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Invariant measures of the geodesic flow and measures at infinity on negatively curved manifolds

by

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ABSTRACT. — Basing upon the correspondence between the invariant measures of the geodesic flow on a negatively curved manifold and the measures on the sphere at infinity, we give new constructions of the maximal entropy and the harmonic invariant measures of the geodesic flow.

Key words : Negatively curved manifold, sphere at infinity, geodesic flow, Patterson measure, Brownian motion, harmonic measure.

RÉSUMÉ. — En partant de la correspondance entre les mesures invariantes pour le flot géodésique sur une variété à courbure négative et les mesures sur la sphère à l'infini, nous donnons des constructions nouvelles des mesures invariantes (la mesure d'entropie maximale et la mesure harmonique) pour le flot géodésique.

Classification A.M.S. : Primary 58 F 11, 58 F 17, 58 G 32, Secondary 28 D 20, 53 C 20, 60 J 45.

0. INTRODUCTION

Let M be a simply connected negatively curved manifold, ∂M – its sphere at infinity. Then every infinite geodesic ξ on M has two endpoints $\xi_\pm = \lim_{t \to \pm\infty} \xi(t) \in \partial M$, and, conversely, every pair of distinct points $(\xi_-, \xi_+) \in \partial^2 M = \partial M \times \partial M \setminus \mathrm{diag}$ determines an infinite geodesic ξ on M. Thus there exists a natural one-to-one correspondence between the Radon invariant measures of the geodesic flow on the unit tangent bundle SM (the measures on the space of parametrized geodesics invariant with respect to the shift along geodesics) and the Radon measures on $\partial^2 M$ (the measures on the pairs of the endpoints of geodesics, *i.e.* the measures on the space of non-parametrized geodesics). If N is a negatively curved manifold with the fundamental group $G = \pi_1(N)$, and M its universal covering space, then Radon invariant measures of the geodesic flow on SN correspond to those invariant measures of the geodesic flow on SM, which are also G-invariant. Hence we get a correspondence between invariant Radon measures of the geodesic flow on SN and G-invariant Radon measures on $\partial^2 M$.

Thus, in order to obtain an invariant measure of the geodesic flow on SN, one can proceed in the following way. Take a quasi-invariant (with respect to G) finite measure ν on ∂M. If there exists a function f on $\partial^2 M$ such that the measure

$$(0.1) \qquad d\Lambda(\xi_-, \xi_+) = f(\xi_-, \xi_+)\, d\nu(\xi_-)\, d\nu(\xi_+)$$

is G-invariant (*i.e.* the square of the Radon-Nikodym cocycle of the measure ν is cohomological to zero), then we get an invariant measure of the geodesic flow on SN. For example, let N be a manifold of dimension $(d+1)$ with the constant sectional curvature -1. Then M is the hyperbolic $(d+1)$-space and ∂M is the d-sphere. Fix a point $x \in M$ and take the visibility Lebesgue measure λ_x on ∂M: for a set $A \subset \partial M$ its measure $\lambda_x(A)$ is the solid angle under which A can be seen from x. Then the measure

$$(0.2) \qquad d\Lambda(\xi_-, \xi_+) = |\xi_- - \xi_+|_x^{-2d}\, d\lambda_x(\xi_-)\, d\lambda_x(\xi_+),$$

where $|\xi_- - \xi_+|_x$ is the angle between the points ξ_- and ξ_+ as seen from the point x, is invariant with respect to the group of the isometries of M and corresponds to the natural Riemannian invariant measure of the geodesic flow on SM. This idea has been extensively used in [P1, P2, S2] (*see* also [S1]) for studying invariant measures of the geodesic flow on non-compact manifolds with constant negative curvature (in this case the dimension d is replaced with the conformal dimension δ).

Here we apply this approach to arbitrary negatively curved manifolds. In this case the theory is non-trivial even for compact manifolds N. Indeed, in the constant curvature case there exists only one natural measure type on the boundary ∂M (M is the universal covering space of N), whereas in general situation we get three different measure types on ∂M (coinciding for the constant curvature): the visibility type (obtained as the image on ∂M of the Lebesgue measure type on the 1-sphere in the tangent space of a certain point $x \in M$), the harmonic type – the type of the hitting distributions on ∂M of the Brownian motion on M, and the Patterson measure type – the type of the limits (as s tends to the critical value) of the probability measures on M obtained by norming the measures

$$(0.3) \qquad \sum \exp(-s \operatorname{dist}(x, gx))\, \delta_{gx},$$

where the sum is taken over all elements g of the fundamental group $\pi_1(N)$ and dist is the Riemannian distance on M. The Riemannian invariant measure of the geodesic flow corresponds to only one of these types – the visibility type.

For all these three cases the corresponding invariant measure of the geodesic flow on SN can be obtained by the formula (0.1) using different weights f. Fix a point $x \in M$ and take in each measure class on ∂M the measure corresponding to this point (visibility measure as seen from x, harmonic distribution with the starting point x, the Patterson measure with the reference point x, respectively). Then it will be natural to consider the weights f as functions on x like in the formula (0.2). These weights can be obtained as the limits

$$(0.4) \qquad f_x(\xi_-, \xi_+) = \lim \varphi(y_-, x)\, \varphi(x, y_+)/\varphi(y_-, y_+),$$

where $y_- \to \xi_-$, $y_+ \to \xi_+$ and φ is a (symmetric) function on M. Namely, for the visibility measures one should take

$$(0.5) \qquad \varphi(x, y) = \det d\exp_x(\exp_x^{-1}(y)),$$

the determinant of the differential of the exponential map $\exp_x$ evaluated at the point $\exp_x^{-1}(y)$ (I owe this remark to J.-P. Otal); for the harmonic measures

$$(0.6) \qquad \varphi(x, y) = 1/G(x, y),$$

where G is the Green kernel of the Brownian motion on M, and for the Patterson measures

$$(0.7) \qquad \varphi(x, y) = \exp(v \operatorname{dist}(x, y)),$$

where v is the growth of M (*i.e.* the critical exponent of the Poincaré series involved in the definition of the Patterson measure). All these weights can be considered as analogues of the weight (0.2), with which they coincide in the constant curvature case.

Comparing with the formula (0.2) we see that the weights arising from the formulas (0.5), (0.6), (0.7) should be uniformly equivalent to the powers of certain metrics on ∂M. The corresponding metric for the visibility weight coincides with the visibility metric, whereas the metric corresponding to the weight (0.7) can be written as

$$(0.8) \qquad \rho_x^\varepsilon(\xi_-, \xi_+) = \exp(-\varepsilon\, l_x(\xi_-, \xi_+)),$$

where $l_x(\xi_-, \xi_+) = t$ is defined by the condition $\operatorname{dist}(\xi_-(t), \xi_+(t)) = 1$ (we identify here $\xi_\pm$ with the corresponding geodesic rays issued from x) and ε is a sufficiently small constant (introduced in order to satisfy the triangle inequality). Remark that it would be interesting to identify the metric corresponding to the harmonic weight.

The metric (0.8) on ∂M turns out to be natural for the correspondence between the invariant measures of the geodesic flow and the measures on $\partial^2 M$. Particularly, the metric entropy of the geodesic flow coincides (up to a constant multiplier) with the Hausdorff dimension of the corresponding measure on $\partial^2 M$ with respect to the product metric obtained from the metric (0.7). On the other hand, this metric is also convenient for the estimations of the dimension of the harmonic measure type on ∂M. Indeed, in the case when M is the universal covering space of a compact manifold

$$(0.8) \qquad \dim = (1/\varepsilon)\, h(M)/l(M),$$

where dim is the Hausdorff dimension of the harmonic measure type with respect to the metric (0.8), $h(M)$ is the entropy of the Brownian motion on M defined in [K] and $l(M)$ is the rate of escape of the Brownian motion (*cf.* [L1], [K]) – this is an analogue of a well known Ledrappier formula from the theory of smooth dynamical systems. Hence the problem of the interrelations of three different measure types on ∂M (essentially solved in [L2], [L3]) can be expressed also in terms of the dimensions of the corresponding measure types with respect to the metric (0.8). For the author this consideration was a leading reason for introducing this metric. Remark that this metric seems also more natural for the estimations of the Hausdorff dimension of the harmonic measure class for arbitrary simply connected negatively curved manifolds. Namely, the estimates for the Hausdorff dimension can be obtained from the formula (0.9) and the estimates for the entropy and the rate of escape (*cf.* [Ki], [KL]).

This paper was conceived simultaneously with the paper [K], but unfortunately could not be prepared in that time. Recently there appeared a number of papers devoted to the related problems for the cocompact case. Mention the papers by Ledrappier [L2], [L3] devoted to construction of the harmonic invariant measure of the geodesic flow and the interrelations between three measure types and the papers by Hamenstädt ([H1], [H2], [H3]) where she, particularly, identify the maximal entropy measure type on ∂M with the Hausdorff measure corresponding to a metric on ∂M similar to our metric (0.8).

Our aim here is to describe a new approach to the construction of the invariant measures of the geodesic flow. So we don't discuss here in details the ergodic properties of the measures obtained in this way in general situation (for non-compact manifolds), these properties in the compact case being already proven in the cited above papers by Hamenstädt and Ledrappier. We shall return to these problems elsewhere. Remark that this approach can be also used for studying the invariant measures of the geodesic flow on hyperbolic groups, where only a definition of the geodesic flow "up to a quasification" is known [G], whereas invariant measures on the square of the hyperbolic boundary can be defined without any quasification.

The structure of the paper is the following.

In Section 1 we give necessary facts and prove auxiliary statements about the negatively curved simply connected Riemannian manifolds. Particularly, we define the metrics ρ_x^ε on the boundary ∂M (Section 1.3) and show that the function l_x on $\partial^2 M$ participating in the definition of this metric has a clear geometric meaning (Proposition 1.4).

In Section 2 we state a natural correspondence between the invariant (σ-finite) measures of the geodesic flow on a simply connected negatively curved manifold M and the (σ-finite) measures on $\partial^2 M$ ("at the square of infinity") – Theorems 2.1, 2.2. For the case of manifolds with compact quotients (*i.e.* universal covering spaces of compact negatively curved manifolds) we prove that the entropy of the geodesic flow case coincides (up to a constant multiplier) with the dimension of the corresponding measure on $\partial^2 M$ with respect to the metric introduced in the Section 1 (Theorems 2.3, 2.4).

