ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 53, nº 3 (1990), p. 319-342
<http://www.numdam.org/item?id=AIHPA_1990__53_3_319_0>

Ann. Inst. Henri Poincaré,

Vol. 53, n° 3, 1990, p. 319-342. *Physique théorique*

Electrodynamics in Robertson-Walker spacetimes

by

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ABSTRACT. — A complete set of electromagnetic eigenmodes of the Einstein Universe is constructed. On Robertson-Walker spacetimes, the inhomogeneous Maxwell equations are decoupled, integral kernels for the resulting scalar equations are given, and the advanced and retarded fields of arbitrary dipoles are determined. For a larger class of conformally static spacetimes, comoving multipoles and their fields are presented.

RÉSUMÉ. — On commence par construire un ensemble complet de modes propres électromagnétiques de l'Univers d'Einstein statique. Puis on découple les équations de Maxwell inhomogènes sur les espace-temps de Robertson-Walker, on donne des noyaux intégraux pour les équations scalaires résultantes et on détermine les champs avancés et retardés d'un dipole quelconque. En outre, on présente des multipoles comouvants et leurs champs pour une classe plus grande d'espace-temps conformément statiques.

I. INTRODUCTION

Originally, the aim of this work was to give an overview of the solutions of the homogeneous as well as of the inhomogeneous Maxwell equations in the static Einstein Universe. This task arose in the context of investigations concerning the arrow of time in the group of Prof. Dr. G. Süßmann. Parts of this paper are based on my Diplomarbeit there.

The eigenmodes of the Einstein Universe have been the subject of various publications—directly or hidden in contributions to spectral geometry or harmonic analysis, but either they were not given explicitly (mathematics) or the proof of completeness was missing (physics). In sections II to VI (in connection with sections VII and VIII) a bridge is built: In 1940, Schrödinger [1] determined a set of eigenmodes of the Einstein Universe, assuming its completeness. To prove that he was right requires some mathematics: using Coulomb gauge, the spectrum of the Laplace-Beltrami operator on the Hilbert space of coexact 1-forms on S^3 must be determined and an orthonormal eigenbasis found. In [9], the spectra of the Laplace-Beltrami operators on the spaces of p-forms on S^n including the dimensions of the eigenspaces are derived, but they do not agree with the ones of [8]. To be sure that the latter are not correct, I calculate them again using orthogonality relations between spaces of Cartesian k-homogeneous polynomial forms of $\mathbb{R}^4$. Then an orthonormal basis of eigen-1-forms is constructed from the well known scalar spherical harmonics on S^3 [7] by evaluating special 1-form valued linear functionals on them. A proof of its completeness is given and a more detailed version of the Hodge decomposition theorem on S^3 [12] is derived. The relation to group theory is discussed in so far as the representations of SO(4) and O(4) induced in the eigenspaces are identified. Here a connection with the works of Kramer [2] and Jantzen [10] becomes obvious.

It seems to me, that electrodynamics on Robertson-Walker spacetimes with negative spatial curvature has been somewhat neglected so far. Therefore in section VII and IX, Maxwell's equations, especially in the inhomogeneous case, are treated on all Robertson-Walker spacetimes simultaneously. This is possible due to their conformal invariance and a theorem of Dodziuk, who showed in particular that on all simply connected complete Riemannian manifolds of dimension n and constant nonpositive sectional curvature, all L^2 harmonic p-forms with $p \neq \frac{n}{2}$ vanish [11]. This result will be applied parallely to the well known vanishing theorem for L^2 harmonic forms on compact oriented Riemannian manifolds [12]. For fields of finite energy and sources that are C^∞ and have compact support on the time slices, Maxwell's equations turn into a divergence equation for an exact 1-form and a wave equation for a coexact 1-form, both forms being uniquely related to the field if the initial data for the latter are chosen in L^2. With generalizations of the functionals already mentioned it is possible to decouple the coexact equation in such a way that there result two conformally invariant scalar wave equations. The local advanced and retarded Green functions for the latter—which are valid globally in the cases of nonpositive curvature—are constructed according to [5], and the Poisson kernels are given. The missing global Green functions for the wave operator on the Einstein Universe are presented; they take into account the refocussing of the light rays. An analysis of the general solution of the homogeneous conformally invariant scalar wave equation in section VIII shows, that Robertson-Walker spacetimes with nonpositive curvature do not have eigenmodes.

Looking for a way to make the abstract formalism come alive, I finally determined the fields of a few localized source distributions: In section X, starting from the dipoles, comoving electric and magnetic multipoles are defined and their unique "comoving" fields determined for a large class of conformally static spacetimes; electric monopoles are discussed separately and some reflections on elementary charges are added. In section XI finally, electric and magnetic dipoles on Robertson-Walker spacetimes are related to each other, whereupon the advanced and retarded fields of the magnetic dipoles are calculated (according to section IX and a generalization of section VII to the case where the source is a distribution with compact support on the time slices) and expressed in terms of their dipole moments. The result is compared with the fields that Kohler [4] assigned to an electric dipole in the Einstein Universe. It turns out that even locally, the comoving fields agree with the advanced and retarded fields of a comoving dipole only if the spacetime has nonpositive curvature.

II. CONVENTIONS AND PRELIMINARIES

R, ρ, ϑ, φ are spherical and x^i Cartesian coordinates on $\mathbb{R}^4$ endowed with the Euclidian metric such that $\sum_{i=1}^{4} (\mathbf{d}x^i)^2 = \mathbf{d}\mathrm{R}^2 + \mathrm{R}^2\, d\sigma^2$, where $d\sigma^2 = d\rho^2 + \sin^2\rho\,(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$ is the metric of $\mathrm{S}^3 = \mathbb{R}^4|_{\mathrm{R}=1}$. The Einstein Universe is represented by $(\mathbb{R}\times\mathrm{S}^3, d\tau^2 - d\sigma^2)$.

$\mathbf{d}$, $\boldsymbol{\delta}$ and $\boldsymbol{\Delta}$ are the standard operators on p-forms on $\mathbb{R}^4$. On p-forms on S^3, $*$ is the Hodge star, $** = 1$, d is the differential, $\delta = (-1)^p * d *$ is the codifferential, and $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator. $-\delta$ and $*d$ on 1-forms on S^3 correspond to the divergence and curl, respectively, on vector fields on $\mathbb{R}^3$. Sometimes, Δ, δ, d or $*$ are taken to act on p-forms of $\mathbb{R}\times\mathrm{S}^3$ or $\mathbb{R}^4$; then they are defined as acting on the pullback of these forms to $\{\tau\}\times\mathrm{S}^3$ or to the sphere of radius R, *i.e.* $\mathrm{R}\cdot\mathrm{S}^3$, respectively. On C^2 functions f and 1-forms $\mathrm{G} = \mathrm{G}_\mathrm{R}\,\mathbf{d}\mathrm{R} + \mathrm{G}_\mu\, d\mu$ on $\mathbb{R}^4$, $\mu \in \{\rho, \vartheta, \varphi\}$, one has

$$\mathbf{d}f = df + f_{,\mathrm{R}}\,\mathbf{d}\mathrm{R}, \tag{2.0}$$

$$\Delta f = \frac{1}{R^2}\Delta f - f_{,RR} - \frac{3}{R} f_{,R}, \tag{2.1}$$

$$\delta G = \frac{1}{R^2}\delta G - G_{R,R} - \frac{3}{R} G_R, \tag{2.2}$$

$$(\Delta G)_\mu = \frac{1}{R^2}(\Delta G)_\mu - G_{\mu,RR} - \frac{1}{R} G_{\mu,R} - \frac{2}{R} G_{R,\mu}, \tag{2.3}$$

$$R(\Delta G)_R = \Delta(RG_R) - 2\delta G. \tag{2.4}$$

A Hilbert product of two square integrable real or complex valued p-forms α and β on $R\cdot S^3$, $\{\tau\}\times S^3$ or S^3 is given by $\langle\alpha,\beta\rangle := \int_{S^3} \bar{\alpha}\wedge *\beta$.

III. EIGENVALUES AND EIGENSPACES OF THE LAPLACE-BELTRAMI OPERATOR ON S^3

The spaces of Cartesian k-homogeneous polynomial p-forms on $\mathbb{R}^4$ will be shown to be spanned by particular subspaces whose nonvanishing pullbacks to S^3 are eigenspaces of Δ. Since $*\Delta = \Delta *$, the eigenspaces of Δ in the spaces of 2-forms and volume forms on S^3 are given as the Hodge duals of the eigenspaces of Δ in the spaces of 1-forms and functions on S^3, respectively. The cases of functions and 1-forms will be treated parallelly, since essential intermediate results can be transferred.

The vector spaces (over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$) needed are:

$$\begin{gathered}
\mathscr{P}^k := \{P^k \mid P^k\ k\text{-homogeneous polynomial: } \mathbb{R}^4 \to \mathbb{K}\},\\
\mathscr{L}^k := \{L^k \in \mathscr{P}^k \mid \Delta L^k = 0\},\\
\mathscr{F}^k := \{F_i^k\, \mathbf{d}x^i \mid F_i^k \in \mathscr{P}^k, x^i \text{ Cartesian coordinates on } \mathbb{R}^4\},\\
\mathscr{H}^k := \{H^k \in \mathscr{F}^k \mid \Delta H^k = 0\},\\
\mathscr{R}^k := L^{k-1}\,\mathbf{d}R^2 \mid L^{k-1}\in\mathscr{L}^{k-1}\}, \qquad \mathscr{D}^k := \{D^k\in\mathscr{H}^k \mid \delta D^k = 0\},\\
\mathscr{S}^k := \{S^k \in \mathscr{H}^k \mid S_R^k = 0\}, \qquad \mathscr{E}^k := \mathbf{d}\mathscr{L}^{k+1}.
\end{gathered} \tag{3.0}$$

For all of them, a subscript $(.)_s$ denotes the pullback to S^3, and omission of the superscript $(.)^k$ indicates $\operatorname{span}_{\mathbb{K}} \bigcup_{k\in\mathbb{N}_0} (.)^k$, where $\operatorname{span}_{\mathbb{K}} U$ means the vector space of all finite linear combinations of elements of a subset U of a vector space over $\mathbb{K}$.

