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On iterations of Misiurewicz's rational maps on the Riemann sphere

by

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ABSTRACT. – This paper concerns ergodic properties of rational maps of the Riemann sphere, of subexpanding behaviour. In particular we prove the existence of an absolutely continuous invariant measure, study its density and prove the metric exactness.

This is not a strictly research paper. Its aim is to present useful facts belonging to the folklore but partially not explicitly published up to now.

INTRODUCTION

Let f be a rational map of a Riemann sphere $\hat{\mathbb{C}}$, let J denote its Julia set, μ denote the normalised, standard Riemann measure on $\hat{\mathbb{C}}$ (*i.e.* $\mu(\hat{\mathbb{C}})=1$), $\text{Crit}=\{z\in\hat{\mathbb{C}}: f'(z)=0\}$, $\text{Crit}^+=\bigcup_{n>0} f^n(\text{Crit})$ and $\omega=\bigcap_{n\geq 0}\text{cl}\, f^n(\text{Crit}^+)$. Derivatives in the paper are considered usually with respect to the Riemann metric.

In this text we verify under suitable assumptions two properties:

(i) Either $\mu(J)=0$ or $J=\hat{\mathbb{C}}$.

(ii) If $A \subset J$ and $\mu(A)>0$ then $\lim_{n \to \infty} \mu(f^n(A))=\mu(J)$.

The property (ii) is called metric exactness. Satisfaction of the both properties without any additional assumptions is a famous open problem; we do not make any progress in solving it.

Consider also the property

(ii') If $A \subset J$ and $\mu(A)>0$ then $\limsup_{n \to \infty} \mu(f^n(A))=\mu(J)$.

In section 2 we prove (i) and (ii') under the assumption that all critical points in J are eventually periodic. In section 3 we consider the case where $\mathrm{Crit} \cap \omega \cap J=\varnothing$ and moreover f is expanding on $J \cap \omega \setminus \{$ the set of neutral rational periodic points $\}$ (*i.e.* there exists $k>0$ such that for every $z \, |(f^k)'(z)|>1$). We call such f Misiurewicz's map by the analogy with maps considered in [M]. In paragraph 4 in the case $J=\hat{\mathbb{C}}$ we prove the existence of an invariant measure equivalent to μ [this allows to pass from (ii') to (ii)] and prove that outside the set $\mathrm{cl}(\mathrm{Crit}^+)$ it has real-analytic subharmonic density.

Everything in this paper is the so called folklore for specialists, but partially not explicitly published. So we decided that this material is worthy of being written down. Lots of facts from this paper rely on Koebe's distortion theorem, see [H], Th. 17.4.6. One version is that for every $t<1$ and $C>0$ there exists $C(t)>0$ such that for every holomorphic univalent function $f: \mathbb{D} \to \hat{\mathbb{C}}$ on the unit disc, such that $\mathrm{diam}(\hat{\mathbb{C}} \setminus f(\mathbb{D}))>C$

$$(*) \qquad \sup\{|f'(x)/f'(y)| : |x|, |y| \leqq t\} \leqq C(t).$$

Contrary to a recently popular custom no involvement of a hyperbolic metric is necessary. Our text is elementary. Let us call the attention of the reader to significant Mary Rees' paper [R]. Our considerations constitute only a starting point to her infinitely more delicate study. A part of our paper overlaps also with M. Lyubich paper [L].

This text grew from discussions at the Cornell conference on iterations in August 1987, a talk of the second author at the Warsaw dynamics seminar and an unpublished preprint by the third author.

Remark on the notation. – The letter C will be used to denote various positive constants either universal or dependent only on f, which may differ from one formula to another even within a single stream of estimates. □ will denote the end of a proof, ■ will denote the end of a statement of a theorem, lemma, etc.

1. BASIC LEMMAS

LEMMA 1. — Let V be an open disc in $\hat{\mathbb{C}}$ such that $V \cap J \neq \varnothing$ and V is disjoint with a periodic orbit Q of period at least 3. Suppose we have a sequence of backward branches of iterates of f^{-1} on V, namely holomorphic, univalent functions $g_n = f_v^{-m(n)}|_V$, where $m(n) \to \infty$ as $n \to \infty$. Then the derivatives g'_n converge to 0 uniformly on every compact set in V. ■

Proof. — If the lemma were false then there would exist a subsequence g_{n_i} and points z_i in a compact subset of V such that $|g'_{n_i}(z_i)| > \varepsilon$ for a positive constant ε. As g_{n_i} is a normal family (the values omit the set Q), there exists a holomorphic function G which is the limit of subsequence of (g_{n_i}), for simplification denote it also by (g_{n_i}), and G is non-constant. Then by the invariance and compactness of the Julia set the point $G(p) = \lim_{n \to \infty} g_{n_i}(p)$ belongs to J, provided $p \in V \cap J$. One of the equivalent definitions of J says that for every point in J and its neighbourhood A no subsequence of $f^n|_A$ is normal. We conclude by Montel's theorem that for an arbitrary open set V' such that $V' \ni G(p)$ and $\mathrm{cl}\, V' \subset G(V)$, for an arbitrary i_0 the set $\bigcup_{i > i_0} f^{m(n_i)}(V')$, covers $\hat{\mathbb{C}}$ except at most 2 points. On the other hand for every i large enough $g_{n_i}(V) \supset V'$ so $f^{m(n_i)}(V') \subset V$, a contradiction. □

COROLLARY 1. — *If* $\mathcal{U}$ *is a neighbourhood of* $\mathrm{cl}(\mathrm{Crit}^+)$, *then on* $J \setminus \mathcal{U}$ *the derivatives* $(f_v^{-n})'$ *converge uniformly to* 0 (*for all backward branches* f_v^{-n} *of* f^{-n}). ■

LEMMA 2. — *Let* $p \in J$ *be a periodic point. Suppose there exists a neighbourhood* V *of* p *such that* $(V \setminus \{p\}) \cap \mathrm{Crit}^+ = \varnothing$. *Then* p *is a source.* ■

Proof. — If we supposed $V \cap \mathrm{Crit}^+ = \varnothing$, *i.e.* $p \notin \omega$, we would deduce that p is a source immediately from Corollary 1. Consider the general case. Let A be an open topological disc containing precisely one point q_A from $O(p)$, the periodic orbit of p, and such that $(A \setminus q_A) \cap \mathrm{Crit}^+ = \varnothing$. Consider B a component of the set $f^{-1}(A)$ intersecting $O(p)$. Because $f|_{O(p)}$ is $1-\text{to}-1$, B contains precisely one point q_B belonging to $O(p)$. We have $f(q_B) = q_A$. The map $f|_{B \setminus f^{-1}(q_A)}$ is a covering map to the annulus $A \setminus q_A$.

