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Singularities of the scattering kernel for generic obstacles

by

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geqq 3$, n odd, be an open connected domain with C^∞ smooth boundary $\partial\Omega$ and bounded complement

$$K = \mathbb{R}^n \setminus \Omega \subset \{ x : |x| \leqq \rho_0 \}.$$

The *scattering kernel* $s(t, \theta, \omega)$ related to the wave equation in $\mathbb{R} \times \Omega$ with Dirichlet boundary conditions on $\mathbb{R} \times \partial\Omega$ has the form (*see* [8])

$$s(t, \theta, \omega) = C_n \int_{\partial K} \partial_\tau^{n-2} \partial_\nu w(\langle x, \theta \rangle - t, x; \omega) \, dS_x. \qquad (1.1)$$

Here $(\theta, \omega) \in S^{n-1} \times S^{n-1}$, $w(\tau, x; \omega)$ is the solution of the problem

$$\begin{gathered} (\partial_\tau^2 - \Delta_x) w = 0 \quad \text{in } \mathbb{R} \times \Omega, \\ w = 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \\ w_{|\tau < -\rho_0} = \delta(\tau - \langle x, \omega \rangle), \end{gathered} \qquad (1.2)$$

ν is the *interior unit normal* to $\partial\Omega$ pointing into Ω, dS_x is the measure induced on $\partial\Omega$, $C_n = (-1)^{(n+1)/2} 2^{-n} \pi^{(1-n)}$ and $\langle \, , \, \rangle$ is the inner product in $\mathbb{R}^n$.

For fixed ω, θ we have $s(t, \theta, \omega) \in \mathscr{S}'(\mathbb{R}_t)$. The analysis of the singularities of $s(t, \theta, \omega)$ for fixed ω, θ is important for some inverse scattering problems.

The aim of this paper is to study sing supp $s(t, \theta, \omega)$ for general (nonconvex) obstacles.

By a *reflecting* (ω, θ)*-ray* in $\bar{\Omega}$ we mean a continuous curve in $\bar{\Omega}$ formed by a finite number of linear segments and two infinite linear segments – an incoming one with direction ω and an outgoing one with direction θ (*cf.* section 2 for a precise definition). If a reflecting (ω, θ)-ray γ in $\bar{\Omega}$ has no segments tangent to $\partial\Omega$, then γ will be called *ordinary*.

By a *generalized* (ω, θ)*-ray* we mean an infinite continuous curve γ in $\bar{\Omega}$ incoming with direction ω and outgoing with direction θ which is a projection on $\bar{\Omega}$ of a generalized bicharacteristic of the wave operator $\square = \partial_t^2 - \Delta$ (*cf.* [9]) and which contains at least one gliding segment which is a geodesic on $\partial\Omega$ with respect to the standard Riemannian metric. Finally, by a (ω, θ)-ray we mean either a reflecting or a generalized (ω, θ)-ray. Throughout this paper we consider only null bicharacteristics of $\square$, *i.e.* bicharacteristics lying in the characteristic set Σ of $\square$ (*see* [9]).

For fixed ω, θ we denote by $\mathscr{L}_{\omega, \theta}$ *the set of all* (ω, θ)*-rays*. For $\gamma \in \mathscr{L}_{\omega, \theta}$ consider the *sojourn time* T_γ of γ (*see* section 2 for a definition). As it was suggested in [4], [11], the singularities of $s(t, \theta, \omega)$ are related to the sojourn times of the (ω, θ)-rays. In [11], [16], [17] for some special classes of obstacles all singularities of $s(t, \theta, \omega)$ have been examined.

According to the geometry of the generalized bicharacteristics of $\square$ (*see* [5], [22]), there could be some points on $T^*(\partial\Omega \times \mathbb{R})$ such that there are more than one generalized bicharacteristic passing through them. We shall say that a generalized bicharacteristic δ of $\square$ is *uniquelly extendible* if for every $z \in \delta$ the only generalized bicharacteristic of $\square$ passing through z is δ. A (ω, θ)-ray γ in $\bar{\Omega}$ will be called *uniquelly extendible* if γ is a projection on $\bar{\Omega}$ of a uniquelly extendible bicharacteristic.

Note that if K is convex or K has a real analytic boundary, then every (ω, θ)-ray in $\bar{\Omega}$ is uniquelly extendible. The same is true if $\partial\Omega$ has no points where the curvature of $\partial\Omega$ vanishes of infinite order along some direction. Another example is the case when K is a finite union of disjoint convex obstacles. We refer to [22] for an example when there exists a bicharacteristic which is not uniquelly extendible.

Let Z_1 be a hyperplane in $\mathbb{R}^n$ orthogonal to ω and such that the open halfspace, determined by Z_1 and having ω as an inward normal, contains $\partial\Omega$. Given $u \in Z_1$, put $\rho_u = (-\rho_0, u, 1, -\omega) \in T^*(\mathbb{R} \times \Omega)$. Denote by $C_t(u)$ the set of those $z \in T^*(\mathbb{R} \times \bar{\Omega})$ such that there exists a generalized bicharacteristic $\gamma(\sigma)$ of $\square$ with $\gamma(-\rho_0) = \rho_u$, $\gamma(t) = z$. For $V \subset Z_1$ set

$$C_t(V) = \bigcup_{u \in V} C_t(u).$$

Our first result is the following.

THEOREM 1. — *Let* $\theta \neq \omega$ *be fixed. Assume that every* (ω, θ)*-ray in* $\bar{\Omega}$ *is uniquelly extendible. Then*

$$\operatorname{sing\,supp} s(t, \theta, \omega) \subset \{ -T_\gamma : \gamma \in \mathscr{L}_{\omega, \theta} \}. \tag{1.3}$$

Remark 1.1. — The assumption of Theorem 1 concerns only the (ω, θ)-rays. Thus for fixed ω the relation (1.3) shows that if K is connected, then the shadow of K with respect to ω does not contribute to $\operatorname{sing\,supp} s(t, \theta, \omega)$, if we make some observations with rays incoming with direction ω. Note that some bicharacteristics of $\square$ which are not related to (ω, θ)-rays can be not uniquelly extendible.

Remark 1.2. — The assumption of Theorem 1 is satisfied also for $(-\theta, -\omega)$-rays. This agrees with the relation $s(t, -\omega, -\theta) = s(t, \theta, \omega)$.

Remark 1.3. — Under stronger assumptions concerning the rays incoming with directions $\pm\omega$, the relation (1.3) was examined in [11].

The inclusion (1.3) is similar to the Poisson relation for the distribution $\sigma(t) = \sum_{j=1}^{\infty} \cos \lambda_j t$, where $\{\lambda_j^2\}_{j=1}^{\infty}$ is the spectrum of the Laplace operator in a bounded domain with smooth boundary (*see* [1], [13]).

From physical point of view it is more interesting to study the obstacles for which (1.3) becomes an equality. This makes it possible to recover all singularities of $s(t, \theta, \omega)$ and to consider them as scattering data (*see* [16] for a result in this direction). One way to attack this problem is to fix $\theta \neq \omega$ and to consider generic obstacles. We follow this way in the present paper and show that generically for some ordinary (ω, θ)-rays γ we have

$$-T_\gamma \in \operatorname{sing\,supp} s(t, \theta, \omega). \tag{1.4}$$

Recently, one of the authors [21] proved that for generic obstacles in $\mathbb{R}^3$, (1.4) holds for any (ω, θ)-ray γ. The proof of this result is based on Theorem 2 stated below and the fact that for fixed $\omega \neq \theta$ and generic obstacles K in $\mathbb{R}^3$ there are no generalized (ω, θ)-rays in the complement of K.

Another way to study (1.3) is to fix K and ω and to consider generic directions θ. For some obstacles K it is known (*see* [16], [12]) that for every fixed $\omega \in S^{n-1}$ there exists a residual subset $\mathfrak{R}(\omega)$ of S^{n-1} such that for every $\theta \in \mathfrak{R}(\omega)$ all (ω, θ)-rays in $\mathbb{R}^n \setminus K$ are ordinary. For such directions we can apply Theorem 1 and obtain (1.4) for all (ω, θ)-rays. We conjecture that for each obstacle and each fixed ω it is possible to find a residual subset $\mathfrak{R}(\omega)$ with the properties mentioned above.

To state our second result we need some notations.

Let $X = \partial\Omega$ and let $C^\infty(X, \mathbb{R}^n)$ be *the space of all* C^∞ *maps* of X into $\mathbb{R}^n$ endowed with *the Whitney* C^∞ *topology* (*cf.* [3], ch. II). The subspace

$C^\infty_{emb}(X, \mathbb{R}^n)$ *of all* C^∞ *embeddings* is open in $C^\infty(X, \mathbb{R}^n)$, hence it is Baire space. A subset $\mathfrak{R}$ of a topological space Z is called *residual* if $\mathfrak{R}$ is a countable intersection of open dense subsets of Z.