Examples, of which the last follows from (2.4) and (2.2):

$$\begin{gathered}
\mathscr{L}_s \text{ is the space of spherical harmonics on } S^3,\\
\mathscr{E}_s \text{ consists of exact 1-forms on } S^3, \textit{i.e. } \mathscr{E}_s = d\mathscr{L}_s,\\
\mathscr{S}_s \text{ consists of coclosed 1-forms on } S^3, \textit{i.e. } \delta\mathscr{S}_s = 0.
\end{gathered} \tag{3.1}$$

In this notation the relations to be established are:

$$\mathscr{P}^k = \bigoplus_{l \in \mathbb{N}_0,\, 2l \leq k} \mathrm{R}^{2l} \mathscr{L}^{k-2l},$$
$$\mathscr{F}^k = \bigoplus_{l \in \mathbb{N}_0,\, 2l \leq k} (\mathrm{R}^{2l} \mathscr{R}^{k-2l} \oplus \mathrm{R}^{2l} \mathscr{E}^{k-2l} \oplus \mathrm{R}^{2l} \mathscr{S}^{k-2l}). \tag{3.2}$$

They immediately imply the essence of this section, namely:

$$\mathscr{P}_s = \mathscr{L}_s \qquad \text{and} \qquad \mathscr{F}_s = \mathscr{E}_s \oplus \mathscr{S}_s. \tag{3.3}$$

In order to prove (3.2), it has to be shown that for $\mathscr{X} \in \{ \mathscr{L}, \mathscr{R}, \mathscr{E}, \mathscr{S} \}$,

$$\operatorname{span}_{\mathbb{K}} \bigcup_{l \in \mathbb{N}_0,\, 2l \leq k} \mathrm{R}^{2l} \mathscr{X}^{k-2l} = \bigoplus_{l \in \mathbb{N}_0,\, 2l \leq k} \mathrm{R}^{2l} \mathscr{X}^{k-2l},$$

and that (3.4)

$$\operatorname{span}_{\mathbb{K}} (\mathscr{R} \cup \mathscr{E} \cup \mathscr{S}) = \mathscr{R} \oplus \mathscr{E} \oplus \mathscr{S}.$$

The inclusions "$\supseteq$" then follow because R^2 is a polynomial, and only the equality of the dimensions will remain to be shown.

Proof of (3.4). — Specialising equation (2.1) to $\mathrm{L}^k \in \mathscr{L}^k$ and equation (2.3) to $\mathrm{E}^k \in \mathscr{E}^k$ and $\mathrm{S}^k \in \mathscr{S}^k$ results in:

$$\Delta \mathrm{L}^k = k(k+2)\,\mathrm{L}^k, \qquad \Delta \mathrm{E}^k = (k+1)(k+3)\,\mathrm{E}^k, \qquad \Delta \mathrm{S}^k = (k+1)^2\,\mathrm{S}^k. \tag{3.5}$$

Thus all L^k, E^k, S^k are eigen-p-forms of Δ, the Laplace-Beltrami operator on $\mathbf{S}^3$, and k is uniquely determined by the corresponding eigenvalue. Moreover, the respective sets of eigenvalues for $\mathscr{E}$ and $\mathscr{S}$ are disjoint. Restricting any L^k, E^k or S^k to $\mathbf{S}^3$, the only information one looses is about powers of R, *i. e.* k, and the R-component of E^k, *i. e.* $\mathrm{E}^k_{\mathrm{R}}$. The former can be recovered from the eigenvalue and the latter by means of (2.2) together with $\boldsymbol{\delta}\mathrm{E}^k \in \boldsymbol{\Delta}\mathscr{L} = \{0\}$. Therefore,

$$\text{the restriction mappings } \mathscr{L} \to \mathscr{L}_s, \mathscr{E} \to \mathscr{E}_s, \mathscr{S} \to \mathscr{S}_s \text{ are bijective.} \tag{3.6}$$

For any three $\mathrm{R} \in \mathscr{R}$, $\mathrm{E} \in \mathscr{E}$, $\mathrm{S} \in \mathscr{S}$, the assumption $\mathrm{R}+\mathrm{E}+\mathrm{S}=0$ implies $\mathrm{E}_s + \mathrm{S}_s = 0$ and by (3.1) also $\Delta \mathrm{E}_s = 0 = \Delta \mathrm{S}_s$. By (3.5) it then follows that $0 = \mathrm{E} = \mathrm{S}$. Therefore $\operatorname{span}_{\mathbb{K}} (\mathscr{R} \cup \mathscr{E} \cup \mathscr{S}) = \mathscr{R} \oplus \mathscr{E} \oplus \mathscr{S}$.

Since Δ is selfadjoint, the eigenspaces $\mathscr{L}^k_s$, $\mathscr{E}^k_s$, $\mathscr{S}^k_s$ are mutually orthogonal, and consequently their respective extensions $\mathscr{L}^k$, $\mathscr{E}^k$, $\mathscr{S}^k$ are linearly independent. Equation (3.4) with $\mathscr{X} = \mathscr{R}$ follows from (3.4) with $\mathscr{X} = \mathscr{L}$.

It will now be shown that the spaces on the left and right hand sides of equations (3.2) have the same dimensions. This could be started with the dimensions of $\mathscr{P}^k$, $\mathscr{L}^k$ and $\mathscr{D}^k$ obtained by counting polynomial coefficients, but the proof of (3.2) can also be completed without using any actual dimension:

From $\dim \mathscr{L}^k = \dim \mathscr{P}^k - \dim \Delta\mathscr{P}^k$ and $\Delta\mathscr{P}^k = \mathscr{P}^{k-2}$ one gets by induction that

$$\dim \mathscr{P}^k = \sum_{l \in \mathbb{N}_0,\, 2l \leq k} \dim \mathscr{L}^{k-2l} = \dim \bigoplus_{l \in \mathbb{N}_0,\, 2l \leq k} \mathrm{R}^{2l} \mathscr{L}^{k-2l}. \tag{3.7}$$

Since $\dim \mathscr{F}^k = 4 \dim \mathscr{P}^k$ and because of (3.4), the second part of (3.2) is satisfied if

$$4 \dim \mathscr{L}^k = \dim \mathscr{R}^k + \dim \mathscr{E}^k + \dim \mathscr{S}^k. \tag{3.8}$$

From $\dim \mathscr{L}^k$ can be derived $\dim \mathscr{R}^k = \dim \mathscr{L}^{k-1}$ and $\dim \mathscr{E}^k = \dim \mathscr{L}^{k+1}$, but with $\dim \mathscr{S}^k$ it is not so easy. In [8] it was claimed that $\dim \mathscr{S}^k = \dim \mathscr{H}^k - \dim \mathscr{E}^k$, probably assuming that the restriction $\mathscr{H} \to \mathscr{H}_s$ is bijective. Actually, this is not the case, because

$$\mathscr{D}^k = \mathscr{E}^k \oplus \mathscr{S}^k. \tag{3.9}$$

Proof. – "$\subseteq$": $\boldsymbol{\delta}\mathscr{E} = 0 = \boldsymbol{\delta}\mathscr{S} \Rightarrow \mathscr{E}^k \oplus \mathscr{S}^k \subseteq \mathscr{D}^k$.
"$\supseteq$": For arbitrary $\mathrm{D}^k \in \mathscr{D}^k$ define

$$\mathrm{P}^{k+1} := \frac{1}{k+1} \mathrm{R}\mathrm{D}^k_{\mathrm{R}}, \qquad \mathrm{F}^k := \mathrm{D}^k - \mathbf{d}\mathrm{P}^{k+1}.$$

The implications

$$(2.4) \quad \Rightarrow \quad \Delta(\mathrm{R}\mathrm{D}^k_{\mathrm{R}}) = 0 \quad \Rightarrow \quad \mathbf{d}\mathrm{P}^{k+1} \in \mathscr{E}^k \quad \Rightarrow \quad \Delta\mathrm{F} = 0$$

and

$$\mathrm{P}^{k+1}_{,\mathrm{R}} = \mathrm{D}^k_{\mathrm{R}} \quad \Rightarrow \quad \mathrm{F}^k_{\mathrm{R}} = 0 \quad \Rightarrow \quad \mathrm{F}^k \in \mathscr{S}^k$$

imply:

$$\mathrm{D}^k = \mathrm{F}^k + \mathbf{d}\mathrm{P}^{k+1} \in \mathscr{S}^k \oplus \mathscr{E}^k.$$

Equation (3.8) now follows from $\dim \mathscr{D}^k = \dim \mathscr{H}^k - \dim \boldsymbol{\delta}\mathscr{H}^k$ together with $\boldsymbol{\delta}\mathscr{H}^k = \mathscr{L}^{k-1}$.

The actual dimensions of all $\mathbb{K}$ vector spaces involved can finally be obtained from

$$\dim \mathscr{P}^k = \frac{1}{6}(k+1)(k+2)(k+3) \qquad \text{(derived using combinatorics)}.$$

There results $\dim \mathscr{L}^k = (k+1)^2$, $\dim \mathscr{R}^k = k^2$, $\dim \mathscr{E}^k = (k+2)^2$, and $\dim \mathscr{S}^k = 2k(k+2)$.