So the fundamental group of $B \setminus f^{-1}(q_A)$ is a subgroup of $\mathbb{Z}$, hence $B \setminus f^{-1}(q_A)$ is an annulus, hence the set $f^{-1}(q_A) \cap B$ consists of one point q_B only. As $q_B \notin \mathrm{Crit}$ [otherwise $O(p)$ would be a sink, so $p \notin J$], f is a covering map from B to the topological disc A. So $f|_B$ is invertible. We have a holomorphic univalent branch $f_v^{-1} : A \to B$. Observe that similarly to A also B has the property $(B \setminus q_B) \cap \mathrm{Crit}^+ = \varnothing$. Indeed if $z \neq q_B$, $z \in f^n(\mathrm{Crit}) \cap B$, $n > 0$, then $f(z) \in f^{n+1}(\mathrm{Crit}) \cap A$ so the only possibility is $f(z) = q_A$ hence $z = q_B$, a contradiction.

Now we can suppose that V is an open topological disc containing precisely one point from $O(p)$. Define by induction branches $g_n = f_V^{-n}$ on V. Namely having defined g_n we set $A = g_n(V)$ and define $g_{n+1} = f_V^{-1} \circ g_n$, where $f_V^{-1} : A \to B$ is a map considered above. Now Lemma 2 follows from the Lemma 1. □

COROLLARY 2. — *For every neutral periodic point p* (i.e. *such that* $|(f^n)'(p)| = 1$, *where n is a period of p*), *there exist a critical point c and a sequence of integers* $n_i \to \infty$ *such that* $f^{n_i}(c) \to p$ *and* $f^{n_i}(c) \neq p$. ■

LEMMA 3. — *Suppose we have a measurable set* $E \subset J$ *and* $\varepsilon > 0$, *such that for a.e.* $z \in E$ *there exist a sequence of positive integers* (n_i), *points* $z_i \in \hat{\mathbb{C}}$ *and branches* $f_{v_i}^{-n_i}$ *of* f^{-n_i} *(holomorphic, univalent) defined on the respective discs* $B(z_i, \varepsilon)$ *which map* z_i *to* z. *Then either* $\mu(E) = 0$ *or there exists* $N \geq 0$ *such that* $\mu(f^{n_i + N}(E)) \to \mu(\hat{\mathbb{C}}) = 1$ *(for one of the above sequences* (n_i)*).* ■

Proof. — Suppose $\mu(E) > 0$. Then there exists a point of density $z \in E$, for which the assumptions of the lemma are applicable. Take a finite family of open discs $D_1, \ldots, D_m$, of radius $\varepsilon/4$, intersecting J, such that $\bigcup_{j=1}^{m} D_j \supset J$. For every i there exists $D_{j(i)}$ for which $B(z_i, \varepsilon/2) \supset D_{j(i)}$. It is known (topological exactness) that there exist a neighbourhood $\mathscr{U}$ of J and an integer $N > 0$ such that for every j we have $f^N(D_j) \supset \mathscr{U}$.

Now by Lemma 1, definition of the density point and Koebe's distortion theorem we have $\mu(f^{-n_i}(D_{j(i)}) \cap E)/\mu(f^{-n_i}(D_{j(i)})) \to 1$. Hence again by Koebe's theorem $\mu(D_{j(i)} \cap f^{n_i}(E))/\mu(D_{j(i)}) \to 1$, hence $\mu(\mathscr{U} \cap f^{n_i + N}(E))/\mu(\mathscr{U}) \to 1$. This implies in particular that J cannot be nowhere dense, so $J = \hat{\mathbb{C}}$. We obtain $\mu(f^{n_i + N}(E)) \to 1$. □

DEFINITION. — Let us call the set $J \setminus \{ z \in J : \mathrm{dist}(f^n(z), \omega) \to 0 \}$ the *transverse limit set* and denote it by $\omega^\perp$. Observe that $f^{-1}(\omega^\perp) = \omega^\perp$. ■

LEMMA 4. — *Either* $\mu(\omega^\perp) = 0$ *or* $\mu(\omega^\perp) = 1$. *In the latter case for every measurable set* $E \subset \omega^\perp$ *with* $\mu(E) > 0$ *we have* $\limsup_{n \to \infty} \mu(f^n(E)) = 1$. ■

Proof. — Let $E \subset \omega^\perp$ and $\mu(E) > 0$. Then there exist $E' \subset E$ with $\mu(E') > 0$ and a number $\delta > 0$ such that for every $z \in E'$ there exists a sequence $n_i \to \infty$ such that $\mathrm{dist}(f^{n_i}(z), \omega) > \delta$. Let N be such an integer that $f^N(\mathrm{Crit}^+) \subset B(\omega, \delta/2)$. So $\mathrm{dist}(f^{n_i - N}(z), \mathrm{Crit}^+) \geq \mathscr{L}^{-N} \delta/2$, where $\mathscr{L}$ is the Lipschitz constant for f. Fix $\varepsilon = \mathscr{L}^{-N} \delta/2$ and apply Lemma 3. □