Given $f \in C^\infty_{emb}(X, \mathbb{R}^n)$, denote by Ω_f the *unbounded domain with boundary* $f(X)$ and by $\mathscr{L}_{\omega, \theta, f}$ *the set of all* (ω, θ)*-rays in* $\bar{\Omega}_f$. Let $L_{\omega, \theta, f}$ (resp. $\mathscr{L}^g_{\omega, \theta, f}$) be *the set of all ordinary* (resp. *generalized*) (ω, θ)*-rays in* $\bar{\Omega}_f$. The results of section 4, combined with those in [14], [15], imply the existence of a residual subset $\mathfrak{R}$ of $C^\infty_{emb}(X, \mathbb{R}^n)$ such that for each $f \in \mathfrak{R}$ we have

$$\mathscr{L}_{\omega, \theta, f} = L_{\omega, \theta, f} \cup \mathscr{L}^g_{\omega, \theta, f}.$$

In particular, if $\mathscr{L}^g_{\omega, \theta, f} = \varnothing$, then every (ω, θ)-ray is an ordinary one.

If γ is an ordinary (ω, θ)-ray, we denote by x_γ (resp. y_γ) the first (resp. the last) reflection point of γ. Let m_γ be the number of reflections of γ and let $dJ_\gamma(u_\gamma)$ be the differential of the map J_γ introduced in section 2. Here u_γ is the orthogonal projection of x_γ on Z_1. Finally, set

$$\mathfrak{G}_f = \{ T_\gamma : \gamma \in \mathscr{L}^g_{\omega, \theta, f} \}.$$

Our second result is the following.

THEOREM 2. — *Let* $\theta \neq \omega$ *be fixed. Then there exists a residual subset* $\mathscr{A}$ *of* $C^\infty_{emb}(X, \mathbb{R})$ *such that for each* $f \in \mathscr{A}$

$$\{ -T_\gamma : \gamma \in L_{\omega, \theta, f}, T_\gamma \notin \mathfrak{G}_f \} \subset \operatorname{sing\,supp} s_f(t, \theta, \omega) \tag{1.5}$$

holds, where $s_f(t, \theta, \omega)$ *is the scattering kernel related to* Ω_f. *Moreover, for* t *sufficiently close to* $-T_\gamma$ *with* $\gamma \in L_{\omega, \theta, f}$, $T_\gamma \notin \mathfrak{G}_f$, *we have*

$$s_f(t, \theta, \omega) = C \left| \frac{\det dJ_\gamma(u_\gamma) \langle \nu(x_\gamma), \omega \rangle}{\langle \nu(y_\gamma), \theta \rangle} \right|^{-1/2} \delta^{(n-1)/2}(t + T_\gamma) + \textit{smoother therms}, \tag{1.6}$$

where $C = (2\pi)^{(1-n)/2} (-1)^{m_\gamma - 1} i^{\sigma_\gamma}$ *and* $\sigma_\gamma \in \mathbb{N}$ *is related to a Maslov index.*

For the proof of Theorem 1 we use the results in [9] for propagation of C^∞ singularities. The crucial point is the application of Proposition 3.1, where we generalize an idea used previously in [11].

Given $\rho(t + t_0) \in C^\infty_0(\mathbb{R}^n)$ with support in a small neighbourhood of $-t_0$, we need to examine the asymptotic of

$$I(\lambda) = (s(t, \theta, \omega), \rho(t + t_0) e^{-i\lambda t}).$$

The results for propagation of singularities of the solution of (1.2) are not sufficient since some critical points of the phase of $I(\lambda)$ make contributions which must be cancelled from physical point of view. Thus we are going to use a stationary approach connected with the $(i\lambda)$-outgoing Green function.

The proof of Theorem 2 is based essentially on some generic properties of (ω, θ)-rays with linear segments. These properties are obtained in section 4 following the approach in [13], [19]. Some of these properties have been previously announced in [20], [15]. The formula (1.6) has been obtained in [11].

The paper is organized as follows. In section 2 we collect some notations and definitions. Theorem 1 is proved in section 3. In section 4 we consider several generic properties of reflecting (ω, θ)-rays and prove Theorem 2.

2. PRELIMINARIES

2.1. By a *segment* in $\mathbb{R}^n$ we mean either a finite segment $[x, y]$ or an infinite one, that is a straightline ray starting at some point and having a given direction.

Let X be a smooth compact $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geqq 2$. If l_1 and l_2 are two segments in $\mathbb{R}^n$ with a common end $x \in X$, we say that l_1 and l_2 *satisfy the law of reflection at* x (with respect to X) if l_1 and l_2 make equal acute angles with a normal vector $\nu_x \neq 0$ to X at x and l_1, l_2 and ν_x lie in a common two-dimensional plane.

2.2. DEFINITION. – Let ω and θ be two fixed unit vectors in $\mathbb{R}^n$. Consider a curve $\gamma = \bigcup_{i=0}^{k} l_i$, where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, \ldots, k-1$ $(k \geqq 1)$, $x_i \in X$ for all i, l_0 (resp. l_k) is the infinite segment starting at x_1 (resp. x_k) and having direction $-\omega$ (resp. θ). Then the curve γ is called a *reflecting* (ω, θ)-*ray* on X if the following conditions are satisfied:

(i) the open segments $\mathring{l}_i$ do not intersect transversally X;

(ii) either $l_i \cap l_{i+1} = \{x_{i+1}\}$ for every $i = 0, 1, \ldots, k-1$ or $k = 2m+1$ $(m = 0, 1, \ldots)$, $l_i \cap l_{i+1} = \{x_{i+1}\}$ for $i = 0, 1, \ldots, m$ and $l_{m-i} = l_{m+i-1}$ for $i = 0, 1, \ldots, m$;

(iii) for every i the segments l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to X.

The points $x_1, \ldots, x_k$ will be called *reflection points of* γ. If γ is of the same form and has the above properties except (i) for $i = k$, we shall say that γ is a (ω, θ)-*trajectory on* X. Note that every reflecting (ω, θ)-ray is a (ω, θ)-trajectory, but the converse is not true in general since the last segment (which is infinite and has direction θ) of a (ω, θ)-trajectory could intersect X. Mention also that the second part of (ii) is only possible for $\theta = -\omega$.

2.3. Suppose $\mathfrak{R} \subset C^{\infty}_{\text{emb}}(X, \mathbb{R}^n)$ and $(U_k)_{k=1}^{\infty}$ is a sequence of open subsets of $\mathbb{R}^n$ with $\bigcup_{K} U_k = \mathbb{R}^n$ and $U_k \supset X$ for every k. Assume in addition that $\mathfrak{R}$ contains a residual subset of $C^{\infty}_{\text{emb}}(X, U_k)$ for every k. Then it is easily seen that $\mathfrak{R}$ contains a residual subset of $C^{\infty}_{\text{emb}}(X, \mathbb{R}^n)$.

2.4. Let ω, $\theta \in S^{n-1}$ be fixed and U_0 be an open ball with radii $a>0$ containing X. Let Z_1 and Z_2 be the hyperplanes tangent to U_0 such that Z_1 (resp. Z_2) is orthogonal to ω (resp. θ) and the halfspace H_1 (resp. H_2), determined by Z_1 and ω (resp. by Z_2 and $-\theta$) contains U_0. Given a reflecting (ω, θ)-ray γ on X with successive reflection points $x_1, \ldots, x_k$, the *sojourn time* T_γ of γ (*cf.* Guillemin [4]) is defined by

$$T_\gamma = \| \pi_1(x_1) - x_1 \| + \sum_{i=1}^{k-1} \| x_i - x_{i+1} \| + \| x_k - \pi_2(x_k) \| - 2a,$$

where $\pi_i : \mathbb{R}^n \to Z_i$ are the orthogonal projections. Clearly, $T_\gamma + 2a$ is the length of this part of γ which lies in $H_1 \cap H_2$. We define T_γ when γ is a (ω, θ)-trajectory or a generalized (ω, θ)-ray so that $T_\gamma + 2a$ is the length of this part of γ which lies in $H_1 \cap H_2$. It is known [4] that the definition of T_γ does not depend on the choice of the ball U_0. Set $u_\gamma = \pi_1(x_1)$ and assume that γ is a (ω, θ)-trajectory which has no segments tangent to X. Then there exists a neighbourhood W_γ of u_γ in Z_1 such that for every $u \in W_\gamma$ there are unique $\theta(u) \in S^{n-1}$ and points $x_1(u), \ldots, x_k(u) \in X$ which are the successive reflection points of a $(\omega, \theta(u))$-trajectory on X with $\pi_1(x_1(u)) = u$. We set $J_\gamma(u) = \theta(u)$, thus obtaining a map

$$J_\gamma : \quad W_\gamma \to S^{n-1}.$$

This map was also introduced by Guillemin [4].

Given a set A and an integer $s \geqq 2$, we set

$$A^{(s)} = \{ (a_1, \ldots, a_s) \in A^s : a_i \neq a_j \text{ whenever } i \neq j \}.$$

If $f : X \to Y$ is a map, by $f^s : X^s \to Y^s$ we denote the map given by $f^s(x_1, \ldots, x_s) = (f(x_1), \ldots, f(x_s))$.