In Section 3 we consider the case when our manifold M has a compact quotient N and construct a measure class at infinity having the maximal dimension with respect to the metrics ρ_x^ε on ∂M introduced in Section 1. Actually we construct a measure which is an analogue of the Patterson measure well known in the constant curvature case ([P2], [S2]). Then, taking the square of this measure on $\partial^2 M$ and multiplying it by the weight (0.7), we get a Radon measure invariant with respect to the action of the fundamental group $\pi_1(N)=G$ and hence an invariant measure of the geodesic flow on SN. Calculating its entropy using the Theorem 2.4 proves that this measure is really the maximal entropy measure of the geodesic flow. This gives yet another construction of the maximal entropy measure (the Bowen-Margulis measure) for the geodesic flow on negatively curved compact manifolds (*cf.* [B1], [M1], [H1]). Our exposition here closely follows the paper [S2] by D. Sullivan. For the sake of simplicity we consider here only the compact quotient case, but this approach can be also applied in the general situation.

In Section 4 we construct an invariant measure of the geodesic flow on a negatively curved manifold N with the harmonic type conditional distributions on ∂M (M-universal covering space of N). For the compact case the Ledrappier's construction [L2] uses the Hölder continuity of the Green kernel of the Brownian motion and strongly relies upon the theory of the Gibbs measures for Anosov flows on compact manifolds. Later another and more straightforward construction of the harmonic invariant measure has been proposed by Hamenstädt [H3], but also using an ergodic theory approach and strongly dependent on the compactness of N. Our idea is based upon a direct probabilistic approach and roughly speaking we simply substitute every two-sided Brownian path with the corresponding geodesic joining the limit points of this path at $-\infty$ and at $+\infty$. So, excluding the time shift (exactly the same trick as with the invariant measures of the geodesic flow in Section 2) we get from the σ-finite measure in the space of two-sided Brownian paths with one-dimensional distribution m (the Riemannian volume on M) a certain natural Radon measure on ∂^2 M belonging to the square of the harmonic measure class on ∂M (Theorem 4.1). This construction is general and can be applied to any simply connected negatively curved manifold with the uniformly bounded sectional curvatures. The obtained measure on ∂^2 M is invariant with respect to the group of isometries of M, so that by Theorem 2.2 we get a locally finite invariant measure of the geodesic flow on any quotient of M. Particularly, for the manifolds with a compact quotient we get exactly the harmonic invariant measure constructed in [L2], [H3]. Remark that this measure can be also obtained directly from the harmonic measures on ∂M using the weight (0.6) – the Naim kernel of the Brownian motion on M ([Ko1], [Ko2]).

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1. NEGATIVELY CURVED BACKGROUND

1.1. Sphere at infinity

Let M be a simply connected negatively curved Riemannian manifold (*Cartan-Hadamard manifold*) with the *pinched* sectional curvatures:

$$-b^2 \leqq \mathrm{K} \leqq -a^2, \qquad 0 < a \leqq b. \tag{1.1}$$

As it is well known, for any point $x \in \mathrm{M}$ the exponential map is a diffeomorphism between the tangent space $\mathrm{T}_x \mathrm{M}$ and the manifold M. By dist we shall always denote the Riemannian distance on M. Say that two geodesic rays on M are *asymptotic* if they lie within a bounded distance one from the other and denote by $\partial \mathrm{M}$ the *space of asymptotic classes of geodesic rays* on M ([BGS], [EO'N]). For any point $\gamma \in \partial \mathrm{M}$ and any $x \in \mathrm{M}$ there exists a unique geodesic ray starting from x and belonging to the class γ. We shall denote this ray (x, γ). By (γ_-, γ_+) denote the unique infinite geodesic belonging to the class $\gamma_- \in \partial \mathrm{M}$ at $-\infty$ and to the class γ_+ at $+\infty$.

The "*sphere at infinity*" $\partial \mathrm{M}$ can be considered as the boundary of M in the *visibility compactification* $\mathrm{M}^- = \mathrm{M} \cup \partial \mathrm{M}$: a sequence (tending to infinity) $x_n \in \mathrm{M}$ is convergent in M^- iff for a certain reference point $x \in \mathrm{M}$ the directions of the vectors $(\exp_x)^{-1}(x_n)$ in $\mathrm{T}_x \mathrm{M}$ converge (this compactification doesn't depend on the choice of the reference point x). The resulting topology on $\partial \mathrm{M}$ is defined by the *cone neighbourhoods*

$$\mathscr{C}_{x,\alpha}(\gamma) = \left\{ \gamma' \in \partial \mathrm{M} : \angle_x(\gamma, \gamma') < \alpha \right\}, \tag{1.2}$$

where $\angle_x(\gamma, \gamma')$ is the angle between the directing vectors of the geodesics (x, γ) and (x, γ') in the tangent space $\mathrm{T}_x \mathrm{M}$, *i.e.* the angle between the points γ and γ' as seen from the point $x \in \mathrm{M}$ [EO'N].

Every linear element ξ from the unit tangent bundle SM can be identified with the two-sided infinite geodesic issued from ξ and endowed with the duly parametrization. Taking the endpoints of this geodesic on $\partial \mathrm{M}$ at $+\infty$ and at $-\infty$ we get the maps

$$\pi^{\pm} : \xi \mapsto \xi_{\pm} = \lim_{t \to \pm\infty} \xi(t) \tag{1.3}$$

from SM to $\partial \mathrm{M}$. Denote by $\pi_x^{\pm}$ the restrictions of these maps to $\mathrm{S}_x \mathrm{M}$, which are homeomorphisms of $\mathrm{S}_x \mathrm{M}$ and $\partial \mathrm{M}$.

Denote by $\partial^2 \mathrm{M}$ the space $\partial \mathrm{M} \times \partial \mathrm{M} \setminus \mathrm{diag} = \{(\gamma_1, \gamma_2) : \gamma_i \in \partial \mathrm{M}, \gamma_1 \neq \gamma_2\}$ endowed with natural *locally compact* topology. This space coincides with the space of all infinite geodesics on M (considered as subsets of M, without any parametrization) with the usual pointwise convergence topology. Adding the natural parametrization along geodesics we get the fibration

$$\pi=(\pi^-, \pi^+): \xi \mapsto (\xi_-, \xi_+) \in \partial^2 M \qquad (1.4)$$

of SM over $\partial^2 M$ with the fibres $\mathbb{R}$. Slightly abusing we shall speak below about the measures on the space $\partial^2 M$ as about the "*measures at the square of infinity*", the measures on ∂M being the "*measures at infinity*".

1.2. Divergence of geodesics

Below we shall use the following statement.

ALEKSANDROV TRIANGLE COMPARISON THEOREM [Al]. – *Let* M *and* M′ *be two Cartan-Hadamard manifolds with the separated sectional curvatures,* i.e. *there exists a constant* $K_0 \leqq 0$ *such that* $K \leqq K_0 \leqq K' \leqq 0$ *for all sectional curvatures* K *and* K′ *in manifolds* M *and* M′, *respectively. Let the points* $x_i \in M$ *and* $x'_i \in M'$ $(i=0, 1, 2)$ *satisfy the condition*

$$\operatorname{dist}(x_i, x_j)=\operatorname{dist}'(x'_i, x'_j), \qquad 0 \leqq i < j \leqq 2, \qquad (1.5)$$

where dist *and* dist′ *are the Riemannian metrics on* M *and* M′, *respectively. If the points* p_i *and* p'_i $(i=1, 2)$ *belong to the geodesic segments* (x_0, x_i) *and* (x'_0, x'_i), *respectively, and satisfy the condition*

$$\operatorname{dist}(x_0, p_i)=\operatorname{dist}'(x'_0, p'_i), \qquad i=1, 2, \qquad (1.6)$$

then

$$\operatorname{dist}(p_1, p_2) \leqq \operatorname{dist}'(p'_1, p'_2). \qquad (1.7)$$

Let α and β be two geodesic rays with common origin and $d(t)=\operatorname{dist}(\alpha(t), \beta(t))$. Comparing M with the hyperbolic plane with the constant curvature $-a^2$ and using the hyperbolic sine theorem one gets that

$$\operatorname{sh}(a.d(t')/2)/\operatorname{sh}(a.t') \leqq \operatorname{sh}(a.d(t)/2)/\operatorname{sh}(a.t) \qquad (1.8)$$

whenever $0 \leqq t' \leqq t$. From the inequality (1.8) follows

PROPOSITION 1.1. – *There exists a constant c depending on the upper bound $-a^2$ of the curvature on* M *only, such that for any two geodesic rays α and β with common origin*

$$\operatorname{dist}(\alpha(t-\tau), \beta(t-\tau)) \leqq \exp(-c\tau), \qquad \forall\, 0 \leqq \tau \leqq t, \qquad (1.9)$$

where t is (uniquely) determined by the relation

$$\operatorname{dist}(\alpha(t), \beta(t))=1. \qquad (1.10)$$

1.3. Metric on the boundary

For any reference point $x \in M$ define the function l_x on $\partial^2 M$ by the relation

$$l_x(\gamma_1, \gamma_2) = t \quad \Leftrightarrow \quad \operatorname{dist}(\alpha_1(t), \alpha_2(t)) = 1, \tag{1.11}$$

where α_i are the geodesic rays (x, γ_i). The neighbourhoods

$$\mathscr{L}_{x,t}(\gamma) = \{\gamma' \in \partial M : l_x(\gamma, \gamma') > t\} \tag{1.12}$$

arising from the function l have the following simple geometrical sense. Take the intersection of the 1-ball centered at the point $\alpha(t)$ of the geodesic ray $\alpha = (x, \gamma)$ with the t-sphere centered at x. Then $\mathscr{L}_{x,t}(\gamma)$ is the "shadow" of this intersection on ∂M if the light propagates from a source at the point x.

From the comparison with the constant curvature case follows that there exist constants A, B > 0 depending on the curvature bounds $-a^2$, $-b^2$ only such that

$$\left.\begin{array}{c} \mathscr{C}_{x, \exp(-t\mathrm{B})}(\gamma) \subset \mathscr{L}_{x,t}(\gamma) \subset \mathscr{C}_{x, \exp(-t\mathrm{A})}(\gamma), \\ \forall x \in M, \qquad \gamma \in \partial M, \qquad t \geqq 1. \end{array}\right\} \tag{1.13}$$

Hence for every point $x \in M$ the neighbourhoods $\mathscr{L}_{x,t}(\gamma)$ determine the visibility topology on ∂M.

For any reference point $x \in M$ and $\varepsilon > 0$ let

$$\rho_x^\varepsilon(\gamma_1, \gamma_2) = \exp(-\varepsilon l_x(\gamma_1, \gamma_2)), \qquad (\gamma_1, \gamma_2) \in \partial^2 M \tag{1.14}$$

and $\rho_x^\varepsilon(\gamma_1, \gamma_2) = 0$ for $\gamma_1 = \gamma_2$ (below we shall sometimes omit the reference point x).