2. THE CASE WHERE ALL CRITICAL POINTS IN J ARE EVENTUALLY PERIODIC

THEOREM 1. — *Suppose that every point* $c \in \mathrm{Crit} \cap \mathrm{J}$ *is eventually periodic*, i.e. *there exist* $n(c) \geqq 0$ *such that* $f^{n(c)}(c)$ *is periodic. Then*

(*a*) *every periodic point in* J *which is not neutral rational* (i.e. *if* $(f^n)'(x)$ *is not a root of unity, where n is a a period*) *is a source.*

(*b*) *the properties* (i) *and* (ii') *from the introduction hold.* ■

Proof. — For every periodic point p in J, not neutral rational, the assumptions of Lemma 2 are satisfied. Indeed we may bother only about critical points not in J. If $c \notin \mathrm{J}$ and for a sequence (n_i), $\operatorname{dist}(f^{n_i}(c), \mathrm{J}) \to 0$ then $(f^{n_i}(c))$ converges to neutral rational periodic point (by the classification of components of $\hat{\mathbb{C}} \setminus \mathrm{J}$, the theory of Julia, Fatou and Sullivan). This proves the assertion (*a*).

To prove (*b*) observe that if a periodic point $p \in \omega$ is a source then for $z \in \mathrm{J}$, $\operatorname{dist}(f^n(z), f^n(p)) \to 0$ implies $f^n(z) = f^n(p)$ for n large enough. The same is true if p is neutral rational. Indeed $\operatorname{dist}(f^n(z), f^n(p)) \to 0$ and $f^n(z) \neq f^n(p)$ for every n imply that for n sufficiently large $f^n(z)$ belongs to one of the "petals" of $f^n(p)$ (*see* [DH], Exposé IX). So z is in the domain of normality of iteration of f, not in J, a contradiction. (We owe the last argument to M. Lyubich.) So $\omega^{\perp} = \mathrm{J} \setminus \mathrm{A}$ where $\mathrm{A} = \bigcup_{n \geqq 0} f^{-n}(\omega)$ is countable, hence of measure 0. The theorem follows now from Lemma 4. □

PROPOSITION 1. — *If all critical points in* $\hat{\mathbb{C}}$ *are eventually periodic, then there are no neutral periodic points. If all critical points are eventually periodic but not periodic, then* $\mathrm{J} = \hat{\mathbb{C}}$ *and all periodic points are sources.* ■

Proof. — If all critical points are eventually periodic then by Corollary 2 there are no neutral points in J. The center of a Siegel disc S is also excluded because $\partial \mathrm{S} \subset \omega$ and $\partial \mathrm{S}$ is uncountable, whereas ω is finite. If additionally no critical point is periodic, if p were a sink we would construct a normal family of branches $g_n = f_{\nu_n}^{-n}$ with p fixed, as in the proof of the Lemma 2. This contradicts $|g_n'(p)| = |f^n(p)|^{-1} \to \infty$. $\mathrm{J} = \hat{\mathbb{C}}$ follows from the Fatou-Julia theory (lack of critical points to serve periodic components of $\hat{\mathbb{C}} \setminus \mathrm{J}$) and Sullivan's: no wandering domains. □

3. THE CASE WHERE CRITICAL POINTS ARE NOT RECURRENT

Let us start with a general lemma.

LEMMA 5. — *For every real numbers* $\varepsilon > 0$, $\lambda > 1$ *and an integer* $k > 0$ *there exists* $\delta > 0$ *such that for every* n *and trajectory*

$\gamma=\{x,\ldots,f^n(x)\}\subset\hat{\mathbb{C}}\setminus\mathrm{B}(\mathrm{Crit},\varepsilon)$ *if* $|(f^k)'(f^m(x))|\geqq\lambda$ *for every* $m=0,\ldots,n-k$, *then there exists a holomorphic, univalent branch* f_ν^{-n} *on the disc* $\mathrm{B}(f^n(x),\delta)$ *which maps* $f^n(x)$ *to* x. ■

Proof. – We can suppose that λ is arbitrarity large and $k=1$, if not, take an iterate instead of f.

Take $\hat{\delta}<(\varepsilon/2)\inf\{|f'(z)|:z\in\hat{\mathbb{C}}\setminus\mathrm{B}(\mathrm{Crit},\varepsilon/2)\}$. Then for every $m=1,\ldots,n$ there exists a univalent branch $f_{\nu_m}^{-1}$ of f^{-1} on $\mathrm{B}(f^m(x),\hat{\delta})$ such that $f_{\nu_m}^{-1}(f^m(x))=f^{m-1}(x)$. Then by Koebe's distortion theorem there exists $\delta>0$, $\delta<\hat{\delta}$ depending only on f, $\hat{\delta}$ and λ, such that for every m and $z_1, z_2\in\mathrm{B}(f^m(x),\delta)$ we have $|(f_{\nu_m}^{-1})'(z_1)/(f_{\nu_m}^{-1})'(z_2)|<\lambda$. We conclude that for every $z\in\mathrm{B}(f^m(x),\delta)$ we have $|(f_{\nu_m}^{-1})'(z)|<1$, so

$$f_{\nu_m}^{-1}(\mathrm{B}(f^m(x),\delta)\subset\mathrm{B}(f^{m-1}(x),\delta)$$

and we can compose the branches. □

THEOREM 2. – *Suppose that on the set* $\mathrm{L}=\omega\cap\mathrm{J}\setminus\mathrm{N}\mathscr{R}$, *where* $\mathrm{N}\mathscr{R}$ *is the set of all periodic neutral rational points*, f *is expanding*, i.e. *there exist* $\lambda>1$ *and* k *such that on* L *we have* $|(f^k)'|>\lambda$. *Then the properties* (i) *and* (ii') *from introduction are satisfied.* ■