3. SINGULARITIES OF THE SCATTERING KERNEL

Let $\rho(t) \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} \rho \subset (-1, 1)$, $\rho(t) = 1$ for $|t| \leqq 1/2$. Set $\rho_\delta(t) = \rho(t/\delta)$, $0 < \delta \leqq 1$. Let $v \in \bar{\mathscr{D}}'(\mathbb{R} \times \Omega)$ be the solution of the problem

$$\begin{gathered} \Box v = F \quad \text{in } \mathbb{R} \times \Omega, \\ v = h \quad \text{on } \mathbb{R} \times \partial\Omega, \\ v_{|t<\tau} = 0, \end{gathered}$$

where $\tau < -\rho_0$ is fixed. Here $F \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$, $h \in H_{\text{loc}}^s(\mathbb{R} \times \partial\Omega)$ with some $s < 0$ and $F = 0$, $h = 0$ for $t < \tau$. By $\bar{\mathscr{D}}'(\mathbb{R} \times \Omega)$ we denote the space of all distributions in $\mathbb{R} \times \Omega$ admitting extensions as distributions on $\mathbb{R}_t \times \mathbb{R}_x^n$.

Then the traces $\left.\frac{\partial^j v}{\partial \nu^j}\right|_{\mathbb{R}\times\partial\Omega} \in \mathscr{D}'(\mathbb{R}\times\partial\Omega)$, $j=0, 1$, exist since $\mathbb{R}\times\partial\Omega$ is non-characteristic for $\square$ (*see* [5]). Let

$$T_1=\sup\left\{ t : t \leqq \rho_0+|t_0|+\delta, \text{there exists } y\in\partial K \text{ with } (t, y)\in(\text{sing supp } h)\cup\left(\text{sing supp}\left(\left.\frac{\partial v}{\partial \nu}\right|_{\mathbb{R}\times\partial K}\right)\right)\right\}.$$

Consider the integral

$$I(\lambda)=\int_{\mathbb{R}}\int_{\partial K} e^{i\lambda(t-\langle y, \theta\rangle)}\rho_\delta(\langle y, \theta\rangle - t+t_0)\left(\frac{\partial}{\partial \nu}-\langle \nu, \theta\rangle\frac{\partial}{\partial t}\right) v\, dt\, dS_y.$$

For the proof of Theorem 1 we need the following

PROPOSITION 3.1. – *Assume that for some* ε, $0<\varepsilon\leqq 1$, *we have*

$$\mathrm{WF}(v)\cap\{(t, y, 1, -\theta)\in T^*(\mathbb{R}\times\bar{\Omega}) : T_1+\varepsilon\leqq t\leqq T_1+2\varepsilon,\ |y|\leqq\tau_1+T_1+2\varepsilon\}=\varnothing, \quad (3.1)$$

where $\tau_1=\rho_0-\tau$. *Then*

$$I(\lambda)=O(|\lambda|^{-m}) \quad \textit{for all } m\in\mathbb{N}.$$

Proof. – Choose two functions $\alpha(t)\in C_0^\infty(\mathbb{R})$, $\beta(x)\in C_0^\infty(\mathbb{R}^n)$ such that:

$$\alpha(t)=\begin{cases}1 & \text{for } \ t\leqq T_1+\varepsilon,\\ 0 & \text{for } \ t\geqq T_1+2\varepsilon,\end{cases}$$

$$\beta(x)=\begin{cases}1 & \text{for } \ |x|\leqq\tau_1+T_1+2\varepsilon,\\ 0 & \text{for } \ |x|\geqq\tau_1+T_1+3\varepsilon.\end{cases}$$

For the distribution $\tilde{v}(t, x)=\alpha(t)\beta(x)v(t, x)$ we obtain the problem

$$\begin{gathered}\square\tilde{v}=\tilde{F} \quad \text{in } \mathbb{R}\times\Omega,\\ \tilde{v}=\alpha\beta h \quad \text{on } \mathbb{R}\times\partial\Omega,\\ \tilde{v}_{|t<\tau}=0\end{gathered}$$

with

$$\tilde{F}=2\alpha_t\beta v_t+\alpha_{tt}\beta v-2\alpha\langle\nabla\beta, \nabla v\rangle-\alpha(\Delta\beta)v+\alpha\beta F.$$

By a finite speed of propagation argument we conclude that $v\in C^\infty$ for $t\leqq T_1+2\varepsilon$, $|x|\geqq\tau_1+T_1+2\varepsilon$. This shows that $\tilde{F}$ is singular only for $T_1+\varepsilon\leqq t\leqq T_1+2\varepsilon$. Then the assumption (3.1) implies

$$\mathrm{WF}(\tilde{F})\cap\{(t, y, 1, -\theta)\in T^*(\mathbb{R}\times\bar{\Omega})\}=\varnothing. \quad (3.2)$$

Since

$$\mathrm{WF}(v_{|\mathbb{R}\times\Omega})\subset\{(t, x, \tau, \xi)\in T^*(\mathbb{R}\times\Omega)\setminus\{0\} : \tau^2=|\xi|^2\},$$

by a standard argument we deduce that for each $m>0$ there exists $s(m)<0$ so that

$$v \in \mathrm{H}^{s(m)}_{\mathrm{loc}}(\mathbb{R}_t; \mathrm{H}^m_{\mathrm{loc}}(\Omega)).$$

We can take the partial Fourier transformation with respect to t of $\tilde{v}$ and $\tilde{\mathrm{F}}$. Put

$$\begin{aligned} \mathrm{V}(x,\lambda) &= (\tilde{v}(t,x), e^{-i\lambda t}), \\ f(x,\lambda) &= (\tilde{\mathrm{F}}(t,x), e^{-i\lambda t}), \\ g(x,\lambda) &= (\alpha\beta h(t,x), e^{-i\lambda t}). \end{aligned}$$

The existence of the Fourier transformation of $h(t,x)$ follows from the fact that $\mathrm{WF}(v_{|\mathbb{R}\times\partial\mathrm{K}})$ is contained in the set of hyperbolic and glancing points of $\square$ (*see* [5], [9]). We obtain the problem

$$\begin{gathered} (\Delta+\lambda^2)\mathrm{V}(t,x) = -f(x,\lambda) \quad \text{in } \Omega, \\ \mathrm{V}=g \quad \text{on } \partial\mathrm{K}, \\ \mathrm{V} \text{ is a } i\lambda\text{-outgoing solution.} \end{gathered}$$

The latter condition means that for $|x| \to \infty$ we have the representation

$$\mathrm{V}(x,\lambda) = \int_{\partial\mathrm{K}} \left[\frac{\partial \mathrm{V}}{\partial \nu}(y,\lambda)\,\mathrm{G}^+_\lambda(x-y) - \mathrm{V}(y,\lambda)\frac{\partial}{\partial\nu}\mathrm{G}^+_\lambda(x-y) \right] d\mathrm{S}_y - \int_\Omega \mathrm{G}^+_\lambda(x-y) f(y,\lambda)\,dy. \quad (3.3)$$

Here the integrals are taken in the sense of distributions and $\mathrm{G}^+_\lambda(x)$ is the $(i\lambda)$-outgoing Green function of the operator $\Delta+\lambda^2$ (*cf.* [7]). More precisely,

$$\mathrm{G}^+_\lambda(x) = \frac{(-1)^{(n-1)/2}}{2(2\pi)^{(n-1)/2}}((1/r)\,\partial_r)^{(n-3)/2}(e^{-i\lambda r}/r), \qquad r=|x|.$$

Notice that for $|x| \to \infty$ we have

$$\mathrm{G}^+_\lambda(x) = \mathrm{Const.}\,\lambda^{(n-3)/2} e^{-i\lambda|x|}/|x|^{(n-1)/2} + O(1/|x|^{(n+1)/2}).$$

We set in (3.3) $x=r\theta$, $r=|x|$, and multiply (3.3) by $r^{(n-1)/2}e^{i\lambda r}$. Taking the limit as $r\to\infty$, we get

$$\int_{\partial\mathrm{K}} e^{i\lambda\langle y,\theta\rangle}\left[\frac{\partial\mathrm{V}}{\partial\nu}(y,\lambda) - i\lambda\langle \nu,\theta\rangle \mathrm{V}(y,\lambda)\right] d\mathrm{S}_y = \int_{\mathbb{R}}\int_\Omega e^{-i\lambda(t-\langle y,\theta\rangle)}\tilde{\mathrm{F}}(t,y)\,dt\,dy, \quad (3.4)$$

where the integrals are taken in the sense of distributions. The condition (3.2) shows that the right-hand side of (3.4) can be estimated by $O(|\lambda|^{-m})$

for all $m \in \mathbb{N}$. Thus we deduce

$$(2\pi)^{-1}\int_{\mathbb{R}}\left(\int_{\partial K} e^{i\lambda\langle y,\theta\rangle}\left[\frac{\partial V}{\partial \nu}(y,\lambda)-i\lambda\langle \nu,\theta\rangle V(y,\lambda)\right]dS_y\right)e^{i\lambda t}\,d\lambda$$
$$=\int_{\partial K}\left(\frac{\partial \tilde{v}}{\partial \nu}-\langle \nu,\theta\rangle\frac{\partial \tilde{v}}{\partial t}\right)(t+\langle y,\theta\rangle, y)\,dS_y \in C_0^\infty(\mathbb{R}).$$