Proposition 1.2. — *There exists $\varepsilon_0 > 0$ (depending on the upper bound $-a^2$ of the curvature on M only) such that ρ_x^ε is a metric on the space ∂M for all $\varepsilon \leqq \varepsilon_0$ and $x \in M$. The topology on ∂M determined by any metric ρ_x^ε coincides with the visibility topology. For the same value of ε the metrics ρ_x^ε and ρ_y^ε corresponding to different points $x, y \in M$ are uniformly equivalent,* i. e. $1/C \leqq \rho_x^\varepsilon/\rho_y^\varepsilon \leqq C$ *for a certain constant* $C = C(\varepsilon, x, y)$.

Proof. — Coincidence of the topologies (and even the Hölder equivalence of the visibility metric on ∂M and ρ_x^ε) follows from the formula (1.13). The uniform equivalence is a corollary of the Proposition 1.4 below. Hence we need to prove only the triangle inequality. Denote by α_i the geodesic rays (x, γ_i) corresponding to the points $\gamma_i \in \partial M$ $(i = 1, 2, 3)$ and let $l_{ij} = l_x(\gamma_i, \gamma_j)$. Suppose $l_{13} = \min\{l_{ij} : 1 \leqq i < j \leqq 3\}$. Then from the Proposition 1.1

$$\begin{aligned} 1 &= \operatorname{dist}(\alpha_1(l_{13}), \alpha_3(l_{13})) \\ &\leqq \operatorname{dist}(\alpha_1(l_{13}), \alpha_2(l_{13})) + \operatorname{dist}(\alpha_2(l_{13}), \alpha_3(l_{13})) \\ &\leqq \exp(-c(l_{12} - l_{13})) + \exp(-c(l_{23} - l_{13})), \end{aligned} \tag{1.15}$$

whence we get the desired triangle inequality for all $\varepsilon \leqq \varepsilon_0 = c$.

Remarks. — 1. In the definition of the metrics ρ^ε one could take an arbitrary radius r instead of $r=1$. The resulting metrics are uniformly equivalent to ρ^ε — *see* Section 1.6 below.

2. We didn't try to find the best possible estimate for ε_0 in terms of the curvature bounds. U. Hamenstädt [H1] has defined an analogous metric on the *punctured space* $\partial M \setminus \{\gamma_0\}$ considering the infinite geodesics issued from γ_0 instead of the geodesic rays issued from a point $x \in M$. In this case the triangle inequality for the corresponding metric is satisfied for all $\varepsilon \leqq a$.

1.4. The Busemann cocycles

For any point $\gamma \in \partial M$ introduce the *Busemann function* on M [BGS]

$$b_\gamma(x) = \lim_{t \to \infty} (\operatorname{dist}(x, \alpha(t)) - t), \tag{1.16}$$

where α is the geodesic ray (x_0, γ) connecting a fixed reference point $x_0 \in M$ with γ. This function is defined up to a constant depending on the reference point x_0, so it will be more convenient to speak about the *Busemann cocycles* on M

$$\beta_\gamma(x, y) = b_\gamma(x) - b_\gamma(y) \tag{1.17}$$

associated with the points $\gamma \in \partial M$. The level sets of the Busemann functions are the *horospheres* on M centered at γ, so that $\beta_\gamma(x, y)$ is the signed distance between the horospheres passing through the points x and y and centered at γ. In other words, $\beta_\gamma(x, y)$ can be considered as a "regularization" of the formal expression $\operatorname{dist}(x, \gamma) - \operatorname{dist}(y, \gamma)$ with γ being a "point at infinity".

For any point $x \in \partial M$ define the following function $\mathscr{B}_x$ on $\partial^2 M$:

$$\mathscr{B}_x(\gamma_1, \gamma_2) = b_{\gamma_1}(x, y) + b_{\gamma_2}(x, y), \tag{1.18}$$

where y belongs to the geodesic (γ_1, γ_2). It is clear that the right-hand side of the formula (1.18) doesn't depend on the choice of y. In other words, $\mathscr{B}_x(\gamma_1, \gamma_2)$ is the length of the segment cut out on the geodesic (γ_1, γ_2) by the horospheres passing through x and centered at γ_1 and γ_2. This function can be considered as a "regularization" of the expression

$$\operatorname{dist}(x, \gamma_1) + \operatorname{dist}(x, \gamma_2) - \operatorname{dist}(\gamma_1, \gamma_2) \tag{1.19}$$

with γ_1, γ_2 being points at infinity. When x varies the values of $\mathscr{B}_x$ satisfy the identity

$$\mathscr{B}_x(\gamma_1, \gamma_2) - \mathscr{B}_y(\gamma_1, \gamma_2) = \beta_{\gamma_1}(x, y) + \beta_{\gamma_2}(x, y). \tag{1.20}$$

Hence we have

PROPOSITION 1.3. – *The square of the Busemann cocycle on any negatively curved simply connected manifold is cohomological to zero.*

Remark. – The "*cross ratio*" $\mathscr{B}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ on ∂M introduced by Otal [O] as a regularization of the expression

$$\operatorname{dist}(\gamma_1, \gamma_3) + \operatorname{dist}(\gamma_2, \gamma_4) - \operatorname{dist}(\gamma_1, \gamma_4) - \operatorname{dist}(\gamma_2, \gamma_3) \tag{1.21}$$

can be written in terms of the functions $\mathscr{B}_x$ as

$$\mathscr{B}(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = -\mathscr{B}_x(\gamma_1, \gamma_3) - \mathscr{B}_x(\gamma_2, \gamma_4) + \mathscr{B}_x(\gamma_1, \gamma_4) - \mathscr{B}_x(\gamma_2, \gamma_3) \tag{1.22}$$

with the right-hand side of the formula (1.22) being independent on x.

1.5. Negatively curved manifold as a hyperbolic metric space

The Cartan-Hadamard manifolds with the bounded away from zero curvature are a particular case of *hyperbolic metric* spaces and the notions of the boundary ∂M and the compactification M^- can be also defined in this more general context ([G], [GH], [CDP]).

For any reference point $x \in M$ the *Gromov product*

$$\left.\begin{array}{c}(x_1 \mid x_2)_x = (1/2)(\operatorname{dist}(x, x_1) + \operatorname{dist}(x, x_2) - \operatorname{dist}(x_1, x_2)), \\ x_1, x_2 \in M\end{array}\right\} \tag{1.23}$$

satisfies the δ-*ultrametric inequality*

$$(x_1 \mid x_2)_x \geqq \min((x_1 \mid x_3)_x, (x_2 \mid x_3)_x) - \delta, \qquad \forall x_1, x_2, x_3 \in M \tag{1.24}$$

for a certain constant δ depending on the upper bound $-a^2$ of the sectional curvature on M only.

The ultrametric inequality (1.24) is equivalent to the following property, which we shall use in the sequel. For any geodesic triangle with the vertices $x_1, x_2, x_3 \in M$ take the points p_1, p_2, p_3 on the sides of this triangle in such a way that for every vertex x_i the distances to the points p_j lying on the sides of the triangle adjacent to x_i are equal, *i. e.* $\operatorname{dist}(x_1, p_1) = \operatorname{dist}(x_1, p_3)$, $\operatorname{dist}(x_2, p_2) = \operatorname{dist}(x_2, p_1)$, $\operatorname{dist}(x_3, p_3) = \operatorname{dist}(x_3, p_2)$ (we assume $p_i \in (x_i, x_{i+1})$). We shall say that the triangle (p_1, p_2, p_3) is the *inner triangle* of the initial triangle (x_1, x_2, x_3). Then there exists an absolute constant D such that

$$\operatorname{dist}(p_i, p_j) \leqq D, \qquad \forall i, j = 1, 2, 3, \tag{1.25}$$

i. e. *the diameter of the inner triangle is less than* D. This statement remains true for the triangles with one, two or three vertices belonging to the boundary ∂M. In this case instead of the condition $\operatorname{dist}(\gamma, p_1)=\operatorname{dist}(\gamma, p_2)$ with $\gamma\in\partial M$ one should take its "regularization" $\beta_\gamma(p_1, p_2)=0$.

A sequence of points $x_n\in M$ converges in the visibility topology iff $(x_n \mid x_m)_x$ tends to infinity as $n, m\to\infty$ for a certain (or, equivalently, for every) reference point x. The boundary ∂M can be identified with the space of the equivalence classes of convergent sequences with respect to the equivalence relation $\{x_n\}\sim\{y_n\}\Leftrightarrow(x_n \mid y_n)_x\to\infty$. The visibility topology on ∂M coincides with the "hyperbolic topology" defined by the neighbourhoods

$$\mathscr{H}_{x,t}(\gamma)=\{\gamma'\in\partial M : (\gamma \mid \gamma')_x > t\}, \tag{1.26}$$

where

$$(\gamma \mid \gamma')_x=\lim_{t\to\infty}(\alpha(t) \mid \alpha'(t))_x=\lim_{t\to\infty}(t-\operatorname{dist}(\alpha(t), \alpha'(t))/2) \tag{1.27}$$

and α (resp., α') is the geodesic ray (x, γ) [resp., (x, γ')]. One can easily see that actually

$$(\gamma \mid \gamma')_x=(1/2)\mathscr{B}_x(\gamma_1, \gamma_2), \tag{1.28}$$

where $\mathscr{B}_x$ is the function on $\partial^2 M$ introduces in Section 1.4.

A metric equivalent to ρ_x^ε from Section 1.3 also can be defined in the context of arbitrary hyperbolic metric spaces. Namely, if X is a hyperbolic metric space with the hyperbolicity constant δ, then for every $\varepsilon\leqq\varepsilon_0$ (with ε_0 depending on δ only) there exists a metric d_ε on the hyperbolic boundary ∂X uniformly equivalent to $\rho_\varepsilon(\gamma_1, \gamma_2)=\exp(-\varepsilon(\gamma_1 \mid \gamma_2))$, where $(\gamma_1 \mid \gamma_2)$ is the Gromov product on ∂X [GH].

1.6. Comparison of functions on $\partial^2 M$

Introduce the last function on $\partial^2 M$ – the distance $d_x(\gamma_1, \gamma_2)$ from a point $x\in M$ to the geodesic (γ_1, γ_2) and prove that all the three functions $l_x(\gamma_1, \gamma_2)$ (Section 1.3), $(\gamma_1 \mid \gamma_2)_x=(1/2)\mathscr{B}_x(\gamma_1, \gamma_2)$ (Sections 1.4, 1.5) and $d_x(\gamma_1, \gamma_2)$ are essentially coincident.

PROPOSITION 1.4. – *There exists a constant* C *depending on the curvature bounds* $-a^2$, $-b^2$ *on the manifold* M *only, such that for all* $x\in M$ *and* $\gamma_1, \gamma_2\in\partial M$ *the difference between any two of the quantities* $l_x(\gamma_1, \gamma_2)$, $(\gamma_1 \mid \gamma_2)_x=(1/2)\mathscr{B}_x(\gamma_1, \gamma_2)$ *and* $d_x(\gamma_1, \gamma_2)$ *doesn't exceed* C.