Proof. – As in section 2 due to Lemma 4 it is sufficient to prove that $\mu(\mathrm{J}\setminus\omega^\perp)=0$. But $\mathrm{J}\setminus\omega^\perp$ is the union of the set $\bigcup_{n>0}f^{-n}(\mathrm{N}\mathscr{R})$ countable so of measure 0, and of the set

$$\mathrm{L}'=\{z\in\mathrm{J}:f^n(z)\to\mathrm{L}\}.$$

Write

$$\hat{\mathrm{L}}_\lambda=\{z\in\mathrm{J}:|(f^k)'(z)|\geqq\lambda\ \&\ \mathrm{dist}(z,\omega)\leqq 1/2\,\mathrm{dist}(\mathrm{Crit},\omega)\}.$$

$\hat{\mathrm{L}}_\lambda$ is a neighbourhood of L. Write $\mathrm{L}_\lambda=\bigcap_{n\geqq 0}f^{-n}(\hat{\mathrm{L}}_\lambda)$. Of course $\mathrm{L}'\subset\bigcup_{n\geqq 0}f^{-n}(\mathrm{L}_\lambda)$. Due to Lemma 5 applied to every point $x\in\hat{\mathrm{L}}_\lambda$ we can apply the Lemma 3 and obtain $\mu(\mathrm{L}_\lambda)=0$. So $\mu(\mathrm{L}')=0$. □

Remark 1. – In view of Lemma 2, our Theorem 2 is more general than Theorem 1. One would like to prove the expanding property on $\omega\cap\mathrm{J}\setminus\mathrm{N}\mathscr{R}$ instead of assuming it, but we are not able to do it. This is true for maps of the interval with negative Schwarzian derivative provided $\mathrm{Crit}\cap\omega=\varnothing$ and there are no sinks and neutral points, *see* [M]. The reason is that for every $\varepsilon>0$ there exist $k>0$, such that if $\mathrm{dist}(\{x,\ldots,f^k(x)\},\mathrm{Crit})\geqq\varepsilon$ then $|(f^k(x)'|>1$.

Here however the latter assertion is false even in absence of neutral points in J. A counterexample is provided by Michel Herman's example of a Siegel disc S not containing critical points in the boundary, *see* [D].

In this example $f|_{\partial S}$ is not expanding because

$$\int_{\partial S} \log |f'(z)| \, d\Omega(z) = \log |f'(p)| = 0$$

where p is the neutral point in S and Ω is the harmonic measure on ∂S viewed from p. From this example by Shishikura's surgery a counterexample with Herman ring can be derived.

Another examples are provided by the maps of the form $f(z) = \lambda z \dfrac{z-r}{1-rz} \dfrac{1-\bar{\gamma} rz}{z-\gamma r}$, with r real close to 0 and $|\gamma|$, $|\lambda| = 1$, $\gamma \neq 1$, *see* [Her]. Such a map is of degree 3, $f|_{\{|z|=1\}}$ is a diffeomorphism and for a suitable λ, $J = \hat{\mathbb{C}}$.

Let us call a closed invariant set in J disjoint with Crit, on which $|(f^k)'| > 1$ fails for every k, a neutral set. The question arises what neutral sets are possible?

Maybe $\mathrm{Crit} \cap \omega \cap J = \varnothing$ implies that $f|_{\omega \setminus N\mathscr{R}}$ is expanding? This question is related to the question whether a periodic neutral irrational point can attract a critical point whose trajectory never hits it and to the similar question about the boundary of a Siegel disc, told us by M. Herman.

If in the case $\mathrm{Crit} \cap \omega \cap J = \varnothing$ a nonempty neutral set $L \subset \omega$ different from $N\mathscr{R}$ exists the question is whether always $\mu(\{x \in J : f^n(x) \to L\} = 0$ or not?

Remark finally that for every $A \subset J$ such that $f^n(A) \cap B(\mathrm{Crit}, \varepsilon) = \varnothing$ for every $n = 0, 1, \ldots$, if for every $x \in A$

$$\chi(x) = \limsup_{n \to \infty} (1/n) \log |(f^n)'(x)| > 0,$$

then $\mu(A) = 0$, *see* [L]. (The proof is similar to that of Lemma 5 and Theorem 2, only backward iteration should be replaced by the forward one.) So Theorem 2 holds if the "expanding" assumption is replaced by the formally weaker one: $\forall x \in L' \, \exists k > 0$ such that $\chi(f^k(x)) > 0$. We do not know however of any example exhibiting the difference between these assumptions. ■

4. ABSOLUTELY CONTINUOUS INVARIANT MEASURE

THEOREM 3. – *Under the assumptions of Theorem* 1 *or* 2, *if* $\mu(J) \neq 0$ (i. e. $J = \hat{\mathbb{C}}$), *there exists an invariant measure* η *equivalent to* μ *on* $\hat{\mathbb{C}}$. ■

Proof (relying on the ideas from [K-S], [M] and [S]).

From the sequence of measures $\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* (\mu)$ we choose a weakly $-*$ convergent subsequence ν_{n_i} and consider the limit measure η.

For every $z \in \hat{\mathbb{C}} \setminus \mathrm{cl}(\mathrm{Crit}^+)$, every small positive $\varepsilon \leqq \mathrm{dist}(z, \mathrm{Crit}^+)$, by Koebe's distortion theorem applied to the branches f_ν^{-n} on the disc $B(z, \varepsilon)$, for every $A \subset B(z, \varepsilon/2)$ we have

$$C^{-1}((f_\nu^{-n})'(z))^2 \leqq \mu(f_\nu^{-n}(A))/\mu(A) \leqq C((f_\nu^{-n})'(z))^2 \tag{1}$$

where C is a constant depending only on f. This follows precisely from

$$\mu(f_\nu^{-n}(A)) = \int_A |(f_\nu^{-n})'(z)|^2 \, d\mu(z) \in [(C(1/2))^{-2}, (C(1/2))^2]\, \mu(A),$$

cf. $(*)$ from the introduction.