Next,

$$\int_{-\infty}^{\infty}\left(\int_{\partial K}\left(\frac{\partial \tilde{v}}{\partial \nu}-\langle \nu,\theta\rangle\frac{\partial \tilde{v}}{\partial t}\right)(t+\langle y,\theta\rangle, y)\,dS_y\right)e^{i\lambda t}\rho_\delta(-t+t_0)\,dt$$
$$=\int_{-\infty}^{\rho_0+|t_0|+\delta}\int_{\partial K} e^{i\lambda(t-\langle y,\theta\rangle)}\rho_\delta(\langle y,\theta\rangle-t+t_0)\left(\frac{\partial \tilde{v}}{\partial \nu}-\langle \nu,\theta\rangle\frac{\partial \tilde{v}}{\partial t}\right)dt\,dS_y$$
$$=I(\lambda)+O(|\lambda|^{-m})\quad \text{for all } m\in\mathbb{N}.$$

The left-hand side can be estimated by $O(|\lambda|^{-m})$ and this completes the proof of Proposition 3.1.

Proof of Theorem 1. — We shall recall some properties of the generalized Hamiltonian flow established by Melrose and Sjöstrand [9]. Our assumption implies that if there exists a (ω, θ)-ray γ passing through ρ_u, then $C_t(u)=\gamma(t)$, where $\gamma(t)$ is the generalized bicharacteristic the projection of which on $\bar{\Omega}$ is γ.

Consider the map $Z_1 \times \mathbb{R} \ni (u, t) \to C_t(u)$. Melrose and Sjöstrand proved (*cf.* Theorem 3.22 in [9], II) that $C_t(u)$ is continuous with respect to the metric $D(\rho, \mu)$ (*cf.* section 3 in [9], II for the definition of $D(\rho, \mu)$). In particular, for fixed $\varepsilon>0$ and $T>0$ there exists a neighbourhood U of u_0 in Z_1 such that for each $u \in U$ and each $t\in[-\rho_0, T]$ we have

$$\max\{D(\rho,\mu):\rho\in C_t(u), \mu\in C_t(u_0)\}<\varepsilon.$$

Let $-t_0$ be fixed so that

$$-t_0\notin\{-T_\gamma:\gamma\in\mathscr{L}_{\omega,\theta}\}.$$

Choose $T>0$ with $|t_0|<T$. Since the set

$$\{T_\gamma:|T_\gamma|\leqq T, \gamma\in\mathscr{L}_{\omega,\theta}\}$$

is closed, we can find $\varepsilon_0>0$ such that

$$T_\gamma\notin[t_0-\varepsilon_0, t_0+\varepsilon_0]\quad\text{for all } \gamma\in\mathscr{L}_{\omega,\theta}. \tag{3.5}$$

We shall study sing supp $s(t, \theta, \omega)$ for $|t|\leqq T$ and fixed $\theta\neq\omega$. Let $0<\delta\leqq\varepsilon_0/2$, then

$$(s(t,\theta,\omega), \rho_\delta(t+t_0)e^{-i\lambda t})=J(\lambda)$$
$$=\sum_{k=0}^{n-2}c_k(-i\lambda)^{n-2-k}\int_{\mathbb{R}}\int_{\partial K} e^{i\lambda(t-\langle y,\theta\rangle)}\rho_\delta^{(k)}(\langle y,\theta\rangle-t+t_0)\frac{\partial w}{\partial \nu}(t,y;\omega)\,dt\,dS_y$$

with $c_k = \text{Const.}$, $c_0 = C_n$, $\rho_\delta^{(k)} = \dfrac{d^k \rho_\delta}{dt^k}$. We shall examine the integral for $k=0$; the analysis of the others is completely analoguous.

Obviously, we have to study the singularities of w for $|t| \leqq \rho_0 + T + \delta$. Without loss of generality we may assume that $\omega = (0, \ldots, 0, 1)$. Consider the hyperplane

$$Z_1 = \{ x \in \mathbb{R}^n : x_n = \tau \},$$

where $\tau < -\rho_0$ is fixed. For $\varphi_j(x') \in C_0^\infty(\mathbb{R}^{n-1})$, $x' = (x_1, \ldots, x_{n-1})$, consider the Cauchy problem

$$\begin{aligned} \square\, v_j &= 0 \quad \text{in } \mathbb{R}_\tau^+ \times \mathbb{R}_x^n, \\ v_j |_{t=\tau} &= \varphi_j(x')\, \delta(\tau - x_n), \\ \left.\frac{\partial v_j}{\partial t}\right|_{t=\tau} &= \varphi_j(x')\, \delta'(\tau - x_n), \end{aligned} \tag{3.6}$$

where $\mathbb{R}_\tau^+ = \{ t \in \mathbb{R} : t > \tau \}$, and the mixed problem

$$\begin{aligned} \square\, W_j &= 0 \quad \text{in } \mathbb{R} \times \Omega, \\ W_j &= 0 \quad \text{on } \mathbb{R} \times \partial\Omega, \\ W_j |_{t=\tau} &= \varphi_j(x')\, \delta(\tau - x_n), \\ \left.\frac{\partial W_j}{\partial t}\right|_{t=\tau} &= \varphi_j(x')\, \delta'(\tau - x_n). \end{aligned}$$

Clearly, there exists a compact set $F_0' \subset \mathbb{R}^{n-1}$ such that if $\operatorname{supp} \varphi_j \cap F_0' = \varnothing$, then

$$WF\left(\left.\frac{\partial W_j}{\partial \nu}\right|_{\mathbb{R} \times \partial K} \right) \cap \{ (t, y, 1, -\theta_{|T_y(\partial K)}) : y \in \partial K \} = \varnothing. \tag{3.7}$$

Then we obtain

$$\int_{\mathbb{R}} \int_{\partial K} e^{i\lambda(t - \langle y, \theta \rangle)} \rho_\delta(\langle y, \theta \rangle - t + t_0) \frac{\partial W_j}{\partial \nu} dt\, dS_y = O(|\lambda|^{-m}), m \in \mathbb{N}. \tag{3.8}$$

Set $F_0 = \{ x \in \mathbb{R}^n : x' \in F_0', x_n = \tau \}$. For $u_0 \in F_0$ denote by $l(u_0)$ the straight-line ray issued from u_0 in direction ω. Let $l(u_0)$ has a direction ω for $0 \leqq t \leqq T$. Assume that

$$\varnothing \neq l(u_0) \cap K \subset \partial K,$$

that is $l(u_0)$ meets ∂K only at points, where $l(u_0)$ is tangent to ∂K. Then $l(u_0)$ is the projection on $\bar{\Omega}$ of a uniquelly extendible bicharacteristic $\gamma_0(t)$ of $\square$ which is determined uniquelly by the Hamiltonian flow of $\square$. Consequently, $C_t(u_0) = \gamma_0(t)$. Choosing a small neighbourhood $\mathcal{O}(u_0)$ of u_0 and φ_j with $\operatorname{supp} \varphi_j \subset \mathcal{O}(u_0)$, the results on propagation of singularities [9] and the continuity of the $C_t(u)$, discussed above, imply (3.7) for $|t| \leqq T$. Thus for such W_j we have (3.8).

If the case described above does not occur, then $l(u_0)$ has common points with the interior of K. Denote by $x_1(u_0)$ the point on $l(u_0)$ such that the segment $[u_0, x_1(u_0)]$ is the maximal one which has no common points with the interior of K. There are two possibilities:

(1) $l(u_0)$ meets transversally ∂K at $x_1(u_0)$;

(2) $l(u_0)$ is tangential to ∂K at $x_1(u_0)$ and ω is an asymptotic direction for ∂K at $x_1(u_0)$.

Let $t_1(u_0)=|u_0-x_1(u_0)|$. It is easy to show that

$$\mathrm{WF}(v_j)\subset\{(t,x,\pm\sigma,\mp\sigma\omega)\in T^*(\mathbb{R}^{n+1})\setminus\{0\}:\sigma>0,\text{ there are }\hat{x}\in Z_1,\hat{x}'\in\operatorname{supp}\varphi_j\text{ and }s\geqq 0\text{ with }t=\tau\pm s,\ x=\hat{x}\pm s\omega\}.$$

In the case (1) we modify v_j in the interior of K in a small neighbourhood of $x_1(u_0)$, provided supp φ_j is sufficiently small. We denote the modified v_j by $\tilde{v}_j$ and arrange $\tilde{v}_j=0$ for $t>t_1+\varepsilon_1$, where $t_1=\max\{t_1(u):u\in\mathcal{O}(u_0)\}$, while $\mathcal{O}(u_0)$ and ε_1 are chosen sufficiently small. In the case (2) we repeat the same procedure modifying v_j in the interior of K. This is possible since $l(u_0)$ enters the interior of K.