Proof. – Consider the geodesic triangle (x, γ_1, γ_2) and its inner triangle (p, p_1, p_2) with $p\in(\gamma_1, \gamma_2)$ and $p_i\in(x, \gamma_i)$ $(i=1, 2)$. Then

$$\operatorname{dist}(x, p_1)=\operatorname{dist}(x, p_2)=(\gamma_1 \mid \gamma_2)_x,$$

hence

$$d_x(\gamma_1, \gamma_2)\leqq(\gamma_1 \mid \gamma_2)_x+D, \tag{1.29}$$

where D is the constant from the inequality (1.25).

The quantity $l_x(\gamma_1, \gamma_2)=l$ is defined by the condition $d(l)=\operatorname{dist}(\alpha_1(l), \alpha_2(l))=1$ with $\alpha_i=(x, \gamma_i)$. At the same time $d((\gamma_1 \mid \gamma_2)_x)=\operatorname{dist}(p_1, p_2) \leqq \mathrm{D}$. Hence from the Aleksandrov comparison theorem follows that

$$(\gamma_1 \mid \gamma_2)_x \leqq l_x(\gamma_1, \gamma_2)+\mathrm{C} \tag{1.30}$$

for a certain constant C,

We have to prove now that $l_x(\gamma_1, \gamma_2)$ can't be substantially larger than $d_x(\gamma_1, \gamma_2)$. Let q be the point on the geodesic (γ_1, γ_2) nearest to x. Then in the triangle (x, q, γ_1) the angle at the vertex q equals $\pi/2$, hence all the vertices of the corresponding inner triangle are close to q. Particularly, the distance from the point q to the ray (x, γ_1) [as well as to the ray (x, γ_2)] is bounded. Take the points $q_i \in (x, \gamma_i)$ such that $\operatorname{dist}(x, q_i)=l_x(\gamma_1, \gamma_2)$, so that $\operatorname{dist}(q_1, q_2)=1$. Then the distance from the points q_i to the geodesic (γ_1, γ_2) is also bounded. Hence if $l_x(\gamma_1, \gamma_2)$ is substantially larger than $d_x(\gamma_1, \gamma_2)$, the short cut from γ_1 to γ_2 going through the points p_1 and p_2 is shorter than (γ_1, γ_2), which is impossible. This argument can be easily made constructive providing a constant K such that

$$l_x(\gamma_1, \gamma_2) \leqq d_x(\gamma_1, \gamma_2)+\mathrm{K}, \tag{1.31}$$

which in combination with the inequalities (1.29) and (1.30) gives the desired result.

2. INVARIANT MEASURES OF THE GEODESIC FLOW AND THE MEASURES "AT THE SQUARE OF INFINITY"

2.1. General correspondence

THEOREM 2.1. — *Let* M *be a simply connected negatively curved manifold with the pinched sectional curvatures:* $-b^2 \leqq \mathrm{K} \leqq -a^2$ $(0 < a \leqq b)$. *Then there exists a natural convex isomorphism between the cones of the Radon invariant measures of the geodesic flow on* SM *and the Radon measures on* $\partial^2 \mathrm{M}$.

Proof. — Let Λ be a Radon measure on $\partial^2 \mathrm{M}$, *i.e.* a measure on the space of ends of geodesics in M. Integrating it with respect to the Lebesgue measure along the geodesics (*i.e.* lifting it to SM using the fibration $\pi: \mathrm{SM} \to \partial \mathrm{M}$) we get an invariant measure λ of the geodesic flow on SM. The measure λ is Radon, since for any compact subset of SM its image in $\partial^2 \mathrm{M}$ is also compact as it follows from the results of Section 1.

Conversely, let λ be an invariant measure of the geodesic flow. Now, in order to obtain Λ we have to disintegrate the measure λ excluding the action of the geodesic flow (the shift along the geodesics in M). Let K be a compact subset of ∂^2 M. Take the set

$$K^{\sim} = \{\xi \in SM : \pi(\xi) \in K\} \tag{2.1}$$

and consider the *dissipative decomposition*

$$K^{\sim} = \bigcup_n K_n, \tag{2.2}$$

where the sets K_n are (mod 0) disjoint and $K_n = T^n K_0$ (T is the geodesic flow). Now put

$$\Lambda(K) = \lambda(K_0). \tag{2.3}$$

It is clear that the value of $\lambda(K_0)$ doesn't depend on the decomposition (2.2) and that Λ can be extended to a measure on ∂^2 M. We have to prove only that $\Lambda(K)$ is finite for any compact set $K \subset \partial^2 M$. As it follows from the results of Section 1, for any compact $K \subset \partial^2 M$ there exists a compact set $A \subset M$ such that every geodesic with the ends from K intersects the set A. Now we can take the decomposition (2.2) with

$$K_n = \{\xi \in SM : \pi(\xi) \in K,\ \tau_A(\xi) \in [n, n+1)\}, \tag{2.4}$$

where

$$\tau_A(\xi) = \min\{t : \xi(t) \in A\} \tag{2.5}$$

is the time of the first intersection of the geodesic ξ with the set A. The set K_0 has a compact closure in SM, hence $\Lambda(K)$ is finite and Λ is Radon.

If M is the universal covering space of a negatively curved manifold N, then the invariant measures of the geodesic flow on SN are in natural one-to-one correspondence with those invariant measures of the geodesic flow on SM which are also invariant with respect to the action of the fundamental group π_1 (N). Hence we get the following result.

THEOREM 2.2. — *Let* N *be a negatively curved manifold with the pinched sectional curvatures and* M *be its universal covering manifold. Then there exists a natural convex isomorphism between the cones of the Radon invariant measures of the geodesic flow on* SN *and the Radon measures on* ∂^2 M *invariant with respect to the action of the fundamental group* π_1 (N). *Particularly, in the case when* N *is compact we have an isomorphism between the cones of finite invariant measures of the geodesic flow on* SN *and of the* π_1 (N)*-invariant Radon measures on* ∂^2 M.

Remarks. — 1. The correspondence constructed in the Theorem 2.2 is convex, hence it preserves the ergodicity. Namely, ergodic invariant measures of the geodesic flow on SN are in correspondence with ergodic (with respect to the group action) G-invariant measures on ∂^2 M.

2. For the more general situation of arbitrary hyperbolic metric spaces only a definition of the geodesic flow "up to a quasification" given in [G] is known. Nonetheless our approach permits one to consider the measures on the square of the hyperbolic boundary (defined in canonical way without any quasification) as natural counterparts of invariant measures of any reasonably defined geodesic flow. It would be especially interesting to study the ergodic properties of group invariant σ-finite measures on the square of the boundary of hyperbolic groups. Our approach for constructing the maximal entropy measure and the harmonic measure for the geodesic flow on Riemannian negatively curved manifolds can be also applied in this situation (see below Sections 3 and 4, respectively).

2.2. Measures and measure classes at infinity

Introduce now several measures and measure classes connected with the invariant measures of the geodesic flow (and with the corresponding measures "at the square of infinity"). Let λ be an invariant Radon measure of the geodesic flow on SM and Λ the corresponding Radon measure on $\partial^2 M$.

We can take an arbitrary finite measure ν on SM equivalent to λ and then apply to ν the maps π^+ and π^-. Denote the types of the resulting measures (which are independent on the choice of ν) as λ^+ and λ^-, respectively.

Further, fixing one of the coordinates (γ_-, γ_+) one can define for any $\gamma \in \partial M$ the conditional measures Λ_γ^+ and Λ_γ^- of the measure Λ with respect to the conditions $\gamma_- = \gamma$ and $\gamma_+ = \gamma$. Since the measure Λ is σ-finite, these conditional measures are defined only up to a constant multiplier. One can easily see that actually they can be identified with the conditional measures on the strong stable and strong unstable horospheres of the geodesic flow. The measures Λ_γ^+ (resp., Λ_γ^-) aren't mutually equivalent in general situation, but the projection of a certain finite measure equivalent to Λ onto the second and the first coordinates in $\partial^2 M$, respectively, gives the measure classes Λ^+ and Λ^-, which coincide with the measure classes λ^+ and λ^-, respectively.

Remark that if λ is an invariant measure of the geodesic flow, then the *reflected* measure $\lambda^\wedge$ obtained from λ using the map $\xi \mapsto -\xi$ on SM is also invariant. The measure $\Lambda^\wedge$ on $\partial^2 M$ corresponding to $\lambda^\wedge$ can be obtained from the measure Λ corresponding to λ by reversing the order of the coordinates on $\partial^2 M$. For the reflected measures $\lambda^\wedge$ and $\Lambda^\wedge$ the measures and measure classes introduced above can be obtained from the corresponding measures and measure classes for the measures λ and Λ by substituting the sign $-$ with the sign $+$ and *vice versa*.

2.3. Entropy and dimension

From now on until the end of Section 2 let M be the universal covering space of a *compact* negatively curved manifold N with the fundamental group $G = \pi_1(N)$. In this case the correspondence stated in Section 2.1 can be expressed also in terms of quantitative characteristics of the measures involved: namely, the metric entropy of the geodesic flow on SN coincides (up to a constant multiplier) with the dimension of the corresponding measure on $\partial^2 M$ with respect to the metric introduced in Section 1.

In the sequel we shall use the notation $\xi'(t)$ for the tangent vector to a geodesic ξ at the point $\xi(t)$. Denote by dist' the natural Riemannian metric on SM lifted from the Riemannian metric dist on M [Pe].

LEMMA 2.1. – *There exists a constant* ε_0 (*depending on the manifold* N) *such that for all* $\varepsilon \leqq \varepsilon_0$ *and all natural n the following conditions on geodesics* ξ, η *on* M *are equivalent:*

(i) $\forall i \in [-n, n]\ \exists g_i \in G : \mathrm{dist}'(\xi'(i), g_i \eta'(i)) \leqq \varepsilon$;

(ii) $\exists g \in G : \forall t \in [-n, n]\ \mathrm{dist}'(\xi'(t), g\eta'(t)) \leqq \varepsilon$;

(iii) $\exists g \in G : \mathrm{dist}'(\xi'(-n), g\eta'(-n)) \leqq \varepsilon$, $\mathrm{dist}'(\xi'(n), g\eta'(n)) \leqq \varepsilon$.

In other words, (*the tangent vectors to*) *two geodesic segments on* N *are close in integer points* (i) *if and only if there exist their liftings to* M *which are close* (ii), *and the latter is equivalent to their endpoints being close* (iii).

Proof. – The equivalence of (i) and (ii) follows from the uniform continuity of the time one geodesic shift, whereas the equivalence of (ii) and (iii) follows from the convexity of the function $\mathrm{dist}'(\xi'(t), \eta'(t))$ for any two geodesics ξ, η on M. ■

The next two Lemmas essentially follow from the estimates of the Jacobi fields given in [Pe].