For $A = B(z, \varepsilon/2)$ the inequality (1) (the left hand side) summed up over all branches gives

$$\sum ((f_\nu^{-n})'(z))^2 \leqq C\, \mu(\hat{\mathbb{C}})/\mu(B(z, \varepsilon/2)) \leqq C\, \varepsilon^{-2}, \tag{2}$$

the estimate independent of n.

Now for every n we sum up the right hand side inequality of (1) over all branches and deduce from the resulting estimate for $f^n_*(\mu)(A)/\mu(A)$, from (2) and from the definition of η that η is absolutely continuous with respect to μ (write $\eta \ll \mu$) on $\hat{\mathbb{C}} \setminus \mathrm{cl}(\mathrm{Crit}^+)$.

Moreover

$$(d\eta/d\mu)(z) \leqq C(\mathrm{dist}(z, \mathrm{Crit}^+))^{-2} \tag{3}$$

We want to know that $\eta \ll \mu$ on $\mathrm{cl}(\mathrm{Crit}^+)$ as well. We have $\mu(\mathrm{cl}(\mathrm{Crit}^+)) = 0$ (*cf.* the proof of Theorem 2). So we need to prove $\eta(\mathrm{cl}(\mathrm{Crit}^+)) = 0$ or equivalenty $\eta(\omega) = 0$. This will be done in Lemma 6.

Meanwhile observe that $\eta(\mathrm{cl}(\mathrm{Crit}^+)) = 0$ implies also the equivalence of η and μ. Indeed, take a disc $B_0 = B(z_0, \varepsilon_0)$ such that $\eta(B_0) > 0$ and $B(z_0, 2\varepsilon_0) \cap \mathrm{Crit}^+ = \varnothing$. Then by (1) the function $|\log d\eta/d\mu|$ is bounded on B_0, so η and μ are equivalent on B_0.

By $\bigcup_{n \geqq 0} f^n(B_0) = \hat{\mathbb{C}}$ and the property $\eta(f(A) \geqq \eta(A)$ for every A, the equivalence of η and μ holds for every $z \in \hat{\mathbb{C}} \setminus \mathrm{cl}(\mathrm{Crit}^+)$ on a disc $B(z, \varepsilon(z))$ hence on all of $\hat{\mathbb{C}}$. □

LEMMA 6. — $\eta(\omega) = 0$. ■

Proof. — By Lemma 5 and the expanding property of f on ω the following is true:

(4) there exist C_0, δ, $\varepsilon \geqq 0$ and $\lambda < 1$ such that for every $n > 0$ and $\gamma = \{x, \ldots, f^n(x)\} \subset B(\omega, \varepsilon)$ there exists a univalent branch f_ν^{-n} on $B(f^n(x), \delta)$, mapping $f^n(x)$ to x, having distortion on $B(f^n(x), \delta)$ less than 2 and such that $|(f^{-n})'| < C_0 \lambda^n$.

As critical points are attracted to ω, there exists $N>0$ such that $\bigcup_{n\geq N} f^n(\mathrm{Crit}) \subset B(\omega, \varepsilon/3)$. This implies that for every $x\in\hat{\mathbb{C}}$ if $\operatorname{dist}(x,\omega)\geqq\varepsilon/2$, then $\operatorname{dist}(x, \bigcup_{n\geq N} f^n(\mathrm{Crit}))\geqq\varepsilon/6$. Then

$$\operatorname{dist}(f^{-N+1}(x), \mathrm{Crit}^+)\geqq \mathscr{L}^{-N+1}\varepsilon/6=\delta_1 \tag{5}$$

($\mathscr{L}$ is the Lipschitz constant for f).

Take $\delta_2<\delta$ such that for every disc D in $\hat{\mathbb{C}}$ of radius $2C_0\delta_2$:

(6) every component of $f^{-1}(D)$ has diameter less than $\varepsilon/2$

and

(7) every component of $f^{-N}(D)$ has diameter less than $\delta_1/2$.

(Clearly it suffices to take $\delta_2\leqq\mathrm{Const}\,\delta_1^{dN}$, where d is degree of f.) Now take an arbitrary disc $B(z,\delta_2)$ with $z\in\omega$. Since $\mu(\omega)=0$ there exists a set $A\subset B(z,\delta_2/2)$ containing $B(z,\delta_2/3)\cap\omega$ with $\mu(A)$ arbitrarily small and $\eta(\mathrm{Fr}\,A)=0$ hence $\eta(A)=\lim \nu_{n_i}(A)$.

Let us estimate $\mu(f^{-n}(A))$. Consider the set $\mathscr{T}$ of all components of $f^{-n}(B(z,\delta_2))$. Divide $\mathscr{T}$ into n families of sets as follows: for every $k=0,1,\ldots,n$ let $\mathscr{T}_k=\{T\in\mathscr{T}: k$ is the maximal integer such that for every $i\geqq n-k$ we have $f^i(T)\cap f^{-(n-i)}(\{z\})\cap B(\omega,\varepsilon)\neq\varnothing\}$.

Consider first $k=n$. Then for every $T\in\mathscr{T}_n$

$$\mu(T\cap f^{-n}(A))/\mu(T)\leqq 4\,\mu(A)/\mu(B(z,\delta_2/2)). \tag{8}$$

We used the fact (4), namely that $f^n(T)$ is $1-\mathrm{to}-1$ and the respective branch $f_\nu^{-n}: B(z,\delta_2)\to T$ has the distortion bounded by 2.

Fix now $k<n$ and divide $\mathscr{T}_k$ into two families

$$\mathscr{T}_{k,\,\mathrm{reg}}=\{T\in\mathscr{T}_k: \bigcup_{n\geqq 0} f^n(T)\cap\mathrm{Crit}=\varnothing\},\qquad \mathscr{T}_{k,\,\mathrm{sing}}=\mathscr{T}_k\setminus\mathscr{T}_{k,\,\mathrm{reg}}.$$

For every $\mathscr{T}_{k,\,\mathrm{reg}}$, every $T\in\mathscr{T}_{k,\,\mathrm{reg}}$ we have again the inequality (8), only 4 may change to another constant.