Clearly, $h_j=\tilde{v}_{j\,|\,\mathbb{R}_\tau^+\times\partial\Omega}=0$ for t sufficiently close to τ. Extending h_j as 0 for $t<\tau$, denote by w_j the solution of the problem

$$\begin{aligned}\Box w_j&=0 \quad \text{in } \mathbb{R}\times\Omega,\\ w_j+h_j&=0 \quad \text{on } \mathbb{R}\times\partial\Omega,\\ w_{j\,|\,t<\tau}&=0.\end{aligned}\tag{3.9}$$

Since $\frac{\partial}{\partial t}(w_j+\tilde{v}_j)_{|\,\mathbb{R}_\tau^+\times\partial K}=0$, we have to study the integrals

$$\mathrm{I}_{j,\delta}(\lambda)=\int_{\mathbb{R}}\int_{\partial K}e^{i\lambda(t-\langle y,\theta\rangle)}\rho_\delta(\langle y,\theta\rangle-t+t_0)\left(\frac{\partial}{\partial\nu}-\langle\nu,\theta\rangle\frac{\partial}{\partial t}\right)w_j\,dt\,dS_y,$$

$$\mathrm{J}_{j,\delta}(\lambda)=\int_{\mathbb{R}}\int_{\partial K}e^{i\lambda(t-\langle y,\theta\rangle)}\rho_\delta(\langle y,\theta\rangle-t+t_0)\left(\frac{\partial}{\partial\nu}-\langle\nu,\theta\rangle\frac{\partial}{\partial t}\right)\tilde{v}_j\,dt\,dS_y.$$

It is easy to see that

$$\mathrm{J}_{j,\delta}(\lambda)=O(|\lambda|^{-m})\quad\text{for all } m\in\mathbb{N}.\tag{3.10}$$

Indeed, observe that for small $\varepsilon>0$ we have $v_j=\tilde{v}_j$ for $\tau\leqq t\leqq\tau+\varepsilon<-\rho_0$. Then $\theta\neq\omega$ yields

$$\mathrm{WF}(\tilde{v}_j)\cap\{(t,y,1,-\theta)\in T^*(\mathbb{R}^{n+1}):\tau\leqq t\leqq\tau+\varepsilon\}=\varnothing.$$

Choose a function $\alpha_1(t)\in C^\infty(\mathbb{R})$ such that

$$\alpha_1(t)=\begin{cases}0 & \text{for } t\leqq\tau+\varepsilon/2,\\ 1 & \text{for } t\geqq\tau+\varepsilon.\end{cases}$$

Then we obtain (3.10) applying the argument of the proof of Proposition 3.1 for $\alpha_1(t)\tilde{v}_j(t,x)$.

Thus it remains to study $I_{j,\delta}(\lambda)$. Next, for each $u_0 \in F_0$, satisfying (1) or (2), we introduce a sufficiently small neighbourhood $\mathcal{O}(u_0) \subset Z_1$, and we take supp $\varphi_j \subset \mathcal{O}(u_0)$. Thus the singularities of w_j are localized along the generalized rays $\gamma(u_0)$ issued from $u_0 \in F_0$ in direction ω.

There are two cases.

Case A. – For all $\sigma > \rho_0 + T + 1$ we have

$$C_\sigma(u_0) \cap \{(\sigma, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) : \rho_0 \leqq |x| \leqq \tau_1 + \sigma + 1\} = \varnothing. \quad (3.11)$$

Then for all $t \geqq \tau$ we obtain

$$C_t(u_0) \cap \{(t, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) : |x| \geqq \rho_0\} = \varnothing.$$

Indeed, assume that for some $\tau \leqq \hat{t}$ we can find a generalized bicharacteristic $\gamma(\hat{t}; u_0) \subset C_{\hat{t}}(u_0)$ such that

$$(\hat{t}, \hat{x}, 1, -\theta) \in \gamma(\hat{t}; u_0) \qquad \text{with} \quad |\hat{x}| \geqq \rho_0.$$

Then $\gamma(\sigma; u_0)$ has direction θ for all $\sigma \geqq \hat{t}$, and we obtain a contradiction with (3.11).

By using the continuity of $C_t(u_0)$ with respect to t and u_0, we can find a small neighbourhood $\mathcal{O}(u_0)$ so that for all $u \in \mathcal{O}(u_0)$ and all $t \in [\tau, \rho_0 + T + 2]$ we have

$$C_t(u) \cap \{(t, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) : \rho_0 \leqq |x| \leqq \rho_0 + 2\} = \varnothing. \quad (3.12)$$

Now let $\beta(x) \in C_0^\infty(\mathbb{R}^n)$ be a function such that

$$\beta(x) = \begin{cases} 1 & \text{for} \quad |x| \leqq \rho_0, \\ 0 & \text{for} \quad |x| \geqq \rho_0 + 1. \end{cases}$$

For supp $\varphi_j \subset \mathcal{O}(u_0)$ and for w_j we obtain

$$\square(\beta w_j) = -2\langle \nabla_x \beta, \nabla_x w_j \rangle - (\Delta\beta) w_j = F_j.$$

Applying the results for propagation of singularities and (3.12), we conclude that

$$WF(F_j) \cap \{(t, x, 1, -\theta) \in T^*(\mathbb{R} \times \bar{\Omega}) : \tau \leqq t \leqq \rho_0 + T + 2\} = \varnothing. \quad (3.13)$$

It is easy to see that the Fourier transform

$$\tilde{w}_j(x, \lambda) = F_{t \to \lambda}(\beta w_j)$$

exists. To check this it is sufficient to use the $(i\lambda)$-outgoing condition and to prove that the solution of the problem

$$\begin{gathered} (\Delta + \lambda^2) W_j = 0 \quad \text{in } \Omega, \\ W_j = -F_{t \to \lambda}(h_j) \quad \text{on } \partial\Omega, \\ W_j \text{ is } (i\lambda)\text{-outgoing}, \end{gathered}$$

is a tempered distribution with respect to λ.

Setting $\tilde{F}_j(x, \lambda) = F_{t \to \lambda}(F_j)$, as in the proof of Proposition 3.1 we obtain

$$\int_{\partial K} e^{i\lambda \langle y, \theta \rangle} \left(\frac{\partial \tilde{w}_j}{\partial \nu}(y, \lambda) - i\lambda \langle \nu, \theta \rangle \tilde{w}_j(y, \lambda) \right) dS_y = \int_{\Omega} e^{i\lambda \langle y, \theta \rangle} \tilde{F}_j(y, \lambda)\, dy.$$

Taking the inverse Fourier transform, we deduce

$$\int_{\partial K} \left(\frac{\partial w_j}{\partial \nu} - \langle \nu, \theta \rangle \frac{\partial w_j}{\partial t} \right)(t + \langle y, \theta \rangle, y)\, dS_y = \int_{\Omega} F_j(t + \langle y, \theta \rangle, y)\, dy.$$

Then the relation (3.13) leads to

$$I_{j,\delta}(\lambda) = \int_{\mathbb{R}} \int_{\Omega} e^{i\lambda (t - \langle y, \theta \rangle)} \rho_\delta(\langle y, \theta \rangle - t + t_0) F_j(t, y)\, dt\, dy$$
$$= O(|\lambda|^{-m}) \quad \text{for all } m \in \mathbb{N}.$$

Case B. – For some $\sigma > \rho_0 + T + 1$ we have

$$C_\sigma(u_0) \cap \{ (\sigma, x, 1, -\theta) \in T^*(\mathbb{R} \times \Omega) : \rho_0 \leqq |x| \leqq \tau_1 + \sigma + 1 \} \neq \varnothing.$$

Then there exists a generalized bicharacteristic $\gamma(t; u_0)$ issued from u_0 in direction ω passing through some point y for $t = \sigma$, $|y| \geqq \rho_0$, with direction θ. The projection of $\gamma(t; u_0)$ on $\bar{\Omega}$ is a (ω, θ)-ray γ, and our assumption yields $C_t(u_0) = \gamma(t; u_0)$. Let T_γ be the sojourn time of γ and let

$$\gamma(t; u_0) = (t, x(t), 1, -\xi(t)) \in T^*(\mathbb{R} \times \bar{\Omega}),\ |\xi(t)| = 1,\ t \geqq \tau.$$

Introduce the numbers

$$T_2 = \inf\{ \sigma : \sigma \geqq \tau, \xi(t) = \theta \text{ for } t \geqq \sigma \},$$
$$T_3 = \inf\{ \sigma : \sigma \geqq \tau, x(t) \notin \partial K \text{ for } t > \sigma \}.$$

Notice that $T_2 \leqq T_3$. Then

$$I_{j,\delta}(\lambda) = \int_{-\infty}^{s} \int_{\partial K} + \int_{s}^{\infty} \int_{\partial K} = I'_{j,\delta}(\lambda) + I''_{j,\delta}(\lambda),$$

where $s < T_2$ will be chosen below. A simple geometrical argument yields $t - \langle x(t), \theta \rangle = T_\gamma$ for $T_2 \leqq t \leqq T_3$. By (3.5) we obtain

$$|\langle x(t), \theta \rangle - t + t_0| \geqq \varepsilon_0 \qquad (T_2 \leqq t \leqq T_3).$$