LEMMA 2.2. – *For any* $\varepsilon > 0$ *there exist a constants* $\delta > 0$ (*depending only on* ε *and the curvature bounds of the manifold* M) *such that for every* $t \geqq 1$ *and geodesics* ξ, η *on* M *if*

$$\mathrm{dist}(\xi(-n), \eta(-n)), \ \mathrm{dist}(\xi(n), \eta(n)) \leqq \delta, \tag{2.6}$$

then

$$\mathrm{dist}'(\xi'(-n), \eta'(-n)), \mathrm{dist}'(\xi'(n), \eta'(n)) \leqq \varepsilon. \tag{2.7}$$

LEMMA 2.3. – *For any* $\varepsilon > 0$ *there exists a constant* $c > 0$ (*depending on* ε *and the curvature bounds of the manifold* M *only*) *such that for every* $n \geqq 1$ *and geodesics* ξ, η *on* M *if*

$$\mathrm{dist}'(\xi'(-n), \eta'(-n)), \ \mathrm{dist}'(\xi'(n), \eta'(n)) \leqq \varepsilon, \tag{2.8}$$

then there exists τ *such that* $|\tau| \leqq \varepsilon$ *and*

$$\mathrm{dist}'(\xi'(t), \eta'(t+\tau)) \leqq \varepsilon \exp(-c(n-|t|)) \tag{2.9}$$

for all $t \in [-n+\varepsilon, n-\varepsilon]$.

Remark. — The role of τ in this Lemma is to exclude the tangential component of the Jacobi fields along ξ. It is clear that for τ one can take the difference $t_2 - t_1$, where $\xi(t_1)$ and $\eta(t_2)$ are the points realizing the distance between ξ and η on M.

The following result is probably known to the specialists (*cf.* [LY]), but I couldn't find it in an explicit form in the literature. In order to formulate it introduce the following notation:

$$B(\xi, \varepsilon, k, m) = \{\eta \in SN : \operatorname{dist}'(\xi'(i), \eta'(i)) \leqq \varepsilon \ \forall k \leqq i \leqq m\}, \quad (2.10)$$

where $\xi \in SN$, $k \leqq m$ are integers and $\varepsilon > 0$. By $B(\xi, \varepsilon) = B(\xi, \varepsilon, 0, 0)$ denote the ε-ball around a point $\xi \in SN$ in the metric dist'.

LEMMA 2.4. — *Let* N *be a compact negatively curved Riemannian manifold,* SN *its unit tangent bundle with the canonical metric* dist' *induced by the Riemannian metric,* λ_0 *an ergodic invariant probability measure of the geodesic flow on* SN. *Then for all* $\varepsilon \leqq \varepsilon_0$ *(where* ε_0 *is taken from the Lemma* 2.1) *the function*

$$\varphi_n(\xi) = (-1/2n) \log \lambda_0 B(\xi, \varepsilon, -n, n) \quad (2.11)$$

converges to the metric entropy $h(T, \lambda_0)$ *almost everywhere and in the space* $L^1(SN, \lambda_0)$.

Proof. — Taking such a partition of SN that the diameters of its elements are less than ε one can prove using the Shannon-McMillan-Breiman Theorem that the functions φ_n are dominated by a sequence of functions converging to the entropy of this partition, which doesn't exceed $h(T, \lambda_0)$. Hence we have to estimate the functions φ_n from below only.

From the local entropy theorem [BK] follows that for a.e. $\xi \in SN$

$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \varphi_n(\xi) = h(T, \lambda_0) \quad (2.12)$$

(in [BK] the authors actually consider the functions $(-1/n) \log \lambda_0 B(\xi, \varepsilon, 0, n)$, but their argument can be easily extended for our case).

Take an element $\eta \in B(\xi, \varepsilon, -n, n)$. The Lemma 2.1 means that we can, passing from SN to its universal covering space SM and from the measure λ_0 to its lift λ, consider ξ and η as elements of SM and identify them with geodesics on M. The length of the intersection of η with the ball $B(\xi'(-n), \varepsilon)$ in the metric dist' doesn't exceed 2ε (since the restriction of dist' on geodesics in M coincides with dist). Take a positive Δ. Then from the Lemma 2.3 follows that

$$\operatorname{dist}'(\xi'(-n+\Delta), \eta'(-n+\Delta+\tau)) \leqq \varepsilon \exp(-c\Delta) = \varepsilon', \quad (2.13)$$

and the length of the intersection of η with the ball $B(\xi'(-n+\Delta), 2\varepsilon')$ is not less than $2\varepsilon'$. Repeating this argument for the other end of the geodesic ξ we get the inequality

$$\lambda B(\xi, -n, n, \varepsilon) \leqq (\varepsilon'/\varepsilon) \lambda B(\xi, \varepsilon', -n-\varepsilon+\Delta, n+\varepsilon-\Delta) \quad (2.14)$$

with $\varepsilon' = \varepsilon \exp(-c\Delta)$. Combining the inequality (2.14) with the formula (2.12) we get the desired result.

From the Lemmas 2.2 and 2.4 we immediately get:

THEOREM 2.3. – *Let* N *be a compact negatively curved Riemannian manifold,* M *its universal covering space,* λ_0 *an ergodic invariant probability measure of the geodesic flow on* SN, Λ *the measure on* $\partial^2 M$ *corresponding to* Λ *according to the Theorem* 2.2. *Then for all* $x \in M$ *and* Λ-a.e. $(\gamma_-, \gamma_+) \in \partial^2 M$

$$\lim_{t \to \infty} (-1/t) \log \Lambda(\mathscr{L}_{x,t}(\gamma_-) \times \mathscr{L}_{x,t}(\gamma_+)) = 2h(T, \lambda_0), \quad (2.15)$$

where $\mathscr{L}_{x,t}(\gamma)$ *are the neighbourhoods on* $\partial^2 M$ *introduced in Section* 1.3.

The limit (2.15) up to a constant multiplier coincides with the Hausdorff dimension of the measure Λ with respect to the metric $\rho_x^\varepsilon \times \rho_x^\varepsilon$ on $\partial^2 M$ [Yo], hence we get:

THEOREM 2.4. – *Under the assumptions of the Theorem* 2.3 *the Hausdorff dimension of the measure* Λ *with respect to the metrics* $\rho_x^\varepsilon \times \rho_x^\varepsilon$ *on* $\partial^2 M$ (see *Section* 1.3) *coincides with* $2h(T, \lambda_0)/\varepsilon$, *and for all* $x \in M$ *and* Λ-*a.e.* $(\gamma_-, \gamma_+) \in \partial^2 M$

$$\lim_{\delta \to 0} \log \Lambda(B(\gamma_-, \delta) \times B(\gamma_+, \delta))/\log \delta \to 2h(T, \lambda_0)/\varepsilon, \quad (2.16)$$

where $B(\gamma, \delta)$ *is the* δ-*neighbourhood of* $\gamma \in \partial M$ *in the metric* ρ_x^ε.

Remark. – Results analogical to the Theorems 2.3, 2.4 can be in a similar way obtained also for the a.e. measures $\Lambda_\gamma^\pm$ and the measure types Λ^+ and Λ^-. Namely, their Hausdorff dimension with respect to the metrics ρ_x^ε coincides with $h(T, \lambda_0)/\varepsilon$ (*cf.* [H2]). From this one can deduce the following corollary. Let ν be a probability measure on SM equivalent to λ and decaying at infinity sufficiently fast. Denote by p the canonical map $p: SM \to M$, and let

$$f_t = dp \circ T^t \nu / dp \circ \nu. \quad (2.17)$$

Then

$$(-1/t) \log f_t(T^t \xi) \to h(T, \lambda_0) \quad (2.18)$$

for λ-a.e. $\xi \in SM$ and in the space $L^1(SM, \nu)$ (*cf.* [Bu] for the case when λ_0 coincides with the Riemannian volume on N).

2.4. Examples

Recall that in this Section M is the universal covering space of a compact negatively curved manifold N with the fundamental group $G=\pi_1(N)$. Consider two polar cases: invariant measures of the geodesic flow arising from closed geodesics on N and the Gibbs invariant measures of the geodesic flow.

2.4.1. Closed geodesics

It is well known that the closed geodesics on N are in one-to-one correspondence with the free homotopy classes on M (*i.e.* with the conjugacy classes in G) and simple closed geodesics are in correspondence with the indivisible conjugacy classes in G. Hence every element $g \in G$ defines a closed geodesic on N and thereby an invariant measure λ_0 of the geodesic flow on SN. Lifting it to SM gives us an invariant measure of the geodesic flow on SM. The corresponding measure Λ on $\partial^2 M$ is uniformly distributed on the G-orbit of the point $(g^{-\infty}, g^{\infty}) \in \partial^2 M$, where

$$\begin{aligned} g^{-\infty} &= \lim_{n \to \infty} g^{-n} x \in \partial M \\ g^{\infty} &= \lim_{n \to \infty} g^{n} x \in \partial M \end{aligned} \tag{2.19}$$

(these limits exist and don't depend on the choice of $x M$ – *see* [G], [GH], [CDP] for these facts and other results concerning the action of G on ∂M and $\partial^2 M$). Remark that the G-orbits of the points $(g^{-\infty}, g^{\infty}) \in \partial^2 M$ are in one-to-one correspondence with the indivisible conjugacy classes in G. The measures $\lambda_x^{\pm}$ and $\Lambda_\gamma^{\pm}$ for a. e. x and γ charge only one of the endpoints of that particular geodesic to which belong x or γ. On the other hand the measure classes $\lambda^+=\Lambda^+$ and $\lambda^-=\Lambda^-$ charge the whole G-orbits of the points g^{∞} and $g^{-\infty}$, respectively.

2.4.2. Gibbs measures

Let λ_0 be a Gibbs invariant measure of the geodesic flow on SN. Then the measure Λ belongs to the measure class $\Lambda^- \times \Lambda^+$, because the measure λ can be locally decomposed into the product of the Lebesgue measure along the geodesics and the measures on the strong stable and strong unstable horospheres. Suppose also that λ_0 is quasi-invariant with respect to the strong stable and strong unstable foliations [BR]. Then for a. e. x and j the measures λ_x^+ and Λ_j^+ (resp., λ_x^- and Λ_j^-) belong to the measure classes $\lambda^+=\Lambda^+$ and $\lambda^-=\Lambda^-$, respectively. So we can consider the (Hölder continuous) function.

$$f_x(\gamma_-, \gamma_+) = d\Lambda(\gamma_-, \gamma_+)/d\lambda_x^-(\gamma_-)\, d\lambda_x^+(\gamma_+). \tag{2.20}$$

The properties of this function are closely connected with the properties of the measure λ_0. For example, there exists a constant C such that

$$1/C \leqq f_x(\gamma_-, \gamma_+) \exp(-2h(T, \lambda_0) d_x(\gamma_-, \gamma_+)) \leqq C, \qquad (2.21)$$

for all $x \in M$ and $(\gamma_-, \gamma_+) \in \partial^2 M$, where $d_x(\gamma_-, \gamma_+)$ is the distance from the point x to the geodesic (γ_-, γ_+) (*see* Section 1). Indeed, from a well known property of the Gibbs measures [B2, Si] follows that there exists a constant K such that

$$1/K \leqq \exp(\beta_\gamma(x, y) h(T, \lambda_0)) d\lambda_x^+ / d\lambda_y^+ (\gamma) \leqq K \qquad (2.22)$$

whenever the points $x, y \in M$ and $\gamma \in \partial M$ belong to a same geodesic (and this is also true for the measures λ_x^-). Now (2.22) and the compactness of N imply the inequality (2.21). Remark that the formula (2.21) implies the theorems 2.3 and 2.4 for the Gibbs measures quasi-invariant with respect to the strong stable and strong unstable foliations.