Consider now $T\in\mathscr{T}_{k,\,\mathrm{sing}}$. Write $M=\min(N,n-k)$.

Observe that

$$\begin{aligned}(9)\quad &\mu(f^{n-k-M}(T)\cap f^{-k-M}(A))/\mu(f^{n-k-M}(T))\leqq\\ &C(\mu(f^{n-k}(T)\cap f^{-k}(A))/\mu(f^{n-k}(T)))^{D(f)}\leqq C(\mu(A)/\mu(B(z,\delta_2)))^{D(f)},\end{aligned}$$

where $D(f)$ is reciprocal to the maximal degree of f^M at critical points.

We leave the first inequality as an exercise to the reader; the constant C depends on M (which is bounded by N), so it depends only on f. The second inequality similarly to (8) follows from the bounded distortion property [*see* (4)].

If $M=n-k$ we obtain from (9)

$$\mu(T\cap f^{-n}(A))/\mu(T)\leqq C(\mu(A)/\mu(B(z,\delta_2)))^{D(f)}. \tag{10}$$

Consider now k such that $M=N<n-k$. Fix an arbitrary $T\in\mathscr{T}_{k,\,\mathrm{sing}}$. Write $B'=f^{n-k-N}(T)$ and $A'=B'\cap f^{-k-N}(A)$. By (4) $f^{n-k}(T)$ is contained in a disc of radius $2C_0\delta_2$, so by (7) B' is contained in a disc $B(y,\delta_1/2)$ for a point $y\in B'$. By (6) $\operatorname{diam} f^{n-k-1}(T)\leqq\varepsilon/2$ and by the definition of $\mathscr{T}_k$ we have $f^{n-k-1}(T)\not\subset B(\omega,\varepsilon)$. So by (5) $B(y,\delta_1)\cap \mathrm{Crit}^+=\varnothing$. We apply the bounded distortion property [*cf.* (1)] and obtain with the use of (9)

$$\mu(T\cap f^{-n}(A))/\mu(T)\leqq C\mu(A')/\mu(B')\leqq C(\mu(A)/\mu(B(z,\delta_2)))^{D(f)}. \tag{11}$$

Now for every T multiply the both sides of the respective inequality (8), (10) or (11) by $\mu(T)$, then sum up these inequalities over all T. We obtain

$$\mu(f^{-n}(A))\leqq C(\sum_{T}\mu(T))(\mu(A))^{D(f)}\leqq C(\mu(A))^{D(f)} \tag{12}$$

($\mu(B(z,\delta_2/2))$ is swallowed by C).

Hence $\eta(\omega\cap B(z,\delta_2/3))=0$. As $z\in\omega$ has been chosen arbitrarily, we obtain $\eta(\omega)=0$. □

Remark 3. — Observe that

$$D(f)\geqq 1/\nu_1\ldots\nu_K\geqq 2^{-(2d-2)}, \tag{13}$$

where $d=\deg f$, K denotes the number of critical points in $J=\hat{\mathbb{C}}$ and ν_j-1 are their multiplicities.

Indeed if δ_2 is chosen small enough, then every critical point is contained in at most one set from the trajectory $f^{n-k-N}(T),\ldots,f^{n-k-1}(T)$ for every $T\in\mathscr{T}_{k,\,\mathrm{sing}}$. ■

Remark 4. — From (12) one immediately obtains for every $A\subset\hat{\mathbb{C}}$

$$\eta(A)\leqq C(\mu(A))^{D(f)} \tag{14}$$

Also a related estimate, better than (3), holds

$$(d\eta/d\mu)(x)\leqq C\,\mathrm{dist}(x,\mathrm{Crit}^+)^{2(D(f)-1)}. \tag{15}$$ ■

Proof of (15). — Let $x\in B=B(z,\delta_2/2)$ for $z\in\mathrm{cl}(\mathrm{Crit}^+)$, see the notation from the proof of Lemma 6. We assumed there $z\in\omega$, but the discussion there holds for every z. If $z\notin B(\omega,\varepsilon)$, then for every n we have only decomposition $\mathscr{T}=\mathscr{T}_{\mathrm{reg}}\cup\mathscr{T}_{\mathrm{sing}}$, where $T\in\mathscr{T}_{\mathrm{reg}}$ or $\mathscr{T}_{\mathrm{sing}}$ depending as $\bigcup_{i=0}^{n-1} f^i(T)\cap\mathrm{Crit}=\varnothing$ or $\neq\varnothing$.

Denote $\mathrm{dist}(x,\mathrm{Crit}^+)=\beta$ and suppose $\beta>0$. We want to estimate $\sum_{y\in f^{-n}(x)}|(f^n)'(y)|^{-2}$. For every $T\in\mathscr{T}_n\cup\bigcup_{k=0}^{n-1}\mathscr{T}_{k,\,\mathrm{reg}}$ by the bounded distortion property we have for the branch $f_\nu^{-n}:B\to T$

$$|(f_\nu^{-n})'(x)|^2/(\mu(f_\nu^{-n}(B))/\mu(B))<C.$$

Suppose that $T \in \mathscr{T}_{k,\,\mathrm{sing}}$ and fix $y \in T \cap f^{-n}(\{x\})$. We have

$$\begin{aligned}
&|(f^n)'(y)|^{-2}/(\mu(T)/\mu(f^n(T)))\\
&\qquad = (|(f^k)'(f^{n-k}(y))|^{-2}/(\mu(f^{n-k}(T))/\mu(f^n(T))))\\
&\qquad \times (|(f^M)'(f^{n-k-M}(y))|^{-2}/(\mu(f^{n-k-M}(T))/\mu(f^{n-k}(T))))\\
&\qquad\qquad \times (|(f^{n-k-M})'(y)|^{-2}/(\mu(T)/\mu(f^{n-k-M}(T)))) = \mathrm{I} \times \mathrm{II} \times \mathrm{III}.
\end{aligned}$$

By the bounded distortion property $\mathrm{I} \times \mathrm{III} \leqq C$. To estimate II write $\hat{\delta} = \operatorname{diam}(f^{n-k}(T))$, $\hat{\beta} = \operatorname{dist}(f^{n-k}(y), \mathrm{Crit}^+)$ and denote by D the reciprocal to the degree of f^M on $f^{n-k-M}(T)$.