IV. COMPLETENESS

By the Stone-Weierstrass theorem, $\mathscr{P}_s$ is dense in the Banach space of continuous functions on $\mathbf{S}^3$ with respect to the uniform norm. For any such function f,

$$\|f\|_2^2 \leqq \mathrm{Vol}(\mathbf{S}^3)\cdot\|f^2\|_\infty = 2\pi^2\|f\|_\infty^2. \tag{4.1}$$

Therefore $\mathscr{L}_s=\mathscr{P}_s$ is also $\|.\|_2$-dense in this Banach space. Consequently $\mathscr{L}_s$ is dense in the Hilbert space of square integrable functions on $\mathbf{S}^3$.

The analog for 1-forms: Since $(\mathbf{S}^3, d\sigma^2)$ is homothetic to $(\mathrm{SU}(2), dk^2)$, dk^2 being the Killing metric of this compact Lie group, there exist globally analytic orthonormal cobases of $\mathbf{S}^3$. Let (E^α), $\alpha\in\{1,2,3\}$, be one of them. At the points of $\mathbf{S}^3$, $(\mathbf{d}\mathrm{R}, \mathrm{E}^\alpha)$ and $(\mathbf{d}x^i)$ are two analytic fields of orthonormal cobases of $\mathbb{R}^4$ related to each other by an $\mathrm{O}(4)$ valued field O_j^α, which is also analytic. For any continuous 1-form $\mathrm{G}=\mathrm{G}_\alpha\mathrm{E}^\alpha=\mathrm{G}_\alpha\mathrm{O}_j^\alpha\mathbf{d}x^j=:\mathrm{G}_j\mathbf{d}x^j$ on $\mathbf{S}^3$, all G_j are thus continuous functions on $\mathbf{S}^3$ which consequently have $\|.\|_\infty$-expansions in $\mathscr{P}_s$. Let now the uniform norm of G be defined by

$$\|\mathrm{G}\|_\infty^2 := \|\sum_\alpha \mathrm{G}_\alpha^2\|_\infty. \tag{4.2}$$

The relation

$$\|\mathrm{G}\|_\infty^2 = \|\sum_j \mathrm{G}_j^2\|_\infty \leqq \sum_j \|\mathrm{G}_j\|^2 \tag{4.3}$$

then implies that G has an $\|.\|_\infty$-expansion in $\mathscr{F}_s$. Because in analogy to (4.1),

$$\|\mathrm{G}\|_2^2 \leqq \mathrm{Vol}(\mathbf{S}^3)\cdot\|\sum_\alpha \mathrm{G}_\alpha^2\|_\infty = 2\pi^2\|\mathrm{G}\|_\infty^2, \tag{4.4}$$

the latter is also an $\|.\|_2$-expansion in $\mathscr{F}_s=\mathscr{E}_s\oplus\mathscr{S}_s$. Therefore, $\mathscr{E}_s\oplus\mathscr{S}_s$ is dense in the Hilbert space of square integrable 1-forms on $\mathbf{S}^3$.

The results about functions and 1-forms imply that $\mathscr{S}_s^k$, $\mathscr{E}_s^k$ and $\mathscr{L}_s^k$ together with their Hodge duals form a complete system of eigenspaces of Δ in the Hilbert space of p-forms on $\mathbf{S}^3$.

COROLLARY. — *A decomposition theorem:*

In the next section, analytic orthonormal bases of the $\mathscr{L}_s^k$, the $\mathscr{E}_s^k$ and the Hilbert closures $\hat{\mathscr{S}}_s^k$ of $\mathscr{S}_s^k$ will be derived, the latter consisting of coexact 1-forms only. The span of the resulting orthonormal bases of $\mathscr{L}_s$, $\mathscr{E}_s\oplus\hat{\mathscr{S}}_s$ together with their Hodge duals is also dense in the Fréchet space of C^∞ p-forms on $\mathbf{S}^3$ with respect to its standard topology for the following reasons: Because of (4.3), it suffices to show that all Lie derivatives with respect to Killing vector fields on $\mathbf{S}^3$ of any C^∞ function f on $\mathbf{S}^3$ have expansions in $\mathscr{L}_s$ converging in the uniform norm, and that any such expansion is identical with the respective derivative of the expansion of f in $\mathscr{L}_s$. The first follows from the Stone-Weierstrass theorem, and the second is true because the expansions are unique and all Killing vector fields on $\mathbf{S}^3$ describe infinitesimal rotations.

From this, a more detailed version of the Hodge decomposition theorem for $\mathbf{S}^3$ follows: with the notations of (5.4) and because the differential operators d, $\mathscr{U}$, $\mathscr{T}$, δ appearing there are continuous with respect to the topology of this Fréchet space,

$$\begin{gathered} \mathrm{V}=\delta d(\mathrm{V}) \oplus \{\text{Const.}\}, \\ \text{the space of } \mathrm{C}^\infty \text{ 1-forms on } \mathbf{S}^3 \text{ is } d(\mathrm{V}) \oplus \mathscr{U}(\mathrm{V}) \oplus \mathscr{T}(\mathrm{V}), \\ \text{the space of } \mathrm{C}^\infty \text{ 2-forms on } \mathbf{S}^3 \text{ is } * d(\mathrm{V}) \oplus * \mathscr{U}(\mathrm{V}) \oplus * \mathscr{T}(\mathrm{V}), \\ * \mathrm{V}=d * d(\mathrm{V}) \oplus * \{\text{Const.}\}. \end{gathered} \tag{4.5}$$

An analogous decomposition theorem on $\mathbb{R}^3$ or on the hyperbolic space $\mathbf{H}^3$ for the space of C^∞ p-forms which additionally are in L^2 can be deduced from section VII, in particular (7.6). In the notation of section VII, this space equals

$$\begin{gathered} \mathrm{W} \oplus d(\tilde{\mathrm{W}}) \oplus \mathscr{U}(\mathrm{W}) \oplus \mathscr{T}(\tilde{\mathrm{W}}) \\ \oplus * \mathscr{T}(\tilde{\mathrm{W}}) \oplus * \mathscr{U}(\mathrm{W}) \oplus * d(\tilde{\mathrm{W}}) \oplus * \mathrm{W}, \end{gathered} \tag{4.6}$$

where W and $d\tilde{\mathrm{W}}$ are the spaces of C^∞ functions and exact 1-forms, respectively, or $\mathbb{R}^3$ of $\mathbf{H}^3$ which are in L^2. To deduce (4.6) from (7.6), one only needs to add the well known Hodge decomposition theroem for C^∞ forms on smooth oriented noncompact Riemannian manifolds ([17], article 196 D., p. 628), and the theorem derived by Dodziuk [11].

V. EXPLICIT ORTHONORMAL BASES

According to section III, the supposed bases must satisfy:

$$\begin{aligned} &\Delta \mathrm{L}_s^k = k(k+2)\,\mathrm{L}_s^k \quad \text{for } \mathrm{L}_s^k \in \mathscr{L}_s^k, && \dim \mathscr{L}_s^k=(k+1)^2 \\ &\Delta \mathrm{E}_s^k=(k+1)(k+3)\,\mathrm{E}_s^k \quad \text{for } \mathrm{E}_s^k \in \mathscr{E}_s^k, && \dim \mathscr{E}_s^k=(k+2)^2 \\ &\Delta \mathrm{S}_s^k=(k+1)^2\,\mathrm{S}_s^k \quad \text{for } \mathrm{S}_s^k \in \mathscr{S}_s^k, && \dim \mathscr{S}_s^k=2k(k+2). \end{aligned} \tag{5.1}$$

A well known orthonormal basis of $\mathscr{L}_s$ is the set of ultraspherical harmonics

$$\left\{ \mathrm{Y}_{klm}=c_{kl}\,\mathrm{Y}_{lm}\sin^l \rho \left(\frac{d}{d\cos\rho}\right)^l \left(\frac{\sin(k+1)\rho}{\sin\rho}\right) \middle| \; k\in\mathbb{N}_0,\ 0\leqq l\leqq k,\ -l\leqq m\leqq l \right\}, \tag{5.2}$$

where $\mathrm{Y}_{lm}(\vartheta, \varphi)$ denote the standard spherical harmonics on $\mathbf{S}^2$ and $c_{kl}^2=\dfrac{2}{\pi}(k+1)\dfrac{(k-l)!}{(k+1-l)!}$. This is shown, for example, in [7]. At the two poles $(\rho \to 0, \pi)$, all the Y_{klm} with $l \neq 0$ take the value zero whereas the

Y_{k00} take the values $\dfrac{k+1}{\sqrt{2\pi^2}}$. Similarly,

$$\left\{ E_{klm} = \frac{dY_{klm}}{\sqrt{k(k+2)}} \middle| k \in \mathbb{N}, 0 \leqq l \leqq k, -l \leqq m \leqq l \right\} \tag{5.3}$$

is an orthonormal basis of $\mathscr{E}_s^{k-1} = d\mathscr{L}_s^k$, because

$$\Delta dY_{klm} = d\Delta Y_{klm} = k(k+2)\, dY_{klm},$$
$$\langle dY_{klm}, dY_{klm} \rangle = \langle \Delta Y_{klm}, Y_{klm} \rangle = k(k+2).$$