The Riemannian invariant measure of the geodesic flow satisfies this property. F. Ledrappier mentioned that it is not still clear whether the maximal entropy (Bowen-Margulis) and harmonic invariant measures of the geodesic flow also satisfy this property (*see* Sections 3 and 4 below).

3. PATTERSON MEASURE AND THE MAXIMAL ENTROPY MEASURE

3.1. Conformal density

Let N be a negatively curved manifold (not necessarily compact) with the fundamental group $\pi_1(N) = G$ and M its universal covering space endowed with the canonical action of G. We shall say that a family $\{\mu_x\}_{x \in M}$ of finite measures on the boundary ∂M is a *conformal density of dimension* δ (*cf.* [S2], [P2] for the motivations of this definition) if all these measures are pairwise equivalent and

$$d\mu_x / d\mu_y(\gamma) = \exp(-\delta \beta_\gamma(x, y)), \qquad \forall x, y \in M, \text{ a.e. } \gamma \in \partial M, \qquad (3.1)$$

i.e.

$$\mu_x = \exp(-\delta \beta_\gamma(x, y)) \mu_y, \qquad \forall x, y \in M, \qquad (3.2)$$

where β is the Busemann cocycle (*see* Section 1.4). In other words, this means that the *Radon-Nikodym cocycles* $\log d\mu_x / d\mu_y(\gamma)$ of the family $\{\mu_x\}$ are proportional to the Busemann cocycles of the manifold M.

From the cocycle identity for β follows that taking an arbitrary measure $\mu = \mu_x$ and multiplying it then by the Radon-Nikodym derivatives (3.1) one gets a conformal density. We will be interested in *invariant* (with respect to the group G) conformal densities, *i.e.* the densities $\{\mu_x\}$ satisfying the condition

$$g\,\mu_x = \mu_{gx}, \qquad \forall\, x \in \mathrm{M}, \quad g \in \mathrm{G}. \tag{3.3}$$

The cocycle identity and the group invariance of the Busemann cocycle imply that (3.3) holds for *every* $x \in \mathrm{M}$ iff it holds for a *certain* $x \in \mathrm{M}$. Hence *for any* $x \in \mathrm{M}$ *there exists a natural one-to-one correspondence between the invariant conformal densities of the dimension* δ *and finite measures* $\mu = \mu_x$ *on* $\partial \mathrm{M}$ *such that*

$$dg\,\mu/d\mu\,(\gamma) = \exp(-\delta \beta_\gamma (gx, x)). \tag{3.4}$$

Below it will be more convenient for us to deal with the measures μ satisfying the condition (3.4) rather than with the corresponding families $\{\mu_x\}$.

3.2. Connection with the geodesic flow

Let $\mu = \mu_x$ be a finite measure on $\partial \mathrm{M}$ satisfying the condition (3.4). Consider its square μ^2 on the space $\partial^2 \mathrm{M}$. Then

$$dg\,\mu^2/d\mu^2\,(\gamma_-, \gamma_+) = \exp(-\delta \beta_{\gamma_-}(gx, x) - \delta \beta_{\gamma_+}(gx, x)). \tag{3.5}$$

Since the square of the Busemann cocycle is cohomological to zero (Proposition 1.3), we immediately get that the measure

$$d\Lambda(\gamma_-, \gamma_+) = \exp(\delta \mathscr{B}_x(\gamma_-, \gamma_+))\, d\mu_x(\gamma_-)\, d\mu_x(\gamma_+), \tag{3.6}$$

where $\mathscr{B}_x$ is the function on $\partial^2 \mathrm{M}$ defined in Section 1.4, is invariant with respect to the action of the group G. In its turn the measure Λ determines an invariant measure of the geodesic flow on SN by the Theorem 2.2. Hence we get

PROPOSITION 3.1. — *Let* N *be a negatively curved manifold with the fundamental group* $\mathrm{G} = \pi_1(\mathrm{N})$ *and* M *be the universal covering space of* N. *Then every* G*-invariant conformal density on* $\partial \mathrm{M}$ *determines an invariant measure of the geodesic flow on* SN.

3.3. Patterson measure

Describe now a method for constructing invariant conformal densities ([P1], [P2], [S2]).

Fix a reference point $x \in \mathrm{M}$ and for every positive number s consider the *Poincaré series*

$$\Pi^s = \sum_{g \in \mathrm{G}} \exp(-s\,\mathrm{dist}(x, gx)), \tag{3.7}$$

where dist is the Riemannian distance on M. Denote by δ the critical exponent of divergence of the Poincaré series (which doesn't depend on the choice of the point x and is an invariant of the manifold N). Consider the family of probability measures

$$\mu^s = (1/\Pi^s) \sum_{g \in G} \exp(-s\,\mathrm{dist}(x, gx))\,\delta_{gx}, \qquad s < \delta, \tag{3.8}$$

then for any element $g_0 \in G$

$$\begin{aligned} g_0 \mu^s &= (1/\Pi^s) \sum_{g \in G} \exp(-s\,\mathrm{dist}(x, gx))\,\delta_{g_0 gx} \\ &= (1/\Pi^s) \sum_{g \in G} \exp(-s\,\mathrm{dist}(g_0 x, gx))\,\delta_{gx}. \end{aligned} \tag{3.9}$$

Let μ be a weak limit point on $M^- = M \cup \partial M$ of the family $\{\mu^s\}$ as s tends to δ. Suppose that the Poincaré series diverges for $s = \delta$ (actually this restriction can be eliminated by adding a slowly increasing weight – *see* [P1], [A3]), then μ is concentrated on the boundary ∂M, and from the formula (3.9) follows that

$$\begin{aligned} dg_0\mu/d\mu(\gamma) &= \lim_{y \to \gamma} \exp(\delta(\mathrm{dist}(x, y) - \mathrm{dist}(g_0 x, y)) \\ &= \exp(-\delta\beta_\gamma(g_0 x, x)), \end{aligned} \tag{3.10}$$

where β is the Busemann cocycle. In other words, *every weak limit point of the family* $\{\mu^s\}$ *as s tends to* δ *defines an invariant conformal density of dimension* δ *on* ∂M (*a Patterson measure*).

3.4. Cocompact case

From now on let the manifold N be compact. Connect the exponent δ of an invariant conformal density with its dimension with respect to the metrics ρ^ε on the boundary ∂M.

THEOREM 3.1. – *Let* N *be a compact negatively curved manifold with the fundamental group* $G = \pi_1(N)$ *and* M *be the universal covering space of* N. *Let* $\{\mu_x\}$ *be a* G-*invariant conformal density on* ∂M *with the exponent* δ. *Then for all* $x \in M$ *and* μ_x-*a. e. point* $\gamma \in M$

$$\lim_{t \to \infty} (1/t) \log \mu_x \mathscr{L}_{t,x}(\gamma) = -\delta, \tag{3.11}$$

where $\mathscr{L}_{t,x}(\gamma)$ *are the neighbourhoods of the point* γ *defined in Section* 1.3.

LEMMA 3.1. – *Under the assumptions of the Theorem* 3.1 *there exist a positive constant* C *such that for any point* $x \in M$ *and any* $\gamma \in M$

$$\mu_x \mathscr{C}_{x,\pi/2}(\gamma) > C, \tag{3.12}$$

where $\mathscr{C}_{x,\pi/2}(\gamma) \subset M$ *is the cone neighbourhood of* γ *with the pole at* x *and aperture* $\pi/2$ *defined in Section* 1.1.

Proof of the Lemma 3.1. – Remark first that from the compactness of N follows that it is sufficient to prove (3.12) for a certain fixed point $x \in M$ only. Take two separated open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ of ∂M such that $\mu_x \mathcal{O}_i > 0$ $(i=1, 2)$. Compactness of ∂M implies that there exists a finite number of points $\gamma_i \in \partial M$ and an angle θ such that for every γ the cone neighbourhood $\mathcal{C}_{x, \pi/2}(\gamma)$ contains at least one of the sets $\mathcal{C}_{x, \theta}(\gamma_i)$. From proximality of the action of the group G on ∂M [G] follows that for every γ_i there exists a $g_i \in G$ such that either $g_i \mathcal{O}_1$ or $g_i \mathcal{O}_2$ is contained in $\mathcal{C}_{x, \theta}(\gamma_i)$. Using the uniform boundedness of the derivatives $dg_i \mu_x / d\mu_x$ for any finite set of $\{g_i\}$ we get the desired result.

Proof of the Theorem 3.1. – Take the geodesic ray α connecting the points $x \in M$ and $\gamma \in \partial M$. Put $y = \alpha(t)$. Then $\beta_\gamma(x, y) = t$ and one can easily see that there exists an absolute constant C (depending on the curvature bounds only) such that

$$|\beta_\gamma(x, y) - \beta_{\gamma'}(x, y)| \leqq C \tag{3.13}$$

for all $\gamma' \in \mathcal{L}_{t, x}(\gamma)$. The family $\{\mu_x\}$ is a conformal density, hence we get

$$|\log \mu_x \mathcal{L}_{t, x}(\gamma) - \log \mu_y \mathcal{L}_{t, x}(\gamma) + t\delta| \leqq C\delta. \tag{3.14}$$

So we have to prove that the measure $\mu_y \mathcal{L}_{t, x}(\gamma)$ can't be too small. In order to do it remark that there exist constants N, K > 0 (depending on the curvature bounds only) such that

$$\mathcal{C}_{\alpha(t+K), \pi/2}(\gamma) \subset \mathcal{L}_{t, x}(\gamma), \tag{3.15}$$

whenever $t \geqq N$. Applying the Lemma we get the desired result.

As a corollary we obtain:

THEOREM 3.2. – *Under the assumptions of the Proposition* 3.1 *the Hausdorff dimension* $\dim_\varepsilon \mu$ *with respect to the metrics* ρ_x^ε *of an invariant conformal density* μ *on* ∂M *with the exponent* δ *is equal to* δ/ε.