We have

$$\begin{aligned}
\mathrm{II} &\leqq C(\hat{\beta}^{D-1})^2/(\hat{\delta}^{D-1})^2\\
&\leqq C(((f^k)'(f^{n-k}(y)))^{-1}\beta \times (f^k)'(f^{n-k}(y))\delta_2)^{2(D-1)} \leqq C\beta^{2(D-1)}.
\end{aligned}$$

Summing up the respective inequalities over all T proves (15). $\square$

Remark 5. — One could slightly modify the proofs of Lemma 6 and inequality (15) if one observed that

$$\operatorname{Card}(f^{n-k-M}(\mathscr{T}_{k,\,\mathrm{sing}})) \leqq \sum_{l=0}^{M-1} d^l(2d-2),$$

where $d = \deg f$ (observe that the right hand side constant is independent of k and n).

This estimate follows from the fact that for $T \in \mathscr{T}_{k,\,\mathrm{sing}}$, for $s \geqq n-k$, $f^s(T)$ does not contain any critical point and for every $s: n-k-M \leqq s < n-k$ at most the number $2d-2 \geqq \operatorname{Card}(\mathrm{Crit})$ of components of $f^s(\mathscr{T}_{k,\,\mathrm{sing}})$ contain critical points.

Thus, in the proof of Lemma 6 for example, if instead of (9) we used the weaker inequality

$$\begin{aligned}
&\mu(f^{n-k-M}(T) \cap f^{-k-M}(A))\\
&\qquad\qquad \leqq C\mu(f^{n-k}(T) \cap f^{-k}(A))^{D(f)} \leqq C(\lambda^{2k}\mu(A))^{D(f)},
\end{aligned}$$

we would also succeed:

$$\sum_{k=0}^{n-1}\ \sum_{T \in \mathscr{T}_{k,\,\mathrm{sing}}} \mu(T \cap f^{-n}(A)) \leqq C\sum_k (\lambda^{2k}\mu(A))^{D(f)} \leqq C(\mu(A))^{D(f)}. \quad \blacksquare$$

Remark 6. — There exists $\xi > 0$ such that for every $\varepsilon > 0$,

$$\eta(B(\omega, \varepsilon)) \leqq C\varepsilon^{\xi}.$$

This fact is stronger than Lemma 6. It follows from (14) and the similar fact with η replaced by μ. The latter is a general phenomenon. Namely the following holds:

PROPOSITION 2. — *Let* X *be a closed nowhere dense subset of* $\hat{\mathbb{C}}$ *such that* $f(X) = X$ *and* $f|_X$ *is expanding. Then for upper limit capacity defined* *by* $\overline{\mathrm{Cap}}(\mathrm{X}) = \limsup_{\varepsilon \to 0} \dfrac{\log \mathrm{N}_\varepsilon(\mathrm{X})}{-\log \varepsilon}$, *where* $\mathrm{N}_\varepsilon(\mathrm{X})$ *is the minimal number of disc of radius* ε *covering* X, *we have*

$$\overline{\mathrm{Cap}}(\mathrm{X}) < 2$$

and there exist C, $\kappa > 0$ *such that for every* $\varepsilon > 0$,

$$\mu(\mathrm{B}(\mathrm{X}, \varepsilon) < \mathrm{C}\,\varepsilon^{\kappa}. \quad ■$$

Proof (A hint). – Consider covering by squares rather than disc and observe with the use of Koebe's distortion theorem that there exists an integer M such that for every square K partitioned into a family $\mathscr{K}$ of squares of side r/M, where r denotes the side of K, at least one element of $\mathscr{K}$ is disjoint with X. Next use this observation to subsequent divisions into squares of sides $1/\mathrm{M}^n$. □

Remark 7 [(ii') implies (ii)]. – By the equivalence of μ and η $\lim_{j \to \infty} \mu(f^{n_j}(\mathrm{A})) = \mu(\hat{\mathbb{C}}) = 1$ implies $\lim_{j \to \infty} \eta(f^{n_j}(\mathrm{A}) = 1$. Since the sequence $\eta(f^n(\mathrm{A}))$ is monotone increasing this implies $\lim_{n \to \infty} \eta(f^n(\mathrm{A}) = 1$. So by the equivalence of μ and η $\lim_{n \to \infty} \mu(f^n(\mathrm{A})) = 1$. ■

Remark 8. – Metric exactness obviously implies ergodicity of η. So every weak $-*$ limit of (ν_n) is equivalent to μ and ergodic. We conclude that the whole sequence ν_n is weakly $-*$ convergent to the unique measure η.