The construction of an orthonormal basis of $\hat{\mathscr{S}}_s^k$ needs a little preparation: On the space V of C^∞ functions on S^3,

$$\mathscr{U}(f) := * d(f \sin \rho \, d\rho) \qquad \text{and} \qquad \mathscr{T}(f) := * d\mathscr{U}(f), \tag{5.4}$$

where $\sin \rho \; d\rho$ could be replaced by the restriction to S^3 of an arbitrary unit 1-form of $\mathbb{R}^4$, are two coexact 1-form valued linear functionals. Explicitly,

$$\begin{aligned}
&\mathscr{U}(f) = \sin^3 \rho \sin \vartheta \,(f^{,\varphi}\, d\vartheta - f^{,\vartheta}\, d\varphi) \\
&\quad \Rightarrow \quad \mathscr{T}(f) = \sin \rho \,((\Delta+1) f)\, d\rho + d((f \sin \rho)_{,\rho}) \\
&\qquad \Rightarrow \quad \Delta \mathscr{U}(f) = * d\mathscr{T}(f) = \mathscr{U}((\Delta+1) f) \\
&\qquad\qquad \Rightarrow \quad \Delta \mathscr{T}(f) = * d\Delta \mathscr{U}(f) = \mathscr{T}((\Delta+1) f),
\end{aligned} \tag{5.5}$$

and quite obviously

$$\langle \mathscr{U}(f), \mathscr{T}(g) \rangle = \langle f \sin \rho \, d\rho, \Delta \mathscr{U}(g) \rangle = 0. \tag{5.6}$$

Evaluating these functionals on (Y_{klm}) results for each k in $2k(k+2)$ mutually orthogonal and nonvanishing eigenforms of Δ with eigenvalue $(k+1)^2$: the eigenvalues arise from (5.5) and $\Delta Y_{klm} = k(k+2)\, Y_{klm}$, the orthogonality relations are implied by (5.6) together with

$$\begin{aligned}
\langle \mathscr{T}(Y_{klm}), \mathscr{T}(Y_{k'l'm'}) \rangle &= \langle \Delta \mathscr{U}(Y_{klm}), \mathscr{U}(Y_{k'l'm'}) \rangle \\
&= (k+1)^2\, l(l+1)\, \delta_{kk'}\, \delta_{ll'}\, \delta_{mm'}.
\end{aligned}$$

The latter also imply $\| \mathscr{T}(Y_{klm}) \|_2 = 0 \Leftrightarrow \| \mathscr{U}(Y_{klm}) \|_2 = 0 \Leftrightarrow l = 0$, which fixes the number of linearly independent $\mathscr{U}(Y_{klm})$ and $\mathscr{T}(Y_{klm})$, respectively. For each k, it is equal to $(k+1)^2 - 1$. Although the $\mathscr{U}(Y_{klm})$ and $\mathscr{T}(Y_{klm})$ are analytic, they are generally not restrictions of polynomial 1-forms on $\mathbb{R}^4$. This is due to the use of $*$ in their definition and contrasts with the Y_{klm} and E_{klm}.

From all these preliminaries one concludes that

$$\begin{aligned}
&\{ P_{klm}, N_{klm} \mid k \in \mathbb{N}, 1 \leqq l \leqq k, -l \leqq m \leqq l \}, \\
&P_{klm} := \frac{1}{\sqrt{2l(l+1)}} \left(\frac{1}{k+1} \mathscr{T}(Y_{klm}) + \mathscr{U}(Y_{klm}) \right), \\
&N_{klm} := \frac{1}{\sqrt{2l(l+1)}} \left(\frac{1}{k+1} \mathscr{T}(Y_{klm}) - \mathscr{U}(Y_{klm}) \right),
\end{aligned} \tag{5.7}$$

is an analytic orthonormal basis of $\hat{\mathcal{S}}_s$ (but not of $\mathcal{S}_s$), which has the useful properties

$$\star d\mathrm{P}_{klm} = (k+1)\,\mathrm{P}_{klm}, \qquad \star d\mathrm{N}_{klm} = -(k+1)\,\mathrm{N}_{klm}. \tag{5.8}$$

Of geometric interest could be that − up to signs −

$$\begin{gathered}\text{a change of orientation of the sphere neither}\\ \text{affects the } \mathrm{Y}_{klm} \text{ nor the } \mathrm{E}_{klm},\\ \text{but exchanges the } \mathrm{P}_{klm} \text{ and the } \mathrm{N}_{klm}.\end{gathered} \tag{5.9}$$

The latter happens, because $\star$ appears once in the definition of $\mathcal{U}(f)$, but twice in the definition of $\mathcal{T}(f)$.

VI. GROUP THEORETICAL ASPECTS

By the use of $\sharp$, vector fields $\mathrm{P}_m := \mathrm{P}^\sharp_{11m}$ and $\mathrm{N}_m := \mathrm{N}^\sharp_{11m}$ on S^3 are generated from the 1-forms dual to them with respect to $d\sigma^2$. These here come from the rescaled restrictions Y_{111}, Y_{11-1} and Y_{110} of the polynomials x, y and z of $\mathbb{R}^4$, respectively. Comparable vector fields coming from $\mathrm{Y}_{100} = \dfrac{\sqrt{2}}{\pi}\cos\rho$ do not exist, according to the previous section. With α, β, $\gamma \in \{-1, 0, 1\}$, the above fields satisfy the commutator relations

$$\begin{gathered}[\mathrm{P}_\alpha, \mathrm{N}_\beta] = 0,\\ [\mathrm{N}_\alpha, \mathrm{N}_\beta] = \varepsilon_{\alpha\beta\gamma}\,\mathrm{N}_\gamma \quad \text{and} \quad [\mathrm{P}_\alpha, \mathrm{P}_\beta] = -\varepsilon_{\alpha\beta\gamma}\,\mathrm{P}_\gamma.\end{gathered} \tag{6.1}$$

They can therefore be identified as orthogonal bases of the left and right invariant vector fields on $\mathrm{SU}(2) \simeq \mathrm{S}^3$, respectively. The Killing metric dk^2 induced on SU (2) by either one of them is related to the metric $d\sigma^2$ of S^3 by $dk^2 = 2\,d\sigma^2$. The six vector fields P_m and N_m generate the identity component SO (4) of the isometry group O (4) of S^3. They help to identify the representations of these groups induced in the eigenspaces of Δ by forming

$$\mathrm{C_L} := -\sum_m \mathrm{P}_m \mathrm{P}_m \qquad \text{and} \qquad \mathrm{C_R} := -\sum_m \mathrm{N}_m \mathrm{N}_m, \tag{6.2}$$

Casimir operators with respect to the left and right action, respectively, of SU (2) onto itself (more details about that can be found in [10]) satisfying on S^3:

$$\begin{aligned} &4\,\mathrm{C_L} = \Delta = 4\,\mathrm{C_R} \quad \text{on functions and volume forms},\\ &4\,\mathrm{C_L} = \Delta + 2\,d\star, \quad 4\,\mathrm{C_R} = \Delta - 2\,d\star \quad \text{on 2-forms},\\ &4\,\mathrm{C_L} = \Delta + 2\star d, \quad 4\,\mathrm{C_R} = \Delta - 2\star d \quad \text{on 1-forms}.\end{aligned} \tag{6.3}$$

The importance of C_L and C_R lies in the following: since $SU(2)\times SU(2)$ is the (twofold) universal covering group of $SO(4)$, any irreducible representation of $SO(4)$ can be labelled by the two spin indices j and j' which represent the eigenvalues $j(j+1)$ of C_L and $j'(j'+1)$ of C_R in it. If conversely on a representation space V of $SO(4)$, $C_L=j(j+1)$ and $C_R=j'(j'+1)$ with $2j, 2j', j+j'\in\mathbb{N}_0$, and if additionally $(2j+1)(2j'+1)=\dim V$, then the representation is irreducible. This is shown in chapter 10 of [6], for example.

From (6.3) and the preceding section it follows that all Y_{klm}, E_{klm}, P_{klm}, N_{klm} and their Hodge duals are simultaneous eigenforms of C_L and C_R:

$$\begin{aligned}
&C_L(Y_{klm})=\frac{k}{2}\left(\frac{k}{2}+1\right)Y_{klm}, \qquad C_R(Y_{klm})=\frac{k}{2}\left(\frac{k}{2}+1\right)Y_{klm},\\
&C_L(E_{klm})=\frac{k}{2}\left(\frac{k}{2}+1\right)E_{klm}, \qquad C_R(E_{klm})=\frac{k}{2}\left(\frac{k}{2}+1\right)E_{klm},\\
&C_L(P_{klm})=\frac{k+1}{2}\left(\frac{k+1}{2}+1\right)P_{klm},\\
&C_R(P_{klm})=\frac{k+1}{2}\left(\frac{k-1}{2}+1\right)P_{klm},\\
&C_L(N_{klm})=\frac{k+1}{2}\left(\frac{k-1}{2}+1\right)N_{klm},\\
&C_R(N_{klm})=\frac{k+1}{2}\left(\frac{k+1}{2}+1\right)N_{klm};
\end{aligned}\tag{6.4}$$

the dimensions of the eigenspaces are $(k+1)^2$ for the Y_{klm} and the E_{klm}, but $k(k+2)$ for the P_{klm} and the N_{klm}.

Thus for $k\geq 1$, $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{Y_{klm}\}$ and $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{E_{klm}\}$ are carrier spaces of two equivalent irreps of $SO(4)$ with $(j,j')=\left(\frac{k}{2},\frac{k}{2}\right)$, $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{P_{klm}\}$ carries an irrep with $(j,j')=\left(\frac{k+1}{2},\frac{k-1}{2}\right)$ and $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{N_{klm}\}$ carries one with $(j,j')=\left(\frac{k-1}{2},\frac{k+1}{2}\right)$. Because of (5.9), all $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{Y_{klm}\}$, $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{E_{klm}\}$, $\mathrm{span}_{\mathbb{K}}\bigcup_{l,m}\{P_{klm}, N_{klm}\}$ and their Hodge duals are representation spaces for irreducible representations of $O(4)$.