Remark. – Actually this theorem can be proven under weaker assumptions. Namely, it is sufficient to assume that the action of the group $G = \pi_1(N)$ on the convex hull of its limit set on ∂M has a compact fundamental domain (*i.e.* G is *convex cocompact* – *cf.* [S2]). The compactness condition can be also somewhat weakened and replaced by corresponding measure theoretic conditions.

3.5. The Bowen-Margulis measure

Denote by v the *growth* of the universal covering manifold M – the limit

$$v = \lim_{R \to \infty} \log(m B_R(x)), \tag{3.16}$$

where $m\,B_R(x)$ is the Riemannian volume of the R-ball on M centered at x. This limit exists, doesn't depend on x and coincides with the *topological entropy* of the geodesic flow on SN [Man]. Moreover, actually there exists a constant C such that the area of the R-sphere around x is asymptotically $C\exp(R\,v)$ [M1]. This implies that the critical exponent of the Poincaré series coincides with v and that it diverges for $s=v$.

Applying the Theorems 2.3 and 3.1 we get that in this case the Patterson measure gives rise to an invariant measure of the geodesic flow on SN with the metric entropy coinciding with the topological entropy v. In the case of the geodesic flows on negatively curved compact manifolds this measure (*the maximal entropy measure*) is known to be unique and is called the *Bowen-Margulis measure* ([B1], [M2]). In terms of the invariant conformal densities it means that the dimensions of invariant conformal densities on ∂M don't exceed v and there exists a unique invariant conformal density of dimension v. This result can be also obtained directly by proving that the invariant conformal density of maximal dimension actually coincides with the Hausdorff measure of the metric ρ^ε on ∂M [H1] (it can be also generalized for the more general convex compact case – *cf.* [S2]).

Remark. – This approach can be also used to construct the maximal entropy measure for the geodesic flow on the hyperbolic groups (*cf.* Section 2). I have been informed by A. Papadopoulos about a preprint by M. Koornaert with a proof of the uniqueness of the Patterson measure for the hyperbolic groups.

4. HARMONIC INVARIANT MEASURE OF THE GEODESIC FLOW

4.1. Brownian motion on negatively curved manifolds

The Laplace-Beltrami operator Δ of the Riemannian metric on M is the generating operator of a diffusion process on M which is called the *Brownian motion* on M [E,IW]. Denote by $p(t, x, y)$ the transition densities of the Brownian motion with respect to the Riemannian volume m on M. For every point $x\in M$ denote by P_x the probability measure in the space of the one-sided Brownian paths $\xi=\{\xi_t\}_{0\leqq t<\infty}$ starting from the point x at the time 0, and for any measure λ on M let $P_\lambda=\int P_x\,d\lambda(x)$. Particularly, denote by P_m the σ-finite measure in the path space corresponding to the initial distribution m-the Riemannian volume on M.

The measure m is invariant with respect to the Brownian motion, *i.e.* all the one dimensional distributions of the measure P_m coincide with the measure m, hence one can naturally extend the measure P_m to a measure P^∞ on the space of the *two-sided Brownian paths* $\xi=\{\xi_t\}_{\infty<t<\infty}$, which is invariant with respect to the time shift in the path space. It is also invariant with respect to the group of isometries of M. Moreover, since the Brownian transition density is symmetric, the Brownian motion is *reversible* with respect to the measure m, *i.e.* the measure P^∞ is also invariant with respect to the time reversion.

It is well known [Pr] that for a. e. Brownian path ξ there exists the limit

$$\lim_{t\to\infty} \xi(t)=\xi_\infty\in\partial M. \tag{4.1}$$

By $\xi_{-\infty}\in\partial M$ denote the analogous limit taken for $t\to-\infty$. The *harmonic measure* ω_x on ∂M corresponding to a point $x\in M$ is the distribution of the limit point ξ_∞ for the Brownian paths starting from x, *i.e.* $\omega_x(K)=P_x\{\xi:\xi_\infty\in K\}$ for any subset $K\in M$. The harmonic measures corresponding to different points x are mutually equivalent and determine the *harmonic class of measures* on ∂M. One can *identify the points* $\gamma\in M$ *with the extreme positive harmonic functions on* M by the formula

$$\gamma(x)=d\omega_x/d\omega_{x_0}(\gamma), \tag{4.2}$$

where x_0 is a certain fixed reference point on M (actually the *Martin boundary* of the Brownian motion on M coincides with the boundary ∂M [AS,A1], hence the functions γ are defined individually and below we can speak about *all* rather than *almost all* functions γ).

4.2. Conditional decomposition

Every point $\gamma\in M$ defines the *conditional Brownian motion* with the generating operator $\Delta+2\log\gamma$ and the transition densities (with respect to the Riemannian volume m)

$$p_\gamma(t,x,y)=p(t,x,y)\gamma(y)/\gamma(x) \tag{4.3}$$

such that a.e. path of the conditional Brownian motion converges to γ when t tends to $+\infty$. Denote by $P_{x,\gamma}$ the probability measure in the space of one-sided paths of the γ-conditioned Brownian motion $\xi=\{\xi_t\}_{0\leqq t<\infty}$ starting from the point x at the time 0, then

$$P_x=\int P_{x,\gamma}\,d\omega_x(\gamma) \tag{4.4}$$

and this formula gives the decomposition of the measure P_x into the integral of the conditional measures with respect to the partition of the path space generated by the function ξ_∞ (this partition coincides with the *tail* partition of the Brownian motion – *cf.* [D], [K]).

PROPOSITION 4.1. – *For any reference point $x_0 \in M$ the measure P^∞ can be decomposed as*

$$P^\infty = \iint P_{\gamma_-, \gamma_+} \, d\omega_{x_0}(\gamma_-) \, d\omega_{x_0}(\gamma_+), \tag{4.5}$$

where for every pair $(\gamma_-, \gamma_+) \in \partial^2 M$ the σ-finite measure P_{γ_-, γ_+} is Markov, has the stationary distribution $\gamma_- \gamma_+ m$ and is such that $\xi_\infty = \gamma_+$ and $\xi_{-\infty} = \gamma_-$ for P_{γ_-, γ_+}-a. e. path $\xi = \{\xi_t\}_{-\infty < t < \infty}$.

Proof. – Denote for $x \in M$ by Q_x the probability measure in the space of the paths $\xi = \{\xi_t\}_{-\infty < t \leq 0}$ obtained by the time reversion from the measure P_x. For the measures Q_x we have the decomposition

$$Q_x = \int Q_{\gamma, x} \, d\omega_x(\gamma), \tag{4.6}$$

where $Q_{\gamma, x}$ is the time reversion image of the measure $P_{x, \gamma}$. From the Markov property and the reversibility of the Brownian motion with respect to the measure m follows that

$$P^\infty = \int Q_x \otimes P_x \, dm(x). \tag{4.7}$$

Using the decompositions (4.4) and (4.6) we get

$$\begin{aligned} P^\infty &= \iiint Q_{\gamma_-, x} \otimes P_{x, \gamma_+} \, d\omega_x(\gamma_-) \, d\omega_x(\gamma_+) \, dm(x) \\ &= \iiint Q_{\gamma_-, x} \otimes P_{x, \gamma_+} \, \gamma_-(x) \, \gamma_+(x) \, d\omega_{x_0}(\gamma_-) \, d\omega_{x_0}(\gamma_+) \, dm(x), \end{aligned} \tag{4.8}$$

so that

$$P^\infty = \iint P_{\gamma_-, \gamma_+} \, d\omega_{x_0}(\gamma_-) \, d\omega_{x_0}(\gamma_+), \tag{4.9}$$

where

$$P_{\gamma_-, \gamma_+} = \int Q_{\gamma_-, x} \otimes P_{x, \gamma_+} \, \gamma_-(x) \, \gamma_+(x) \, \mathrm{dm}(x). \tag{4.10}$$

One can easily verify that for every pair $(\gamma_-, \gamma_+) \in \partial M$ the measure P_{γ_-, γ_+} is Markov and corresponds to a Markov process with the forward transition densities

$$p_{\gamma_+}(t, x, y) = p(t, x, y) \, \gamma_+(y) / \gamma_+(x) \tag{4.11}$$

and the backward transition densities

$$p_{\gamma_-}(t, x, y) = p(t, x, y)\,\gamma_-(y)/\gamma_-(x) \tag{4.12}$$

(with respect to the Riemannian volume m) and has the stationary σ-finite measure $\gamma_-\gamma_+ m$. Moreover, $\xi_\infty = \gamma_+$ and $\xi_{-\infty} = \gamma_-$ for P_{γ_-,γ_+}-a. e. path $\xi = \{\xi_t\}_{-\infty < t < \infty}$, so that the decomposition (4.5) can be considered as a decomposition of the σ-*finite* measure P^∞ into the conditional measures conditioned by the tail behaviour of the paths both at $+\infty$ and at $-\infty$.

Remark. — Actually the decomposition (4.5) is general and can be constructed for any Markov process with a σ-finite invariant measure (*cf.* [D]). This decomposition doesn't coincide with the ergodic decomposition of the measure P^∞ with respect to the time shift in the path space (it was mentioned by Y. Guivarc'h), but still it is the maximal decomposition with Markov components. On the other hand for the one-sided path space the analogous one-sided decomposition really coincides with the ergodic decomposition of the measure P^∞ with respect to the one-sided time shift.

LEMMA 4.1. — *For every point* $\gamma \in M$ *the corresponding extreme harmonic function* γ *belongs to* $L^2(M \setminus \mathcal{O}_\gamma, m)$ *for any neighbourhood* $\mathcal{O}_\gamma$ *of the point* γ *in the compactification* $M^- = M \cup \partial M$.

Proof (I due the idea of this proof to A. Ancona). — From the *Harnack inequality at infinity* [AS,A1] follows that there exists a constant C such that $\gamma(x) \leqq CG(z, x)$ for every z belonging to the 1-ball $B_1(x_0)$ centered at the reference point x_0 and every $x \in M \setminus (\mathcal{O}_\gamma \cup B_2(x_0))$. Consider a smooth density φ concentrated on $B_1(x_0)$. Then $\gamma(x) \leqq CG\varphi(x)$ for every $x \in M \setminus (\mathcal{O}_\gamma \cup B_2(x_0))$, where $G\varphi$ is the potential of φ. Now

$$\begin{aligned}\int_M (G\varphi)^2(x)\,dm(x) &\leqq (1/\lambda)\langle G\varphi, -\Delta G\varphi\rangle_m = (1/\lambda)\langle G\varphi, \varphi\rangle_m \\ &= (1/\lambda)\iint G(x, y)\,\varphi(x)\,\varphi(y)\,dm(x)\,dm(y) < \infty, \end{aligned}\tag{4.13}$$

where $-\lambda$ is the non-zero top bound of the spectrum of Δ in the space $L^2(M, m)$ [C].