Moreover, the sequence of the densities $\mathrm{F}_n = d(f^n_*(\mu))/d\mu$ is equicontinuous on every compact set K disjoint with Crit^+. This follows from the theorem of Koebe [H], Lemma 17.4.1 that there exists a universal constant $\mathrm{C} > 0$ such that for the disc $\mathrm{B}(z, \varepsilon)$ disjoint with Crit^+ (*cf.* the begin of Proof of Th. 3), for every branch f_ν^{-n} we have

$$\sup_{x \in (\mathrm{B}(z, \varepsilon/2)} |(f_\nu^{-n})''(x)/(f_\nu^{-n})'(x)| \leqq \mathrm{C}.$$

Indeed this allows to estimate the derivative along an arbitrary vector v of length 1

$$|d(\mathrm{F}_n)/dv| \leqq \sum_\nu |(d|(f_\nu^{-n})'|^2/(dv)(x)$$

$$\leqq \sum_\nu 2|(f_\nu^{-n})''(x)/(f_\nu^{-n})'(x)|.|(f_\nu^{-n})'(x)|^2 \leqq 2\,\mathrm{C}.$$

We conclude that the densities $d\nu_n/d\mu$ converge to $d\eta/d\mu$ uniformly on every K. ■

In fact the above convergence holds even in analytic functions, namely the following holds:

PROPOSITION 3. — *For every* disc $B(z, \varepsilon)$ *disjoint with* Crit^+ *the functions* $F_n = df^n_*(\mu)/d\mu$ *are real analytic and subharmonic. They can be extended to complex analytic functions* $\tilde{F}_n$ *on the ball* $\tilde{B}(z, \varepsilon/4) \subset \mathbb{C}^2$, *the extension of the* disc $B(z, \varepsilon/4) \subset \mathbb{R}^2 = \mathbb{C}$, *uniformly bounded.*

In consequence $d\eta/d\mu$ *is real-analytic and subharmonic.* ∎

Proof. — For every $x \in B(z, \varepsilon/2)$ we have

$$F_n(x) = \sum_\nu |(f_\nu^{-n})'(x)|^2 \leqq \sum_\nu \left| \sum_{k=0}^{\infty} a_{\nu,k} (x-z)^k \right|^2,$$

where

$$|a_{\nu,k}| \leqq \frac{1}{2\pi} \int_{|x-z|=2\varepsilon/3} \frac{|(f_\nu^{-n})'(x)|}{(2\varepsilon/3)^{k+1}} d|x| \leqq C(2\varepsilon/3)^{-k} |(f_\nu^{-k})'(z)|.$$

We work in some charts in which we denote $x=(x_1, x_2)$, $z=(z_1, z_2)$. For each branch f_ν^{-n} we estimate, using the above,

$$|(f_\nu^{-n})'(x_1, x_2)|^2 \leqq \sum_{s,t=0}^{\infty} b_{\nu,s,t} |(x_1-z_1)^s (x_2-z_2)^t|, \tag{16}$$

where

$$b_{\nu,s,t} = C\, 2^{s+t} st (2\varepsilon/3)^{-(s+t)} |(f_\nu^{-n})'(z)|^2.$$

So we have convergence in the polydisc in $\mathbb{C}^2$ of radius $\varepsilon/3$. Summing of (16) over all branches f_ν^{-n} gives a bound independent of n on $\tilde{B}(z, \varepsilon/4)$. So the family $\tilde{F}_n$ is equicontinuous on compact sets in $\tilde{B}(z, \varepsilon/4)$ and the limit functions in the uniform convergence topology are analytic. Subharmonicity follows from the fact that every $|(f_\nu^{-n})'|^2$, as the composition of the harmonic function $2\log|(f_\nu^{-n})'|$ with the convex increasing function exp, is subharmonic, *see* [Hö].

Note: Already after the completion of this paper our attention was drawn to the fact that some of our assertions are explicitly contained in Lyubich's papers other than [L], namely in

[1] Entropy properties of rational endomorphisms of the Riemann sphere, *Erg. Th. and Dyn. Sys.*, Vol. **3**, 1983, pp. 351-385.

[2] A study of the stability of dynamics of rational functions, *Teoria Funkcij, Funkcj. An. pril.* 42, Kharkov, 1984, pp. 72-90 (in Russian).

[3] Dynamics of rational maps: the topological picture, *Usp. Mat. Nauk.*, Vol. **41**, 4, 1986, pp. 35-95.

Lemma 1 overlaps with [1] Prop. 3, [2] Lemma 1.1 and [3] Prop. 1.10. Corollary 2 can be found in [2] and overlaps with [3] Prop. 1.11. The concept of the "easy" set $\omega^\perp$, present already in [L] is explicit in [3] section

1.19. The part $\mu(\omega^{\perp})=0$ of our Lemma 4 and Theorem 1 [except (ii')] can be found there.

REFERENCES

[D] A. DOUADY, Disques de Siegel et anneaux de Herman, *Séminaire Bourbaki*, No. 677, février 1987.

[D-H] A. DOUADY and J. HUBBARD, Étude dynamique des polynômes complexes, *Publ. Math. Orsay*, (2), Vol. **84**, (4), p. 85.

[H] E. HILLE, *Analytic Function Theory*, Boston-New York-Chicago-Atlanta-Dallas-Palo Alto-Toronto: Ginn and Company, 1962.

[Her] M. HERMAN, Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann, *Bull. Soc. Math. France*, Vol. **112**, 1984, pp. 93-142.

[Hö] L. HÖRMANDER, *An Introduction to Complex Analysis in Several Variables*, D. van Nostrand Company Inc., Princeton, New Jersey, 1966.

[K-S] K. KRZYZEWSKI and W. SZLENK, On Invariant Measures for Expanding Differentiable Mappings, *Studia Math.*, Vol. **33**, 1969, pp. 83-92.

[L] M. Ju. LYUBICH, On a Typical Behaviour of Trajectories for a Rational Map of Sphere, *Dokl. Ak. N. U.S.S.R.*, Vol. **268.1**, 1982, pp. 29-32.

[M] M. MISIUREWICZ, Absolutely Continuous Measures for Certain Maps of an Interval, *Publications Mathématiques I.H.E.S.*, Vol. **53**, pp. 17-51.

[R] M. REES, Positive Measure Sets of Ergodic Rational Maps, *Ann. Sci. Éc. Norm. Sup.*, 4e série, Vol. **19**, 1986, pp. 383-407.

[S] W. SZLENK, Absolutely Continuous Invariant Measures for Rational Mappings of the Sphere S^2, Geometric Dynamics (Rio de Janeiro, 1981), pp. 767-783, *Lect. Notes Math.*, Vol. **1007**, Springer, Berlin-New York, 1983.

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