For small $\mathcal{O}(u_0)$, supp $\varphi_j \subset \mathcal{O}(u_0)$ and $|t| \leqq T_3$ the singularities of w_j are contained in a small neighbourhood of $\gamma(t; u_0)$. This makes it possible to choose $\mathcal{O}(u_0)$ and $T_2 - s$ so small that

$$\xi(s) \neq \theta, \tag{3.14}$$
$$|\langle y, \theta \rangle + t_0 - t| \geqq \varepsilon_0/2 \quad \text{for} \quad t \geqq s \tag{3.15}$$

and

$$(t, y) \in \text{sing supp}(w_j|_{\mathbb{R} \times \partial K}) \cup \text{sing supp}\left(\left. \frac{\partial w_j}{\partial \nu} \right|_{\mathbb{R} \times \partial K} \right).$$

Moreover, we take $s<T_2$ so that either $x(s)\notin\partial K$ or $x(s)\in\partial K$ and $\gamma(s;u_0)$ is a glancing point for $\square$. In the latter case (3.14) implies

$$\xi(s)\neq\theta_{\mid T_{x(s)}(\partial K)}. \tag{3.16}$$

Fixing s, we conclude that

$$I''_{j,\delta}(\lambda)=O(|\lambda|^{-m}) \qquad \text{for all} \quad m\in\mathbb{N}, \tag{3.17}$$

since $\varepsilon_0/2\geqq\delta$ and $\rho_\delta(\langle y,\theta\rangle-t+t_0)=0$ for (t,y) satisfying (3.15).

To deal with $I'_{j,\delta}(\lambda)$, we take $\mathcal{O}(u_0)$ sufficiently small and arrange

$$WF(w_j)\cap\{(s,y,1,-\theta)\in T^*(\mathbb{R}\times\bar{\Omega}):|y|\leqq\tau_1+s+1\}=\varnothing.$$

To do this, we exploit (3.14) and the continuity of $C_s(u)$ for $u\in\mathcal{O}(u_0)$. Since $WF(w_j)$ is closed, we can choose $\varepsilon>0$ so that

$$WF(w_j)\cap\{(t,y,1,-\theta)\in T^*(\mathbb{R}\times\bar{\Omega}):s\leqq t\leqq s+\varepsilon,\ |y|\leqq\tau_1+s+1\}=\varnothing. \tag{3.18}$$

Similarly, we use (3.16) to arrange

$$\left(WF(w_j|_{\mathbb{R}\times\partial K})\cup WF\left(\frac{\partial w_j}{\partial\nu}\Big|_{\mathbb{R}\times\partial K}\right)\right)\cap\{(t,y,1,-\theta_{\mid T_y(\partial K)}):s\leqq t\leqq s+\varepsilon,y\in\partial K\}=\varnothing. \tag{3.19}$$

Next, we take a function $\alpha_2(t)\in C^\infty(\mathbb{R})$ such that

$$\alpha_2(t)=\begin{cases}1 & \text{for} \quad t\leqq T_2-s,\\ 0 & \text{for} \quad t\geqq T_2-s+\varepsilon.\end{cases}$$

By applying (3.19), for $\tilde{w}_j=\alpha_2(t)w_j(t,x)$ we get

$$\tilde{I}_{j,\delta}(\lambda)=\int_{-\infty}^{\infty}\int_{\partial K}e^{i\lambda(t-\langle y,\theta\rangle)}\times\rho_\delta(\langle y,\theta\rangle-t+t_0)\left(\frac{\partial}{\partial\nu}-\langle\nu,\theta\rangle\frac{\partial}{\partial t}\right)\tilde{w}_j\,dt\,dS_y$$
$$=I'_{j,\delta}(\lambda)+O(|\lambda|^{-m}) \quad \text{for all } m\in\mathbb{N}.$$

On the other hand, for $\tilde{w}_j$ we can apply the arguments of the proof of Proposition 3.1, since $\square\tilde{w}_j=\tilde{F}_j$ satisfies (3.2) as a consequence of (3.18) and the finite speed of propagation of singularities. Finally, we conclude that

$$I'_{j,\delta}(\lambda)=O(|\lambda|^{-m}) \quad \text{for all } m\in\mathbb{N}. \tag{3.20}$$

In this way for each $u_0\in F_0$ we have chosen a neighbourhood $\mathcal{O}(u_0)$. We obtain a covering $\{\mathcal{O}(u_0):u_0\in F_0\}$ of F_0, and we may assume

$$F_0\subset\bigcup_{j=1}^{M}\mathcal{O}(u_0^{(j)}).$$

Let for $j=1, \ldots, N$, $N \leqq M$, the points $u_0^{(j)} \in F_0$ satisfy the assumptions in (1) or (2). Choose a partition of unity $\{\varphi_j(x')\}_{j=1}^{\infty}$ of Z_1 so that $\operatorname{supp} \varphi_j \subset \mathcal{O}(u_0^{(j)})$ for $j=1, \ldots, N$ and $(\operatorname{supp} \varphi_j) \cap F_0' = \varnothing$ for $j > M$. Set

$$\tilde{w} = \sum_{j=1}^{N} (w_j + \tilde{v}_j) + \sum_{j>N} W_j.$$

Then

$$\begin{aligned} \square \tilde{w} &= 0 \quad \text{in } \mathbb{R}_\tau^+ \times \Omega, \\ \tilde{w} &= 0 \quad \text{on } \mathbb{R}_\tau^+ \times \partial\Omega, \end{aligned}$$

$$\tilde{w}_{|t=\tau} = \delta(\tau - x_n), \qquad \left.\frac{\partial \tilde{w}}{\partial t}\right|_{t=\tau} = \delta'(\tau - x_n).$$

Consequently, $w = \tilde{w}$ in $\mathbb{R}_\tau^+ \times \bar{\Omega}$ and we can replace w by $\tilde{w}$ in $J(\lambda)$. Then by (3.8), (3.10), (3.17), (3.20) we conclude that $-t_0 \notin \operatorname{sing\,supp} s(t, \theta, \omega)$. This completes the proof of Theorem 1.

4. SOME GENERIC PROPERTIES OF (ω, θ)-TRAJECTORIES

In this section we will use several times the following result of [15].

THEOREM 4.1. – *Let $n \geqq 2$, $s \geqq 2$, p and q be natural numbers and let* U *be an open subset of* $(\mathbb{R}^n)^{(s)}$. *Let*

$$H = (H_1, \ldots, H_p): \quad U \to \mathbb{R}^p$$

be a smooth map such that for every $i=1, \ldots, s$ there exists r_i, $1 \leqq r_i \leqq p$, with $\operatorname{grad}_{y_i} H_{r_i}(y) \neq 0$ for all $y \in U$, $y = (y_1, \ldots, y_s)$. Let $L: U \to \mathbb{R}^q$ be a smooth map such that $dL(y) \neq 0$ for every $y \in U$ with $L(y) = 0$. Denote by T *the set of those $f \in C_{\mathrm{emb}}^{\infty}(X, \mathbb{R}^n)$ such that for every critical point x of $H \circ f^s$ with $f^s(x) \in U$ we have $L(f^s(x)) \neq 0$. Then* T *contains a residual subset of* $C_{\mathrm{emb}}^{\infty}(X, \mathbb{R}^n)$. $\square$

This is Theorem 3.1 (B) of [15], where the assumption for L is stronger, namely, it is required that $dL(y) \neq 0$ for every y in U. However, the proof in [15] holds without any changes if we assume $dL(y) \neq 0$ only for those $y \in U$ with $L(y) = 0$.

Let $\gamma = \bigcup_{i=0}^{k} l_i$ be a (ω, θ)-trajectory on X with $k \geqq 2$. Then l_0 and l_k cannot be orthogonal to X at their end points. If in addition for every $i=1, \ldots, k-1$, $l_i = [x_i, x_{i+1}]$ is not orthogonal to X at x_i and x_{i+1}, then γ will be called a *non-symmetric* (ω, θ)-*trajectory* on X. In this case we set $d(\gamma) = k - s$ (the *defect* of γ), where s is the number of all different reflection points of γ. If some l_i is orthogonal to X at x_i or x_{i+1}, then we must have $\theta=-\omega$, the second part of (ii) in 2.2 is satisfied, and $\gamma=\bigcup_{i=0}^{m} l_i$, where l_m is orthogonal to X at x_{m+1}. In this case γ is a reflecting (ω, θ)-ray, it will be called a *symmetric* ω*-ray* on X, and we set $d(\gamma)=m-s+1$. Note that if γ is a non-symmetric (ω, θ)-trajectory, then $d(\gamma)=0$ means that γ passes only once through each of its reflection points. For symmetric γ, $d(\gamma)=0$ means that γ passes exactly twice through each of its reflection points excluding that of them at which γ is orthogonal to X.