LEMMA 4.2. — *For every pair* $(\gamma_-, \gamma_+) \in \partial^2 M$ *and a point* z *belonging to the geodesic* (γ_-, γ_+) *let* $A = A(\gamma_-, \gamma_+, z)$ *be the hypersurface consisting of all the geodesics passing through* z *and perpendicular to the geodesic* (γ_-, γ_+). *Then for any point* $x \in M \setminus A$

$$\operatorname{dist}(x, z) - C(\alpha) \leqq \operatorname{dist}(x, A) \leqq \operatorname{dist}(x, z), \tag{4.14}$$

where $0 \leqq \alpha < \pi/2$ *is the angle between the geodesic segment* (z, x) *and the geodesic* (γ_-, γ_+), *and* $C(\alpha)$ *is a constant depending on* α *and the curvature bounds for the manifold* M *only.*

Proof. – Let $y \in A$ be such a point that $\operatorname{dist}(x, A) = \operatorname{dist}(x, y)$. Consider the geodesic triangle with the vertices x, y and z and take the inner geodesic triangle with the vertices p_1, p_2 and p_3 on the sides (x, y), (x, z) and (y, z) of the first triangle, respectively, such that

$$\operatorname{dist}(x, p_1) = \operatorname{dist}(x, p_2),$$
$$\operatorname{dist}(y, p_1) = \operatorname{dist}(y, p_3) \quad \text{and} \quad \operatorname{dist}(z, p_2) = \operatorname{dist}(z, p_3).$$

Then all the pairwise distances $\operatorname{dist}(p_i, p_j)$ are less than a constant D, depending on the lower curvature bound of M only (*see* Section 1.5). The angle at the vertex z is bounded off zero and the angle at y is equal to $\pi/2$. Hence, comparing with the zero curvature case we get that the distances

$$\operatorname{dist}(y, p_1) = \operatorname{dist}(y, p_3) \quad \text{and} \quad \operatorname{dist}(z, p_2) = \operatorname{dist}(z, p_3)$$

are uniformly controlled by D, whence the desired result.

LEMMA 4.3. – *There exists a function* Ψ *such that for every* $x \in M$, $\gamma \in M$ *and a positive number* r

$$P_{x, \gamma}\{\xi : \operatorname{dist}(x, \xi_t) \leqq r \,\forall\, t\,[0, 1)\} \geqq 1 - \Psi(r), \tag{4.15}$$

where $\Psi(r) < \exp(-C r^2)$ *for* $r > R$ *with the constants* C *and* R *depending on the curvature bounds of the manifold* M *only.*

Proof. – The drift vector fields $2\nabla \log \gamma$ are uniformly bounded by the upper curvature bound of M [Ya]. Hence the claim follows from the comparison theorem for the Brownian motion [IW] or directly from the Ito lemma [Pr], [E]. ■

4.3. Harmonic invariant measure of the geodesic flow

THEOREM 4.1. – *Let* M *be a simply connected negatively curved manifold with uniformly bounded and bounded off zero sectional curvatures. For every compact set* $K \subset \partial^2 M$ *let*

$$\Lambda(K) = P^{\infty}(K_0), \tag{4.16}$$

where

$$K^{\sim} = \{\xi : (\xi_{-\infty}, \xi_{\infty}) \in K\} = \bigcup K_n \tag{4.17}$$

is a decomposition of the set $K^{\sim}$ *into the union of pairwise disjoint sets* K_n *such that* $K_n = S^n K_0$ (P^{∞}-mod 0), *where* $S^{\tau} \xi(t) = \xi(t - \tau)$ *is the time shift in the path space. Then the value* $\Lambda(K)$ *doesn't depend on the choice of the decomposition* (4.17), *is finite and positive for all compact* K. *The resulting Radon measure* Λ *on* $\partial^2 M$ *belongs to the square of the harmonic measure class and is invariant with respect to the group of isometries of* M.

Proof. – The fact that $\Lambda(K)$ doesn't depend on the choice of the dissipative decomposition (4.17) is general (*cf.* the proof of the Theorem 2.1). We need to prove only the finiteness of Λ on compact subsets $K \subset \partial^2 M$. From the proposition 4.1 follows that

$$P^{\infty}(K_0) = \iint P_{\gamma_-,\gamma_+}(K_0)\, d\omega_{x_0}(\gamma_-)\, d\omega_{x_0}(\gamma_+), \qquad (4.18)$$

hence it is sufficient to construct the decomposition (4.17) only with respect to the conditional measures P_{γ_-,γ_+}. Fix a pair $(\gamma_-, \gamma_+) \in \partial^2 M$ and a point z on the geodesic (γ_-, γ_+) depending on the pair (γ_-, γ_+) measurably (e.g., take for z the point on (γ_-, γ_+) nearest to a certain fixed reference point $x_0 \in M$). Let $A = A(\gamma_-, \gamma_+)$ be the hypersurface in M consisting of all geodesics passing through $z = z(\gamma_-, \gamma_+)$ and perpendicular to the geodesic (γ_-, γ_+). For any path ξ with the limit points (γ_-, γ_+) at $-\infty$ and at $+\infty$, respectively, let

$$\tau_A(\xi) = \inf\{t \in \mathbb{R} : \xi(t) \in A\} \qquad (4.19)$$

be the first time when ξ hits the surface A. Now the sets

$$K_n = \{\xi : (\xi_{-\infty}, \xi_{\infty}) \in K,\ \tau_{A(\xi_{-\infty}, \xi_{\infty})}(\xi) \in [n, n+1)\} \qquad (4.20)$$

give us the decomposition (4.17). Moreover

$$d\Lambda(\gamma_-, \gamma_+)/d\omega_{x_0}(\gamma_-)\, d\omega_{x_0}(\gamma_+) = P_{\gamma_-,\gamma_+}\{\xi : \tau_{A(\gamma_-,\gamma_+)}(\xi) \in [0, 1)\}, \qquad (4.21)$$

and all what we need is to prove the finiteness of the right-hand side of the last formula.

From the Lemma 4.3 follows that

$$\begin{aligned} P_{\gamma_-,\gamma_+}\{\xi : \tau_A(\xi) \in [0, 1)\} &\leqq \int P_{x,\gamma_+}\{\xi : \tau_A(\xi) \in [0, 1)\}\, \gamma_-(x)\, \gamma_+(x)\, dm(x) \\ &\leqq \int \Psi(\operatorname{dist}(x, A))\, \gamma_-(x)\, \gamma_+(x)\, dm(x) \\ &\leqq \left(\int \Psi(\operatorname{dist}(x, A))\, \gamma_-^2(x)\, dm(x) \right)^{1/2} \\ &\quad \times \left(\int \Psi(\operatorname{dist}(x, A))\, \gamma_+^2(x)\, dm(x) \right)^{1/2}. \qquad (4.22) \end{aligned}$$

Each of the factors in the last line is finite, because if $\mathcal{O}_\gamma$ is a cone neighbourhood of a point γ ($=\gamma_-$ or γ_+) with the pole at z, then the integral of $\Psi \gamma^2$ over $M \setminus \mathcal{O}_\gamma$ is finite by the Lemma 4.1, whereas the integral of $\Psi \gamma^2$ over $\mathcal{O}_\gamma$ is finite by the Lemmas 4.2 and 4.3 (the gradient of $\log \gamma$ is uniformly bounded [Ya], hence γ grows at most exponentially with a bounded exponent, the growth of M being also exponential [CE]). ■

As a corollary we get

THEOREM 4.2. — *Let* N *be a negatively curved manifold with pinched sectional curvatures*, M *its universal covering space. Then there exists an invariant Radon measure* λ_0 *of the geodesic flow on* SN *such that all the corresponding conditional measures* λ_x^+ *and* λ_x^- $(x \in M)$ *belong to the harmonic measure class on* ∂M.

Remarks. — 1. We don't discuss here the question about the ergodicity of the resulting invariant measure of the geodesic flow. For the compact case it follows from the results of Ledrappier [L2] (*see* also [H3]) that the measure satisfying the conditions of the Theorem 4.2 is unique, hence ergodic. We shall return to this problem in general situation elsewhere.

2. In the potential theory is known a construction of the symmetric *Naim kernel* on the Martin boundary (coincident in our case with the boundary ∂M) which can be written in our notations as [Ko1, Ko2]

$$\begin{aligned}\Theta(\gamma_-, \gamma_+) &= \liminf_{x \to \gamma_-} \gamma_+(x)/G(x_0, x) \\ &= \liminf_{x \to \gamma_-} \lim_{y \to \gamma_+} G(x, y)/G(x_0, x)\, G(x_0, y) \qquad (4.23)\end{aligned}$$

and gives the *Dirichlet integral* $D(u)$ of harmonic functions u on M with the square integrable boundary values $\hat{u}$ on ∂M by the formula

$$D(u) = \iint |\hat{u}(\gamma_-) - \hat{u}(\gamma_+)|^2 \, d\Lambda^*(\gamma_-, \gamma_+), \qquad (4.24)$$

where

$$d\Lambda^*(\gamma_-, \gamma_+) = \Theta(\gamma_-, \gamma_+)\, d\omega_{x_0}(\gamma_-)\, d\omega_{x_0}(\gamma_+). \qquad (4.25)$$

From the multiplicative inequality for the Green kernel ([A1], Theorem 1) follows that in our situation the Naim kernel is finite for all pairs $(\gamma_-, \gamma_+) \in \partial^2 M$ and the measure Λ^* is a Radon measure on $\partial^2 M$. Moreover, the measure Λ^* is invariant under the action of the group of the isometries of M. In the case when M has a compact quotient using the ergodicity of the harmonic measure class on $\partial^2 M$ one can deduce that the measures Λ and Λ^* are proportional. It seems to be the case also for a general simply connected negatively curved manifold M. It is interesting to provide a direct probabilistic proof of this coincidence and to calculate the ratio Λ/Λ^*.

3. An advantage of our construction is that we actually get the measure Λ dealing with purely probabilistic (measure theoretic) notions only, leaving alone potential theory considerations. Thus one can ask about the finiteness of the density constructed in the Theorem 4.1 for an arbitrary Markov process with a σ-finite stationary measure. This question can be formulated without solving the difficult problem of the identification of

the Martin boundary, but only in the terms of the Poisson boundary as a measure space. For example, one might ask this question for a symmetric random walk on a discrete group with the counting Haar measure as an invariant measure. In this situation the finiteness of Λ would imply that the tensor square of the Radon-Nikodym cocycle of the Poisson boundary (*see* [KV]) is cohomological to zero. This is really the case for the symmetric finite range random walks on hyperbolic groups, because our argument can be reproduced word by word for reversible Markov chains with finite range on hyperbolic graphs (*cf.* [A2]; this situation is even simpler because for finite range chains we don't need the Lemmas 4.2 and 4.3).

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