Remarks on the literature

The last of the relations (6.3) is also described in [10], eqs (5.18) to (6.3). The Casimir operators $C_\pm$ of [2] acting on 1-form potentials in

Coulomb gauge and thus on coclosed 1-forms on S^3 are related to the ones above by $C_+ = -\Delta$, $C_- = 2 * d$.

In [16] it is shown that the complex span of all Y_{klm} together is the representation space of an irreducible unitary representation of $O(1,4)$, the conformal group of S^3. It seems likely that the same is true for all P_{klm}, N_{klm} and E_{klm} together, but this has not been proved.

VII. DECOUPLING OF MAXWELL'S EQUATIONS

Let $RW = (\mathbb{R} \times M_k, \Omega^2(\tau)(d\tau^2 - d\rho^2 - S_k^2 \, d\omega^2))$, $d\omega^2 = d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2$, be an arbitrary Robertson-Walker spacetime: for $k = -1, 0, 1$, $(M_k, S_k) = (\mathbf{H}^3, \sinh \rho)$, $(\mathbb{R}^3, \rho)$, $(\mathbf{S}^3, \sin \rho)$, respectively. Remember that proper time t is connected to the time parameter τ by the relation $dt = \Omega(\tau) \, d\tau$. Let $(*_k, d, \delta, \Delta)$ now refer to $(M_k, d\rho^2 + S_k^2 \, d\omega^2)$.

Maxwell's equations in terms of F and J are invariant under conformal rescalings of the spacetime metric if the 4-current density J is considered to be a 3-form. The continuity equation then means that J is closed, while the 2-form F is closed by the homogeneous Maxwell equations. The inhomogeneous equations require that the differential of $\circledast$ F equals J, where $\circledast$ denotes the Hodge star on RW such that $d\tau \wedge \circledast \, d\tau$ is the volume form on this spacetime, and $\circledast \, d\tau = \sin^2 \rho \sin \vartheta \, d\rho \wedge d\vartheta \wedge d\varphi$. Consequently, F satisfies an inhomogeneous wave equation where the source term is the differential of $\circledast$ J. Analogously, any 1-form potential A whose differential equals F satisfies a conformally invariant wave equation with $\circledast$ J as source. A is taken to be in Coulomb gauge, since this gauge is invariant under conformal rescalings of the spacetime metric with factors $\Omega^2(\tau)$.

If J is C^∞ and $\circledast J|_{\{\tau\} \times M_k}$ has compact support for all times τ, the property of A or F being in $L^2(M_k, d\rho^2 + S_k^2 \, d\omega^2)$ is preserved for all finite values of τ for the following reasons: The fact that RW is locally conformally flat makes the wave equation locally equivalent to a symmetric hyperbolic system of first order and thus the method of energy inequalities (*see* [18], for example) applicable; the global hyperbolicity of RW ensures the existence of advanced and retarded solutions. Therefore, it is reasonable to assume that F and A are C^∞ and in L^2 on all subspaces $\tau = \text{Const.}$ of RW.

By the Hodge decompositions on these subspaces it then follows that

$$\Omega^2 \circledast J = J_0 \, d\tau + J_e + J_s, \qquad A = A_0 \, d\tau + A_s,$$

with J_e exact and J_s, A_s coexact such that J_0, J_e, J_s, A_0, A_s are C^∞ and J_e, J_s, A_s are in L^2 on the time slices. All L^2 harmonic p-forms on the spaces $(M_k, d\rho^2 + S_k^2 \, d\omega^2)$ vanish, according to [11]. J_0, J_e and J_s are conformally invariant because J is, and J_0 uniquely determines J_e via the continuity equation $J_{0,\tau}+\delta J_e=0$.

Similarly, any closed 2-form F consisting of L^2 fields E and B uniquely determines A_s and E_e such that

$$F=B+d\tau\wedge E=dA_s+d\tau\wedge(A_{s,\tau}+E_e), \tag{7.0}$$

where E_e is the exact part of the electric 1-form, $A_{s,\tau}$ its coexact counterpart and dA_s equals the magnetic 2-form B. Since gauge transformations always mean adding a 4-gradient to A, they cannot affect A_s. Gauge freedom here only exists as the choice of mixture between the electric potential and the exact part of the 1-form potential. From A_s and E_e one can obtain a 1-form potential in Coulomb gauge, $A=A_0\,d\tau+A_s$, or in time gauge, $\hat{A}=-A_e+A_s$. A_0 and A_e are then defined by $-dA_0=E_e=A_{e,\tau}$, but not uniquely. To achieve Lorentz gauge, a scalar wave equation on RW must be solved that is not conformally invariant. Instead of the gauge potential A, I will from now on preferably use the 1-forms A_s and E_e.

In terms of J_0, J_s and E_e, A_s the inhomogeneous Maxwell equations on RW become:

$$-\delta E_e=J_0,\qquad \square A_s:=\Delta A_s+A_{s,\tau\tau}=J_s. \tag{7.1}$$

On Robertson-Walker spacetimes with positive curvature, solutions only exist if the total charge on S^3 vanishes, because $\int_{S^3}*J_0=-\int_{\partial S^3}*E_e=0$.

Then the divergence equation has a unique solution such that $E_{e,\tau}=J_e$. This solution is square integrable on the time slices.

The 1-form wave equation in (7.1) will now be decomposed into a system (8.8) of two conformally invariant scalar wave equations and two Poisson equations on S^2:

Let f and g be a real valued C^∞ function on RW, let $v_k^\flat$ denote the exact analytic 1-form $S_k\,d\rho$ on M_k and v_k the vector field dual to it with respect to $d\rho^2+S_k^2\,d\omega^2$. Then define

$$\begin{aligned}\Theta(f)&:=-\left(\frac{\partial^2}{\partial\vartheta^2}+\cot\vartheta\frac{\partial}{\partial\vartheta}+\frac{\partial^2}{\partial\varphi^2}\right),\\ \square_k(f)&:=S_k^{-1}\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial\rho^2}+S_k^{-2}\Theta\right)(S_kf),\\ \mathscr{U}_k(f)&:=*_k\,d(v_k^\flat\cdot f)\qquad\text{and}\qquad \mathscr{T}_k(f):=*_k\,d\mathscr{U}_k(f).\end{aligned} \tag{7.2}$$

In the case that $\Omega^2(\tau)=1$, $\square_k=\dfrac{\partial^2}{\partial\tau^2}+k+\Delta$ is the conformally invariant scalar wave operator on RW. $\mathscr{U}_k$ and $\mathscr{T}_k$ are generalizations of the functionals $\mathscr{U}$ and $\mathscr{T}$ of (5.4), and in analogy to (5.4−6), one deduces with or without coordinate representations:

$$v_k\lrcorner\,\mathscr{U}_k(f)=0,\qquad v_k\lrcorner\,\mathscr{T}_k(f)=\Theta(f), \tag{7.3}$$

$$\mathcal{U}_k(f)+\mathcal{T}_k(g)=0 \quad \Leftrightarrow \quad f, g \in \ker \Theta, \tag{7.4}$$
$$\Box\,(\mathcal{U}_k(f)+\mathcal{T}_k(g))=\mathcal{U}_k(\Box_k f)+\mathcal{T}_k(\Box_k g). \tag{7.5}$$

Next it will be shown that there exist real valued C^∞ functions f, g, $\tilde{f}$, $\tilde{g}$ on RW, by (7.4) unique up to elements of $\ker\Theta$, such that

$$A_s=\mathcal{U}_k(f)+\mathcal{T}_k(g) \qquad \text{and} \qquad J_s=\mathcal{U}_k(\tilde{f})+\mathcal{T}_k(\tilde{g}). \tag{7.6}$$

Using (7.3), they can be determined from A_s and J_s by

$$\begin{aligned} \Theta f&=v_k \neg \star_k dA_s, \qquad & \Theta g&=v_k \neg A_s, \\ \Theta \tilde{f}&=v_k \neg \star_k dJ_s, \qquad & \Theta \tilde{g}&=v_k \neg J_s. \end{aligned} \tag{7.7}$$

These equations can be solved, since the integrals of their right hand sides over the relevant 2-spheres vanish, due to the fact that A_s and J_s are divergence free. The decomposition (7.6) then follows, because all C^∞ 1-forms X on RW (and also all 1-forms distributional on the time slices) with $0=v_k \neg X=v_k \neg \star dX=\delta X$ vanish. It might be illustrative, that J_s and A_s vanish in spherically symmetric cases while for axisymmetric J_s, $\Theta\tilde{f}=0=\Theta f$.

Successive application of (7.6), (7.5), (7.4) to (7.1) results in the decoupled equations $\Theta(\Box_k f-\tilde{f})=0=\Theta(\Box_k g-\tilde{g})$. Commuting Θ and $\Box_k$ and inserting (7.7) one achieves

$$\Box_k \Theta f=v_k \neg \star_k dJ_s, \qquad \Box_k \Theta g=v_k \neg J_s, \tag{7.8}$$

With the solutions E_e of $-\delta E_e=J_0$ to be obtained from $\Delta A_0=J_0$ and with Θf, Θg of (7.8) fixed by initial data, the resulting field is

$$F=d\mathcal{U}_k(f)+d\mathcal{T}_k(g)-(\mathcal{U}_{k,\tau}(f)+\mathcal{T}_{k,\tau}(g)+E_e)\wedge d\tau. \tag{7.9}$$

VIII. THE EIGENMODES

Differing from the preceding section, it is no longer assumed that F and A are C^∞ in all coordinates. On the time slices, they are now allowed to be distributional forms which are in L^2, while the time dependence is C^∞ and such that $F_{,\tau\tau}+\lambda F=0$ with arbitrary $\lambda\in\mathbb{R}$. Any such nonzero F obeying the source free Maxwell equations is called an eigenmode of RW. Its energy is finite and conserved.

