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Non-hyperbolicity and invariant measures for unimodal maps

by

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There are several results dealing with the existence of an invariant probability measure, which is absolutely continuous with respect to the Lebesgue measure (acim in short), see for example [C-E], [K], [N-S]. In this note we want to describe a weak condition which guarantees the existence of such a measure. We believe this condition is even equivalent to the existence of acim's. We say that the critical point $c$ of a smooth map $f$ has order $l$ if there are constants $O_1, O_2$ so that

$$O_1 |x - c|^{-l-1} \leq |Df(x)| \leq O_2 |x - c|^{-l-1}$$

(NF).

As usual let $f^n$ be the $n$-th iterate of $f$ and let $c_1 = f(c)$. Furthermore denote the Lebesgue measure of a measurable set $I$ by $|I|$.

Main theorem. – Suppose that $f$ is unimodal, $C^3$, has negative Schwarzian derivative and that the critical point of $f$ is of order $l \geq 1$. Moreover assume that the growth-rate of $|Df^n(c_1)|$ is so fast that

$$\sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/l} < \infty$$

holds. Then $f$ has a unique absolutely continuous invariant probability measure $\mu$ which is ergodic and of positive entropy. Furthermore there exists a positive constant $K$ such that

$$\mu(A) \leq K |A|^{1/l},$$
for any measurable set $A \subset (0,1)$.

M. Benedicks and L. S. Young announced the existence of acim's for maps for which $|Df^n(c_1)|$ grows at least polynomially.

Of course the estimate $\mu(A) \leq K |A|^{1/l}$ shows that the poles of the invariant measure $\mu$ are at most of the form $|x - x_0|^{1/l - 1}$. It is not hard to show that any absolutely continuous invariant probability measure has a pole of this order at the critical values $f^n(c)$, $n \geq 1$, and therefore this estimate is optimal. Even for maps for which $|Df^n(c_1)|$ grows exponentially this result is new (the results in [C-E] and [N-S] only give some bounds for the order of the poles).

A REFORMULATION OF THE MAIN THEOREM AND AN OUTLINE OF ITS PROOF

In [BL] it is shown that any unimodal map with negative Schwarzian derivative is ergodic (w.r.t. to the Lebesgue measure) and that any absolutely continuous invariant probability measure $\mu$ has positive metric entropy. Therefore, in order to prove the Main Theorem it is enough to establish the existence of an absolutely continuous invariant probability measure $\mu$.

In order to prove the existence of this invariant measure we will use the strategy of [N-S]. Using general arguments one can show that $f$ has an absolutely continuous invariant probability measure provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any measurable set $A$ with $|A| < \delta$ one has that $|f^{-n}(A)| < \varepsilon$ for all $n > 0$. In fact in this paper we will prove the following more precise statement: there exists a constant $K$ such that for every $n$ and every measurable set $A$,

$$|f^{-n}(A)| < K |A|^{1/l}. \quad (1)$$

One of the main results in [N-S] was to show that (1) can be deduced from the following: there exists a constant $K'$ such that for any $n$ and every $\varepsilon > 0$,

$$|f^{-n}(c_1 - \varepsilon, c_1)| < K' \varepsilon^{1/l} \quad (2)$$

where $l$ is the order of the critical point of $f$. Because of the non-flatness condition at the critical point this is equivalent to: there exists a constant $K''$ such that for every $n > 0$ and every $\varepsilon > 0$

$$|f^{-n}(c - \varepsilon, c + \varepsilon)| < K'' \varepsilon.$$

From all this it follows that the Main Theorem can be deduced from

**Theorem.** Suppose that $f$ is unimodal, $C^3$, has negative Schwarzian derivative and that the critical point of $f$ is of order $l \geq 1$. Moreover assume
that
\[ \sum_{n=0}^{\infty} |Df^n(c_1)|^{-1/\ell} < \infty \]
holds. Then there exists a constant \( K < \infty \) such that for each \( \varepsilon > 0 \),
\[ |f^{-n}(c-\varepsilon, c+\varepsilon)| < K \varepsilon. \] (3)

Let us say a few words about the proof of inequality (3). The main idea is to show that each component of \( f^{-n}(c-\varepsilon, c+\varepsilon) \) is either contained in or at least can be compared in size (this process we will call 'sliding') with a set of the form
\[ f^{-(n-k)}\left(c - \frac{\varepsilon}{|Df^k(c_1)|^{1/\ell}}, c + \frac{\varepsilon}{|Df^k(c_1)|^{1/\ell}}\right). \]

Using this and the summability condition, inequality (3) will then be proved by induction.

Details of the proof can be found in a paper which will appear in Inventiones Mathematicae.

REFERENCES


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