. Then

$$\begin{aligned} h(a \wedge b) = h((q_a + q_b)(\{2\})) &= h(q_a + q_b)(\{2\}) \\ &= (h(q_a) + h(q_b))(\{2\}) \\ &= (q_{h(a)} + q_{h(b)})(\{2\}) = h(a) \wedge h(b). \end{aligned}$$

Hence $h(a \wedge b) = h(a) \wedge h(b)$ holds for all a, $b \in L_1$. This proves that h is a lattice σ-homomorphism.

3. APPLICATIONS

PROPOSITION 1. – *Let* H, K *be separable, complex Hilbert spaces* ($\dim H \geqq 3$) *and let* $h: L(H) \to L(K)$ *be a homomorphism of logics. Then* h *is a lattice σ-homomorphism.*

Proof. – *See* [7].

PROPOSITION 2. – *Let* H, K *be separable, complex Hilbert spaces* ($\dim H \geqq 3$) *and let* $h: L(H) \to L(K)$ *be a homomorphism. Then* $\dim h([x]) = \dim h([y])$ *for every* x, $y \in H$ (*where* $[x]$ *denotes the one-dimensional subspace generated by* x). *In particular, there is a finite or countably infinite cardinal* r *such that* $\dim K = r \dim H$.

COROLLARY. – *A homomorphism between two separable complex Hilbert space logics* L(H), L(K) ($\dim H \geqq 3$) *is necessarily injective.*

The proof follows by Theorem 1 and Matolcsi [11], Propositions 3 and 4.

Recall that a set S is of *non measurable cardinality* if there does not exist any probability measure defined on all subsets of S vanishing at all points. In the opposite case we say that S is of measurable cardinality.

PROPOSITION 3. – *Let* H, K *be Hilbert spaces* ($\dim H \geqq 3$). *Then every σ-homomorphism* $h: L(H) \to L(K)$ *is a complete lattice homomorphism if and only if* $\dim H$ *is a non measurable cardinal.*

Proof. – Assume that $\dim H$ is non measurable. Let x be a unit vector of K. Then $m_x^h(M) = \langle h(M)x, x \rangle$, $M \in L(H)$, defines a completely additive state on L(H) (*see*, e.g., [9]). Since x is arbitrary, we conclude that h

is a complete homomorphism of logics. Applying Theorem 1, we obtain that h is a complete lattice homomorphism.

Now suppose that $\dim H$ is a measurable cardinal. Let $(x_t)_{t \in T}$ be an orthonormal basis in H. Then there exists a probability measure μ on 2^T vanishing on all one-point sets.

Define

$$m(M) = \int_T \| Me_t \|^2 \, d\mu(t), \qquad M \in L(H).$$

Then m is a state on $L(H)$ with the properties $m(H)=1$ and $m([e_t])=0$ for all $t \in T$. Indeed,

$$m(H) = \mu(T) = 1,$$
$$m([e_t]) = \int \| e_t \|^2 \, d\mu(t) = \mu(\{t\}) = 0.$$

(*See* also [3]). This implies that m is a σ-additive, but not completely additive measure on $L(H)$. Define

$$K_m(M, N) = \int_T \langle Me_t, Ne_t \rangle \, d\mu(t).$$

For $\alpha_i \in C$, $i \leqq n$ $(n \in N)$ we have

$$\sum \alpha_i \bar{\alpha}_j K_m(M_i, M_j) = \int \sum \alpha_i \bar{\alpha}_j \langle M_i e_t, M_j e_t \rangle \, d\mu(t)$$
$$= \int \| \sum \alpha_i M_i e_t \|^2 \, d\mu(t) \geqq 0.$$

This shows that K_m is a reproducing kernel. By [12], there is a vector-valued measure ξ defined on $L(H)$ with values in a Hilbert space K, such that $m(M) = \| \xi(M) \|^2$ $(M \in L(H))$. By [10], there is a σ-homomorphism $h: L(H) \to L(K)$ of logics and a vector $v \in K$ such that $\xi(M) = h(M)v$ $(M \in L(H))$. This entails that h is not a complete homomorphism.

In a similar manner as Proposition 2, we can prove the following statement.

PROPOSITION 4. — *Every σ-homomorphism $h: L(H) \to L(K)$, where $2 \neq \dim H$ is non measurable, is injective. Moreover,* $\dim h([x]) = \dim h([y])$ *for all $x, y \in H$, and there is a cardinal α $(\alpha = \dim [x])$ such that* $\dim K = \alpha \dim H$.

The following proposition is a generalization of the theorem obtained by Jajte, Paszkiewicz [8]. Using Proposition 1 and results of Aerts, Daubechies ([1], [2]) and Wright [14], we obtain an alternative proof.

PROPOSITION 5. — *Let* H, K *be two separable Hilbert spaces with dimensions greater or equal to three and let* $h: \mathrm{L}(\mathrm{H}) \to \mathrm{L}(\mathrm{K})$ *be a homomorphism. Then there exists a family of maps* $(\varphi_j)_{j \in \mathrm{J}}$ *from* H *to* K *such that*

— *each* φ_j *is an isometry or anti-isometry;*

— K *is the orthogonal sum* $\mathrm{K} = \bigoplus_{j \in \mathrm{J}} \varphi_j(\mathrm{H})$;

— *for all* $\mathrm{M} \in \mathrm{L}(\mathrm{H})$, $h(\mathrm{M}) = \bigvee_{j \in \mathrm{J}} \varphi_j^{-1}(\mathrm{M})$.

PROPOSITION 6 (a generalization of Dye's theorem). — *Let* $\mathcal{M}$ *be a von Neumann algebra with no direct summand of type* I_2. *Then any* σ-*homomorphism of logics between the projection logic of* $\mathcal{M}$ *and that of a von Neumann algebra* $\mathcal{N}$ *is implemented by the direct sum of a* $*$-*isomorphism and a* $*$-*antiisomorphism between* $\mathcal{M}$ *and* $\mathcal{N}$.

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