Fortunately, the Hodge decomposition theorem ([17], article 196 D.) and the theorem of Dodziuk remain applicable. Since there is no source, E_e is L^2 harmonic and thus identically zero. Consequently, the general solution of the source free Maxwell equations is

$$\{F=dA_s+d\tau\wedge A_{s,\tau} \mid \Delta A_s+A_{s,\tau\tau}=0\}. \tag{8.0}$$

According to [11], it does not contain any nonzero 2-forms with $F_{,\tau}=0$.

All F satisfying $F_{,\tau\tau}+\lambda F=0$ with arbitrary $\lambda \in \mathbb{R}$ come from A_s obeying the same equation. In this case, the 1-form distribution $A_s|_{\{\tau\}\times M_k}$ satisfies the homogeneous elliptic equation $(\Delta-\lambda)\, A_s=0$ on M_k. By corollary 8.3.2 of [13], which says that the singular supports of source and solution are identical for elliptic equations on subsets of $\mathbb{R}^n$, A_s and thus F must be C^∞ forms. Hence, all eigenmodes F of RW come from C^∞ potentials A_s which are in L^2 on the time slices, and modes with $\lambda=0$ do not exist.

As described in the preceeding section, the set (8.0) can be derived from the general solution of the scalar equation $\square_k f=0$. According to [15], the space of C^∞ solutions of the equation $(\Delta+k-\lambda)f=0$ is given by

$$\left\{f=Y_{lm}\, S_k^l\left(\frac{1}{S_k}\frac{\partial}{\partial p}\right)^l\left(\frac{\sin\omega\rho}{S_k(\rho)}\right)(c_1\cos\omega\tau+c_2\sin\omega\tau)\,\middle|\,\omega^2=\lambda>0\right\}$$
or
$$\left\{f=Y_{lm}\, S_k^l\left(\frac{1}{S_k}\frac{\partial}{\partial \rho}\right)^l\left(\frac{\sinh\omega\rho}{S_k(\rho)}\right)(c_1 e^{-\omega\tau}+c_2 e^{\omega\tau})\,\middle|\,\omega^2=-\lambda>0\right\}, \tag{8.1}$$

the set of constants $\{(c_1, c_2)\}$ being $\mathbb{R}^2$.

For all of them,

$$\langle \mathscr{T}(f), \mathscr{T}(f)\rangle_{M_k}=\omega^2\langle \mathscr{U}(f), \mathscr{U}(f)\rangle_{M_k}=\omega^2\, l(l+1)\langle f, f\rangle_{M_k}.$$

Since $\langle f, f\rangle_{M_k}$ is infinite for $k\in\{-1, 0\}$, they do not give rise to 1-forms A_s which are in L^2 unless $k=1$. Hence, the Robertson-Walker spacetimes with nonpositive curvature do not have eigenmodes.

On the Einstein Universe, which represents the Robertson-Walker spacetimes with positive curvature, a Fourier transformation with respect to the coordinate τ changes $\Delta A_s+A_{s,\tau\tau}=0$ into the eigenvalue equation of Δ on coexact 1-forms of S^3. The results of section V then lead to a basis of the real vector space of the 1-form potentials satisfying the source free Maxwell equations on the Einstein Universe. It is formed by the

$$p_k P_{klm},\quad n_k P_{klm},\quad p_k N_{klm},\quad n_k N_{klm}, \tag{8.2}$$

where $p_k(\tau):=\cos(k+1)\tau+\sin(k+1)\tau$, $n_k(\tau):=\cos(k+1)\,\tau-\sin(k+1)\tau$. The four elements belonging to the same (k, l, m) can be generated from one of them by time and space inversions in the sense of (5.9). All source free fields are periodic in τ with period 2π, the time a light ray needs to travel once around the sphere. Nonvanishing static solutions and modes with $k=0$ do not exist. A detailed description of the six "ground vibrations" with $k=1$ was given by Schrödinger in [1], App. III. They belong to the right and left invariant 1-forms on SU(2), respectively. The set of eigenmodes of the Einstein Universe presented in [1] is complete in the same sense as solution (8.2): Equations (2.16) of [1], rewritten in the form $0=(\nu-\star d)H$, $H=e^{i(n\varphi+m\psi)}f_{k4}(\omega)$ being a coexact 1-form on S^3, best exhibit the link between the two versions. They are identical with

(5.8). The number of linearly independent solutions Schrödinger found for his equations (2.16), namely ν^2-1 for each positive and negative integer $\nu \neq -1, 0, 1$, is exactly the same as the number of orthogonal P_{klm} or N_{klm} with specific k, namely $k(k+2)$. By section IV, both sets of solutions are therefore dense in the Hilbert space of coexact 1-forms on S^3.

The results of sections IV and V also find an application in the inhomogeneous case: one way of solving (7.3) is to expand both A_0 and J_0 in a series of Y_{klm} and both A_s and J_s in a series of P_{klm} and N_{klm}. The time dependence is then contained in the coefficients only. In the case of A_0 the latter are determined by algebraic equations and in the case of A_s by equations for forced 1-dimensional oscillations.

IX. INTEGRAL KERNELS FOR THE SCALAR POISSON AND CONFORMALLY INVARIANT WAVE EQUATIONS

The local advanced and retarded Green functions of $\Box_k$ are, respectively,

$$G_k^{\pm}(\tau, x; \tau', x') = \frac{\delta(\tau - \tau' \pm d_{xx'})}{4\pi S_k(d_{xx'})}, \tag{9.1}$$

where $d_{xx'}$ denotes the geodesic distance between two points x and x' of $(M_k, d\rho^2 + S_k^2\, d\omega^2)$. In the noncompact cases $k=0, -1$ they are also the global ones, but for $k=1$ they are only valid on a domain where $|\tau - \tau' \pm d_{xx'}| < \pi$. A way to derive them was described by Friedlander in section 4.6 of [5]. Last but not least, the global Green functions for $\Box_1$ will be determined.

First the Poisson equation on S^3 is to be solved: especially in those cases where the source is a distribution, it is useful to know the solutions G_x of $\Delta G_x = \delta_x - \dfrac{1}{2\pi^2}$ for any $x \in S^3$. The term $-\dfrac{1}{2\pi^2}$ occurs because the integral of ΔG_x over S^3 vanishes and $\mathrm{Vol}(S^3) = 2\pi^2$. A_0 is then the convolution of G_x and J_0 on the Lie group SU(2). Since this convolution is generally difficult to perform, I shall consider only charge distributions situated at the two poles $\rho \to 0, \pi$. G_0 and G_π are then of special interest. They will be expressed in terms of nonregular functions and help to describe a nonstatic but spherically (anti-) symmetric solution of Maxwell's equations: each G_x is given as the distributional limit of a series of C^∞ functions G_x^N satisfying $\Delta G_x^N = \delta_x^N - \dfrac{1}{2\pi^2}$, where $\delta_x^N(y)$ denote the C^∞

functions $\sum_{k=0}^{N}\sum_{l=0}^{k}\sum_{m=-l}^{l} Y_{klm}(x)\,Y_{klm}(y)$. This limit exists, because for any C^∞ test function $f(y)=\sum_0^\infty f_{klm}\,Y_{klm}(y)$ on S^3,

$$\delta_x^N[f]=\sum_0^N Y_{klm}(x)\langle Y_{klm}, f\rangle=\sum_0^N f_{klm}\,Y_{klm}(x)$$

converges to $f(x)$. By theorem XIII in chapter III of [14], the δ_x^N then converge as distributions and the limit $\delta_x := \lim_{N\to\infty}\delta_x^N$ is a distribution. Since the inverse of Δ is a Hilbert-Schmidt operator and in particular continuous, as can be seen from its spectrum, the analog holds for the G_x^N and their limit G_x.

From $\Delta G_0=\delta_0-\dfrac{1}{2\pi^2}=\sum_{k=1}^{\infty}\sum_{l=0}^{k}\sum_{m=-l}^{l} Y_{klm}(0)\,Y_{klm}$ together with the remark following the set (5.2) and the eigenvalue equations (5.1) one obtains by comparing coefficients

$$\begin{aligned} G_0&=\frac{1}{2\pi^2}\sum_{k=1}^{\infty}\frac{k+1}{k(k+2)}\,\frac{\sin(k+1)\rho}{\sin\rho}+\text{Const.}\\ &=\frac{1}{2\pi^2}\,\frac{1}{\sin\rho}\left(\frac{\pi}{2}\cos\rho-\frac{\rho}{2}\cos\rho-\frac{1}{4}\sin\rho\right)+\text{Const.}\\ &=\frac{1}{4\pi^2}(\pi-\rho)\cot\rho+c'. \end{aligned}\tag{9.2}$$

This reproduces the result of Stephani presented in [3].

Since in analogy to G_0, $G_\pi=-\dfrac{1}{4\pi^2}\rho\cot\rho+c''$, the unqiue solution for a source $J=\circledast\,(J_0\,d\tau+J_e)$, $J_0=(\delta_0-\delta_\pi)\,h(\tau)$ is therefore $F=d\tau\wedge E_e$ with the electric 1-form

$$E_e=-d(G_0-G_\pi)\cdot h(\tau)=\frac{h(\tau)}{4\pi\sin^2\rho}\,d\rho;\tag{9.3}$$

J_e equals $E_{e,\tau}$ by the continuity equation. Remarks on this solution will follow in the next section. Its analogs (G_0, J_0, E_e) on the noncompact spaces are

$$\left(\frac{1}{4\pi}(\ln S_k)'(\rho),\ \delta_0\cdot f(\tau),\ \frac{f(\tau)}{4\pi S_k^2(\rho)}\,d\rho\right);\tag{9.4}$$

to obtain G_x from G_0, ρ must be replaced by $d_{xx'}$.