The first main result in this section is the following.

THEOREM 4.2. – *Let* $\mathcal{D}$ *be the set of those* $f\in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ *such that every* (ω,θ)*-trajectory on* $f(X)$ *has zero defect. Then* $\mathcal{D}$ *contains a residual subset of* $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.

This theorem can be proved using arguments similar to those in the proof of Theorem A in [19]. Here we proceed in a different way applying Theorem 4.1 above. This way is simpler and shorter, and can also be used to simplify the proofs in [19] and [15].

We begin with a combinatorical classification of (ω,θ)-trajectories, similar to that used in [13], [19] for periodic reflecting rays.

Let $k\geqq s\geqq 2$ be integers and let

$$\alpha:\quad \{1,\ldots,k\}\to\{1,\ldots,s\} \tag{4.1}$$

be a map with

$$\alpha(i)\neq\alpha(i+1)\qquad (i=1,\ldots,k-1).$$

If

$$\{\alpha(i),\alpha(i+1)\}\neq(\alpha(j),\alpha(j+1)\} \tag{4.2}$$

holds whenever $1\leqq i<j\leqq k-1$, then α will be called a *ns*-map. If $k=2m+1$, (4.2) holds for $1\leqq i<j\leqq m$, and

$$x_{m-i+1}=x_{m+i+1}\qquad (i=0,1,\ldots,m),$$

then α will be called a *s*-map.

In this section we will always assume that α is a *ns*-map or a *s*-map, and by definition we set

$$\alpha(0)=0,\qquad \alpha(k+1)=s+1. \tag{4.3}$$

So α will be considered as a map

$$\alpha:\quad \{0,1,\ldots,k+1\}\to\{0,1,\ldots,s+1\}.$$

As in [13], [19] we will use the notation

$$I_i(\alpha)=\{j:\text{there is } t=0,1,\ldots,k \text{ with } \{i,j\}=\{\alpha(t),\alpha(t+1)\}\}$$

for $i=1,2,\ldots,s$.

Fix an open ball U_0 in $\mathbb{R}^n$ containing X, and let Z_i and π_i be as in subsection 2.4. For $y=(y_1, \ldots, y_s) \in (\mathbb{R}^n)^{(s)}$ we set $y_0=\pi_1(y_1)$ and $y_{s+1}=\pi_2(y_{\alpha(k)})$. Denote by U_α the set of those $y \in U_0^{(s)}$ which satisfy the following two conditions:

$$y_i \notin \text{convex hull}\{y_j : j \in I_i(\alpha)\} \qquad (i=1, \ldots, s),$$

and

for every $i=1, \ldots, s$ if m, j, r, t are distinct elements of $I_i(\alpha)$, then either y_i, y_m, y_j or y_i, y_r, y_t are not collinear.

Then U_α is an open subset of $U_0^{(s)}$, and the map

$$F=F_\alpha: \quad U_\alpha \to \mathbb{R}, \tag{4.4}$$

defined by

$$F(y)=\sum_{i=0}^{k} \|y_{\alpha(i)}-y_{\alpha(i+1)}\| \tag{4.5}$$

is smooth. If $y_1, \ldots, y_s$ are all different reflection points of a (ω, θ)-trajectory γ on X such that $y_{\alpha(1)}, \ldots, y_{\alpha(k)}$ are the successive reflection points of γ, then γ will be called a (ω, θ)-trajectory of *type* α. In this case we have $y=(y_1, \ldots, y_s) \in U_\alpha$ and $F(y)$ is just the length of this part of γ which lies in $H_1 \cap H_2$. Moreover, y is a critical point of the map

$$F_{|X^s}: \quad X^s \to \mathbb{R}.$$

It is also clear that for every (ω, θ)-trajectory γ there exists a surjective map α which is either a *ns*-map or a *s*-map such that γ is of type α.

Proof of Theorem 4.2. — Fix an arbitrary surjective *ns*-map (4.1) extended by (4.3), and suppose $k>s$. Denote by $\mathcal{D}_\alpha$ the set of those $f \in C^\infty_{emb}(X, U_0)$ such that there are no (ω, θ)-trajectories of type α on $f(X)$. We are going to prove that $\mathcal{D}_\alpha$ contains a residual subset of $C^\infty_{emb}(X, U_0)$. To this end we will use Theorem 4.1 for $U=U_\alpha$, $p=1$, and $H=F: U_\alpha \to \mathbb{R}$. As in the proof of Lemma 4.3 in [13], one can easily verify that for every $y \in U_\alpha$ and every $i=1, \ldots, s$ there exists $j=1, \ldots, n$ such that $\dfrac{\partial F}{\partial y_i^{(j)}}(y) \neq 0$. Here $y_i^{(j)}$ are the components of the vector $y_i \in \mathbb{R}^n$.

Since $k>s$, there exists $i=1, \ldots, s$ such that $|\alpha^{-1}(i)|>1$. Take two distinct elements j_1, j_2 of $\alpha^{-1}(i)$. Then $m=\alpha(j_1-1)$, $j=\alpha(j_1+1)$, $r=\alpha(j_2-1)$, $t=\alpha(j_2+1)$ are distinct elements of $I_i(\alpha)$. Clearly, $\{m, j\} \neq \{0, s+1\}$, so either m or j is not contained in $\{0, s+1\}$. We may assume $m \notin \{0, s+1\}$ (otherwise we can exchange the notation: $m=\alpha(j_1+1)$, $j=\alpha(j_1-1)$). Similarly, we may assume $r \notin \{0, s+1\}$. Set

$$L_u(y)= \left\langle \frac{y_m-y_i}{\|y_m-y_i\|}+(-1)^u \frac{y_j-y_i}{\|y_j-y_i\|}, \frac{y_r-y_i}{\|y_r-y_i\|}-(-1)^u \frac{y_t-y_i}{\|y_t-y_i\|} \right\rangle \tag{4.6}$$

for $u=1, 2$, $y\in U_\alpha$, and define $L: U_\alpha \to \mathbb{R}^2$ by

$$L(y)=(L_1(y), L_2(y)). \tag{4.7}$$

We have to check that if $L(y)=0$ for some $y\in U_\alpha$, then $dL(y)\neq 0$. Suppose $y\in U_\alpha$ and $L(y)=0$. If $\frac{\partial L_1}{\partial y_m^{(l)}}(y)=0$ for every $l=1, \dots, n$, by direct calculations we find that y_m-y_i is collinear with $v=\frac{y_r-y_i}{\|y_r-y_i\|}+\frac{y_t-y_i}{\|y_t-y_i\|}$. Note that $y\in U_\alpha$ implies $v\neq 0$. Since $L_1(y)=0$ and $\frac{y_m-y_i}{\|y_m-y_i\|}$ and $\frac{y_j-y_i}{\|y_j-y_i\|}$ are unit vectors, we obtain that y_j-y_i is also collinear with v. Therefore the points y_i, y_m and y_j are collinear. Suppose also that $\frac{\partial L_2}{\partial y_r^{(l)}}(y)=0$ for every $l=1, \dots, n$. Then in the same way one gets that y_i, y_r and y_t are collinear which is a contradiction with $y\in U_\alpha$. Hence $dL(y)\neq 0$.

Finally, note that if $y_1, \dots, y_s$ are the reflection points of a (ω, θ)-trajectory of type α, then for $y=(y_1, \dots, y_s)\in U_\alpha$ we have $L(y)=0$. Now, applying Theorem 4.1, we find that $\mathscr{D}_\alpha$ contains a residual subset of $C^\infty_{\text{emb}}(X, U_0)$.

If $\theta=-\omega$ and α is a surjective s-map (4.1) with $k>2s-1$, the argument above with minor changes shows that $\mathscr{D}_\alpha$ again contains a residual subset of $C^\infty_{\text{emb}}(X, U_0)$. We omit the details in this case.

Finally, mention that $\mathscr{D}=\bigcap\limits_\alpha \mathscr{D}_\alpha$, where α runs over the surjective maps (4.1) which are either ns-maps with $k>s$ or s-maps with $k>2s-1$. Therefore $\mathscr{D}$ contains a residual subset of $C^\infty_{\text{emb}}(X, U_0)$ which proves the theorem.

THEOREM 4.3. — *Let $\mathscr{C}$ be the set of those $f\in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that every two different (ω, θ)-trajectories on $f(X)$ have no common reflection points. Then $\mathscr{C}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.*

Proof. — We have to consider pairs of ns- or s-maps. We deal in details only with the case of two ns-maps. The other cases are quite similar.

Let U_0, Z_i and $\pi_i (i=1,2)$ be as above. For a given $Y=f(X)$, $f\in C^\infty_{\text{emb}}(X, U_0)$, suppose γ_1 and γ_2 are two different non-symmetric (ω, θ)-trajectories on Y, and let $y_1, \dots, y_s$ be all reflection points of γ_1 and γ_2 taken together. Then there exist integers k, $l\geqq 1$ and ns-maps (4.1) and

$$\beta: \quad \{1, \dots, l\} \to \{1, \dots, s\} \tag{4.7}$$

such that

$$\operatorname{Im}\alpha \cup \operatorname{Im}\beta=\{1, \dots, s\}, \tag{4.8}$$

$$\{\alpha(i), \alpha(i+1)\}\neq\{\beta(j), \beta(j+1)\} \qquad (1\leqq i\leqq k, 1\leqq j\leqq l), \tag{4.9}$$

$y_{\alpha(1)}, \ldots, y_{\alpha(k)}$ are the successive reflection points of γ_1 and $y_{\beta(1)}, \ldots, y_{\beta(l)}$ are the successive reflection points of γ_2. In this case we will say that (γ_1, γ_2) is a pair of type (α, β). Set $\beta(0) = -1$ and $\beta(l+1) = s+2$, thus extending β to a map

$$\beta: \ \{0, 1, \ldots, l, l+1\} \to \{-1, 1, \ldots, s, s+2\}.$$

We will use the notation $y_{-1} = \pi_1(y_{\beta(1)})$, $y_{s+2} = \pi_2(y_{\beta(l)})$. Define F by (4.4) and (4.5) and $G: U_\beta \to \mathbb{R}$ by

$$G(y) = \sum_{i=0}^{l} \| y_{\beta(i)} - y_{\beta(i+1)} \|.$$

Then $y = (y_1, \ldots, y_s) \in U = U_\alpha \cap U_\beta$ and y is a critical point for both $F \circ f^s$ and $G \circ f^s$.

Let (α, β) be a pair of maps (4.1) and (4.7) with (4.8), (4.9) and

$$\operatorname{Im} \alpha \cap \operatorname{Im} \beta \neq \varnothing. \tag{4.10}$$

Denote by $\mathscr{C}_{\alpha, \beta}$ the set of those $f \in C^\infty_{\text{emb}}(X, U_0)$ for which there is no pair (γ_1, γ_2) of (ω, θ)-trajectories on $f(X)$ of type (α, β). To prove that $\mathscr{C}_{\alpha, \beta}$ contains a residual subset of $C^\infty_{\text{emb}}(X, U_0)$, we proceed exactly as in the proof of Theorem 4.2. We omit the details.

Denote by $\mathscr{S}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that $T_\gamma \neq T_\delta$ for every two different (ω, θ)-trajectories γ and δ on $f(X)$, and by $\mathscr{P}$ the set of those $f = C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that if γ is a non-symmetric (ω, θ)-trajectory on $f(X)$, then any two different segments of γ are not parallel, and if γ is a symmetric (ω, θ)-trajectory on $f(X)$, then there are no different parallel segments among the first half of the segments of γ.

The following generic properties of (ω, θ)-trajectories will be important.

THEOREM 4.4. – *Each of the sets $\mathscr{S}$ and $\mathscr{P}$ contains a residual subset of* $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$.

Proof. – We deal again with the intersections of $\mathscr{S}$ and $\mathscr{P}$ with $C^\infty_{\text{emb}}(X, U_0)$. where U_0 is a fixed open ball containing X.

If $T_\gamma = T_\delta$ for two different (ω, θ)-trajectories γ and δ on $Y = f(X)$, $f \in \mathscr{C} \cap \mathscr{D}$, there exist different elements $y_1, \ldots, y_s$ of Y such that $y_1, \ldots, y_k$ are the successive reflection points of γ for some $k < s$, while $y_{k+1}, \ldots, y_s$ are the successive reflection points of δ. Moreover, $F(y) = G(y)$, where F, $G: U \to \mathbb{R}$ are defined by

$$F(y) = \| \pi_1(y_1) - y_1 \| + \sum_{i=1}^{k-1} \| y_i - y_{i+1} \| + \| y_k - \pi_2(y_k)) \|, \tag{4.11}$$

$$G(y) = \| \pi_1(y_{k+1}) - y_{k+1} \| + \sum_{i=k+1}^{s-1} \| y_i - y_{i+1} \| + \| y_s - \pi_2(y_s) \|. \tag{4.12}$$

Here U is the set of those $y \in (\mathbb{R}^n)^{(s)}$ such that $y_i \notin [y_{i-1}, y_{i+1}]$ for all $i=2, \ldots, k-1$ and $i=k+1, \ldots, s-1$, $y_1 \notin [\pi_1(y_1), y_2]$, $y_k \notin [y_{k-1}, \pi_2(y_k)]$, $y_{k+1} \notin [\pi_1(y_{k+1}), y_{k+2}]$, and $y_s \notin [y_{s-1}, \pi_2(y_s)]$. Applying Theorem 3.1 for $H=(F, G)$ and $L: U \to \mathbb{R}$, $L(y)=F(y)-G(y)$, we obtain that

$$\mathcal{S}'_{k,s} = \{ f \in C^\infty_{\text{emb}}(X, U_0) : \text{if } \operatorname{grad}_x H \circ f^s(x) = 0, \text{ then } L(f^s(x)) \neq 0 \}$$

contains a residual subset of $C^\infty_{\text{emb}}(X, U_0)$. Since

$$\bigcap_{k<s} \mathcal{S}'_{k,s} \cap \mathcal{C} \cap \mathcal{D} \subset \mathcal{S},$$

we deduce that $\mathcal{S}$ contains a residual subset of $C^\infty_{\text{emb}}(X, U_0)$.

To deal with $\mathcal{P}$ we define F by (4.11) with $k=s$, exchanging U suitably. For fixed i and j with $1 \leq i < j \leq s$ we use the function $L: U \to \mathbb{R}^n$,

$$L(y) = \frac{y_i - y_{i+1}}{\| y_i - y_{i+1} \|} + \varepsilon \frac{y_j - y_{j+1}}{\| y_j - y_{j+1} \|},$$

where $\varepsilon = \pm 1$, $y_0 = \pi_1(y_1)$ and $y_{s+1} = \pi_2(y_2)$, to express the fact that $[y_i, y_{i+1}]$ and $[y_j, y_{j+1}]$ are parallel. We omit the details.

Proof of Theorem 2. – Denote by $\mathcal{T}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that every (ω, θ)-trajectory of $f(X)$ has no segments tangent to $f(X)$ and $\det dJ_\gamma \neq 0$ (*cf.* subsection 2.4). It follows by [14], [15] that if we define $\mathcal{T}'$ in the same way by means of reflecting (ω, θ)-rays instead of (ω, θ)-trajectories, then $\mathcal{T}'$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$. The same argument shows that $\mathcal{T}$ has this property, too.

Next, denote by $\mathcal{K}$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that for every $y \in f(X)$ there are no directions $v \in T_y f(X) \setminus \{0\}$ such that the curvature of $f(X)$ at y with respect to v vanishes of order $2n-3$. It can be derived from the results of Landis [6] that $\mathcal{K}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$. Then $\mathcal{A} = \mathcal{S} \cap \mathcal{P} \cap \mathcal{T} \cap \mathcal{K}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$. We will show that the inclusion (1.4) holds for Ω_f, provided $f \in \mathcal{A}$.

Denote by $\mathcal{L}_{\omega,\theta}(\Omega_f)$ the set of all (ω, θ)-ray in $\bar{\Omega}_f$. Note that the set $\mathcal{G}_f$ is closed. Instead, assume that $\gamma_m \in \mathcal{L}^g_{\omega,\theta,f}$ for every $m \in \mathbb{N}$ and $T_{\gamma_m} \to T_0$. By a standard argument we deduce the existence of a (ω, θ)-ray γ_0 with sojourn time T_0. Moreover, the starting point $z_0 \in Z_1$ of γ_0 is a limit point of the set of starting points $\{ z_m : m \in \mathbb{N} \}$ of the rays γ_m. If γ_0 is formed only by linear segments, then all these segments are not tangent to $f(X)$, since $f \in \mathcal{T}$. On the other hand, if γ_0 is ordinary, then $f \in \mathcal{T}$ shows that the rays starting in a small neighbourhood of z_0 in Z_1 with direction ω are not (ω, θ)-rays. Thus $\gamma_0 \in \mathcal{L}^g_{\omega,\theta,f}$ and $\mathcal{G}_f$ is closed.

Let $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega_f)$ be an ordinary reflecting (ω, θ)-ray with sojourn time T_γ. Since $f \in \mathcal{S} \cap \mathcal{P} \cap \mathcal{T}$ and $T_\gamma \notin \mathcal{G}_f$, a continuity argument implies that for some $\varepsilon_0 > 0$ we have $T_\delta \notin [T_\gamma - \varepsilon_0, T_\gamma + \varepsilon_0]$ for all $\delta \in \mathcal{L}_{\omega,\theta}(\Omega_f) \setminus \{\gamma\}$.

Then we can repeat the localization procedure in the proof of Theorem 1. This procedure shows that the singularities of $s(t, \theta, \omega)$ in a small neighbourhood of $-T_\gamma$ depend only on the ray γ. Since γ is an ordinary (ω, θ)-ray with a non-vanishing differential cross section, we can repeat the arguments in [11], [16] to finish the proof of Theorem 2.

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