Similarly to $\mathbf{G}_x$, the solutions $\tilde{\mathbf{G}}_x$ of $(\Delta+1)\tilde{\mathbf{G}}_x=\boldsymbol{\delta}_x$ on $\mathbf{S}^3$ are obtained from

$$\tilde{\mathbf{G}}_0=\frac{1}{2\pi^2}\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{\sin(k+1)\rho}{\sin\rho}=\frac{1}{4\pi^2}\frac{\pi-\rho}{\sin\rho}.$$

Consequently,
the unique distributional solution $\tilde{\mathbf{G}}_{x\hat{x}}$ of $(\Delta+1)\tilde{\mathbf{G}}_{x\hat{x}}=\boldsymbol{\delta}_x+\boldsymbol{\delta}_{\hat{x}}$ is

$$\tilde{\mathbf{G}}_{x\hat{x}}=(4\pi\sin d_{xx'})^{-1}=(4\pi\sin d_{\hat{x}x'})^{-1}, \tag{9.5}$$

where $\hat{x}$ denotes that point on $\mathbf{S}^3$ for which $d_{x\hat{x}}=\pi$, that is the point antipodal to x.

Now the global advanced and retarded Green functions $\mathbf{G}^{\pm}(\tau,x;\tau',x')$ for the wave operator $\square_1=\frac{\partial^2}{\partial\tau^2}+(\Delta+1)$ on the Einstein Universe can be given: the distributions

$$\mathbf{G}^{\pm}_{\tau x}[f]:=\int_{\mathbb{R}}d\tau'\,\langle\,\mathbf{G}^{\pm}(\tau,x;\tau',x'),f(\tau',x')\,\rangle_{\mathbf{S}^3}, \tag{9.6}$$

$\mathbf{G}^{\pm}(\tau,x;\tau',x')$ being defined as

$$\tilde{\mathbf{G}}_{x\hat{x}}\left(\boldsymbol{\delta}(t_0-d_{xx'})+\sum_{m=1}^{\infty}\boldsymbol{\delta}(t_m-d_{xx'})-\sum_{m=1}^{\infty}\boldsymbol{\delta}(t_m+d_{xx'})\right),$$

with $t_m:=\pm(\tau'-\tau)-2m\pi$, will be shown to satisfy

$$\square_1\,\mathbf{G}^{\pm}_{\tau x}[f]:=\mathbf{G}^{\pm}_{\tau x}[\square_1 f]=f(\tau,x). \tag{9.7}$$

Proof: In analogy to Minkowski spacetime one obtains by some calculus

$$\mathbf{G}^{\pm}_{\tau x}[\square_1 f]=\left\langle\,\tilde{\mathbf{G}}_{x\hat{x}},(\Delta+1)\left(\sum_{m=0}^{\infty}f(\tau\pm(2m\pi+d_{xx'}),x')\right.\right.$$
$$\left.\left.-\sum_{m=1}^{\infty}f(\tau\pm(2m\pi-d_{xx'}),x')\right)\right\rangle_{\mathbf{S}^3}, \tag{9.8}$$

and, since Δ is selfadjoint and $(\Delta+1)\,\tilde{\mathbf{G}}_{x\hat{x}}=\boldsymbol{\delta}_x+\boldsymbol{\delta}_{\hat{x}}$, further

$$\mathbf{G}^{\pm}_{\tau x}[\square_1 f]=f(\tau,x)+\sum_{m=0}^{\infty}f(\tau\pm(2m\pi+\pi),\hat{x})$$
$$-\sum_{m=1}^{\infty}f(\tau\pm(2m\pi-\pi),\hat{x})=f(\tau,x). \tag{9.9}$$

Equation (9.7) can also be deduced from the fact that on each τ' interval such that $t_m\in(-\pi,\pi)$, $m\geqq 1$, and on each τ' interval such that $t_m\in(0,2\pi)$, $m\in\mathbb{N}_0$, $\mathbf{G}^{\pm}(\tau,x;\tau',x')$ is the difference between an advanced and a retarded Green function while it vanishes if $t_0\leqq 0$; the set of all these time intervals is an atlas of $\mathbb{R}\times\mathbf{S}^3\setminus\{\tau\}\times\mathbf{S}^3$, and there exists a partition of unity subordinate to this atlas applicable to all test functions.

The different contributions to the retarded Green function $\mathbf{G}^-$ have the following meaning: the term with t_0 takes into account the light rays which, up to the time τ, did not pass $\hat{x}$ yet, the term with $t_1+d_{xx'}$ stands for those that passed $\hat{x}$ once but have not completed a full circle on $\mathbf{S}^3$ yet, the term with $t_1-d_{xx'}$ includes the ones which crossed each x and $\hat{x}$ at exactly one time before τ, and so on.

The expressions for $\mathbf{G}^\pm(\tau,x;\tau',x')$ were actually obtained by comparing coefficients in the Fourier transform with respect ot τ of the equation $\square_1\mathbf{G}^\pm(\tau,x;\tau',x')=\boldsymbol{\delta}_x\boldsymbol{\delta}(\tau'-\tau)$. There resulted

$$\mathrm{G}^\pm=\frac{\mathrm{H}(\pm\tau)}{2\pi^2\sin\rho}\sum_{k=1}^{\infty}\sin(k+1)\rho\sin(k+1)|\tau|,$$

where H denotes the Heaviside function.

Evaluating the functionals $\mathbf{G}^-_{\tau x}$ on a source function $\Theta\tilde{g}(\tau,x)$ like in (7.8), one achieves a finite result if, for example,

$$\langle\tilde{\mathbf{G}}_{x\hat{x}},\Theta\tilde{g}(\tau-\tau',x')\rangle_{\mathbf{S}^3}\leqq(\tau')^{-1-\varepsilon}\qquad\text{for}\quad\tau'\leqq\tau,\ \varepsilon>0.$$

X. COMOVING MULTIPOLES

The following considerations hold on Robertson-Walker spacetimes in particular, but for the sake of generality, let $(*,d,\delta,\Delta)$ now refer to the 3-dimensional oriented Riemannian manifold $(\mathrm{M},\ ds^2)$, and let $(\circledast,\mathrm{J},\mathrm{J}_0,\mathrm{J}_s,\mathrm{F}=\mathrm{B}-\mathrm{E}\wedge d\tau)$ refer to the spacetime $\mathrm{RM}=(\mathbb{R}\times\mathrm{M},\Omega^2(\tau)(d\tau^2-ds^2))$. M needs to have one Killing vector at least.

Then $\mathrm{J}_{el}=\Omega^{-2}\circledast\mathrm{J}_0\,d\tau$ with $\mathrm{J}_{0,\tau}=0$ is a comoving charge distribution on RM, because the total charge contained in any open subset of M does not depend on τ. $\mathrm{J}_{ma}=\Omega^{-2}\circledast\mathrm{J}_s$ with $\mathrm{J}_{s,\tau}=0$ will be called a comoving current distribution. The current 3-forms J_{el} and J_{ma} satisfy the continuity equation and do not depend on τ.

Let Z, $\mathrm{Z}_1\ldots\mathrm{Z}_n$, $n\in\mathbb{N}_0$ be arbitrary Killing vector fields on M, $\mathrm{Z}^\flat$ the 1-form corresponding to Z by ds^2, L_Z the Lie derivative with respect to Z and $\mathrm{D}:=\mathrm{L}_{\mathrm{Z}_1}\ldots\mathrm{L}_{\mathrm{Z}_n}$. Then for any scalar distribution $\mathbf{G}$ on M, and in particular for $\mathbf{G}=\mathrm{D}\boldsymbol{\delta}_0$,

$$\mathrm{L}_\mathrm{Z}\mathbf{G}=-\delta(\mathrm{Z}^\flat\mathbf{G}).\tag{0.1}$$

Charge distributions J_{el} of comoving electric 2^{n+1}-poles and current distributions J_{ma} of comoving magnetic 2^{n+1}-poles are defined by the relations

$$\mathrm{J}_0=\delta(\mathrm{Z}^\flat\mathrm{D}\boldsymbol{\delta}_0)\qquad\text{and}\qquad\mathrm{J}_s=*\,d(\mathrm{Z}^\flat\mathrm{D}\boldsymbol{\delta}_0).\tag{0.2}$$

In the case $D=1$, $Z(0)$ is the dipole moment. If RM is the Minkowski spacetime, this definition reproduces all the usual multipoles except for the monopole. In order to get multipoles of arbitrary directionality, M must be locally homogeneous. Static dipoles on the Einstein Universe can be obtained with

$$Z=P_0+N_0=\cos\vartheta\frac{\partial}{\partial\rho}-\cot\rho\sin\vartheta\frac{\partial}{\partial\vartheta}.$$

If M admits a global solution $\mathbf{G}_0$ of $\Delta\mathbf{G}_0=\boldsymbol{\delta}_0+\text{Const.}$, comoving fields can be determined [as on Robertson-Walker spacetimes, where

$$\mathbf{G}_0=\frac{1}{4\pi^2}(\pi-\rho)\cot\rho,\quad \frac{1}{4\pi\rho},\quad \frac{1}{4\pi}\coth\rho\qquad\text{for}\quad k=1,0,-1,\text{ respectively,}$$

according to (9.4)]:

Since Δ commutes with D and L_Z,

$$\Delta L_Z D\mathbf{G}_0=L_Z D\boldsymbol{\delta}_0. \tag{0.3}$$

$$E=dL_Z D\mathbf{G}_0\qquad\text{and}\qquad *B=dL_Z D\mathbf{G}_0+Z^\flat D\boldsymbol{\delta}_0 \tag{0.4}$$

are thus "comoving" fields of the multipoles with $J_0=\delta(Z^\flat D\boldsymbol{\delta}_0)$ and $J_s=*\,d(Z^\flat D\boldsymbol{\delta}_0)$, respectively. This means that they satisfy

$$E_{,\tau}=dE=0,\qquad -\delta E=J_0,\qquad B_{,\tau}=dB=0,\qquad \delta B=J_s. \tag{0.5}$$

As distributional solutions both are unique up to arbitrary time independent harmonic distributions $\mathbf{B}_h$ and E_h which, by corollary 8.3.2 of [13], are even C^∞. If M is compact, E and B are thus unique as distributions while for noncompact M, they are the only distributional solutions which are in $L^2(M\setminus\mathscr{B}_\varepsilon(0))$ for all closed ε-balls with $\varepsilon>0$ around the origin of M. In this sense, the electric 1-forms $-d\mathbf{G}_0$ correspond to the comoving monopoles or elementary charges to be described by $J_0=\boldsymbol{\delta}_0-1/\mathrm{Vol}(M)$ in the compact cases and by $J_0=\boldsymbol{\delta}_0$ in the noncompact ones.

For all comoving systems (J, F) the action 4-form, *i.e.* the ⊛-dual of the Lagrange density, is invariant under τ-translations whereas the Lorentz force acting on test particles with instantaneous 4-velocity $(1,0,0,0)$ has a time dependence $\sim\Omega^{-2}$. This describes the fact that geodesic distances – whose squares appear in the denominator of the Lorentz force – gain a factor Ω while M expands. Thus by only looking at the Lorentz force, one cannot decide whether the spacetime RM has a time dependence $\sim\Omega^2(\tau)$ or whether RM is static and all elementary charges depend on τ like $J_0=\Omega^{-2}(\boldsymbol{\delta}_0-1/\mathrm{Vol}(M))$. The solution (9.3) of Maxwell's equations on the Einstein Universe might serve as an example.

XI. TIME DEPENDENT DIPOLES ON ROBERTSON-WALKER SPACETIMES

The notation is as in section VII, but for convenience, the index k will be omitted. The decomposition (7.6) still holds if J_s is distributional on the time slices instead of C^∞. Naturally, $\Theta\tilde{f}$ and $\Theta\tilde{g}$ will then also be distributions. Similarly, initial data for A_s can be transferred into initial data for Θf and Θg. Any distributional solution of (7.8) will again give rise to a well defined field (7.9) which satisfies Maxwell's equations in case $J_0=0$, but the energy of this field will usually be infinite as it would for the field E_e of a J_0 distributional on the time slices. Time independent harmonic terms in the expressions for the electric and magnetic fields must be excluded by the initial data for F.

On RW, a magnetic dipole is described by

$$J_{md}=\Omega^{-2} \circledast J_s, \qquad J_s=\ast d(Z^\flat(\tau)\cdot\delta_0)$$

with $Z=\mathcal{T}^\#(S\cdot h(\tau,\vartheta,\varphi))$ being a time dependent Killing vector field on $(M, d\rho^2+S^2\,d\omega^2)$. It should be assumed that $h=\sum_{m=-1}^{1} h_m(\tau)Y_{1m}(\vartheta,\varphi)$. Then Z and h satisfy the relations $\Delta Z^\flat=4kZ^\flat$, $\Theta h=2h$, and $Z^\flat=\Theta h\cdot d\rho+(S^2)'\cdot dh$ outside the origin.

Suppose, E and B are the retarded (or advanced) fields of this dipole such that

$$\delta E=0, \qquad dB=0, \qquad dE-B_{,\tau}=0, \qquad \delta B+E_{,\tau}=\ast d(Z^\flat\cdot\delta_0). \quad (1.1)$$

Then the retarded (or advanced) fields $\tilde{E}:=\ast B-Z^\flat\cdot\delta_0$, $\tilde{B}:=-\ast E$ satisfy

$$\begin{aligned} d\tilde{B}&=0, & \delta\tilde{E}&=-\delta(Z^\flat.\delta_0),\\ d\tilde{E}-\tilde{B}_{,\tau}&=0, & \delta\tilde{B}+\tilde{E}_{,\tau}&=-(Z^\flat_{,\tau}.\delta_0). \end{aligned} \quad (1.2)$$

This is exactly what one expects from the respective fields of the electric dipole on RW described by

$$J_{ed}=\Omega^{-2}\circledast(J_0\,d\tau+\tilde{J}), \qquad J_0=\delta(Z^\flat\cdot\delta_0), \qquad \tilde{J}=-(Z^\flat_{,\tau}\cdot\delta_0).$$

Hence, it suffices to determine the fields od the magnetic dipoles.

First the sources $v\lrcorner J_s$ and $v\lrcorner \ast dJ_s$ of the wave equations (7.8) shall be determined. For this purpose, the properties $0=v^\flat(0)=dv^\flat$ and $-\delta v^\flat=3S'(\rho)$ of $v^\flat=S\,d\rho$ should be recognized. One obtains:

$$\begin{aligned} v\lrcorner J_s&=v\lrcorner \ast d(\delta_0 Z^\flat)=\delta_0(v\lrcorner \ast dZ^\flat)+v\lrcorner \ast(d\delta_0\wedge Z^\flat)\\ &=Z\lrcorner \ast(v^\flat\wedge d\delta_0)=Z\lrcorner \ast d(\delta_0 v^\flat)-\delta_0(Z\lrcorner \ast dv^\flat)=0, \quad (1.3)\\ v\lrcorner \ast dJ_s&=v\lrcorner \ast d(\delta_0\ast dZ^\flat)+v\lrcorner\,\delta(d\delta_0\wedge Z^\flat)\\ &=\delta_0(v\lrcorner\,\Delta Z^\flat)+\delta((d\delta_0)(v\lrcorner Z^\flat)-Z^\flat(v\lrcorner d\delta_0))\\ &=-\delta(\delta_0\,d(v\lrcorner Z^\flat)+(\delta v^\flat)\delta_0 Z^\flat)=-\delta(2\delta_0 Z^\flat)=-2Z\lrcorner d\delta_0. \quad (1.4) \end{aligned}$$

Convolution with the Green functions of (9.1) leads to $\Theta g=0$ and

$$\Theta f^{\pm}(\tau,x)=-\frac{1}{2\pi}\langle S^{-1}(d_{xx'})\cdot Z^{\flat}(\tau\pm d_{xx'},x'),d\boldsymbol{\delta}_0\rangle_M$$

$$=-\frac{1}{2\pi}\langle \delta(S^{-1}(d_{xx'})\cdot Z^{\flat}(\tau\pm d_{xx'},x')),\boldsymbol{\delta}_0\rangle_M$$

$$=-\frac{1}{2\pi}\Big[\delta(S^{-1}(d_{xx'})\cdot Z^{\flat}(\tau\pm d_{xx'},x'))\Big]_{x'=0}$$

$$=\frac{1}{2\pi}\Big(Z(\tau\pm\rho,0)\lrcorner\,[dS^{-1}(d_{xx'})]_{x'=0}-S^{-1}(\rho)[\delta(Z^{\flat}(\tau\pm d_{xx'},x'))]_{x'=0}\Big)$$

$$=\frac{1}{2\pi}\Big(-Z(\tau\pm\rho,0)\frac{S'(\rho)}{S^2(\rho)}\pm\frac{1}{S(\rho)}Z_{,\tau}(\tau\pm\rho,0)\Big)\lrcorner\,[d(d_{xx'})]_{x'=0}. \quad (1.5)$$

This can be simplified using the fact that the inner product of a Killing vector and the unit tangent of a geodesic is constant along that geodesic, which means here that

$$Z(\tau,0)\lrcorner\,[d(d_{xx'})]_{x'=0}=Z_\rho(\tau,x)=\Theta\,h(\tau,\vartheta,\varphi). \quad (1.6)$$

Consequently,

$$f^{\pm}(\tau,x)=\frac{1}{2\pi S^3}((\ln S)'h\mp h_{,\tau})\ \text{with}\ S(\rho)\ \text{and}\ h(\tau\pm\rho,\vartheta,\varphi),\ \text{and}$$

$$A_s^{\pm}(\tau,x)=\mathscr{U}(f^{\pm})=\ast\,d(f^{\pm}\cdot v^{\flat})=S^3\sin\vartheta\,(f^{\pm,\varphi}d\vartheta-f^{\pm,\vartheta}d\varphi)$$

$$=\frac{\sin\vartheta}{2\pi}((\ln S)'(h^{,\varphi}d\vartheta-h^{,}d\varphi)\mp(h^{,\varphi}_{,\tau}d\vartheta-h^{,}d\varphi))$$

$$=\frac{\sin\vartheta}{2\pi(S^2)'}((\ln S)'(Z^{\varphi}d\vartheta-Z_{\vartheta}d\varphi)\mp(Z^{\varphi}_{,\tau}d\vartheta-Z^{\vartheta}_{,\tau}d\varphi))(\tau\pm\rho,x) \quad (1.7)$$

is the 1-form potential, of which E is the derivative $A_{s,\tau}$. For $B=dA_s$ there results

$$\ast\,B^{\pm}(\tau,x)=\mathscr{T}(f^{\pm})=S^{-1}\Theta f^{\pm}d\rho+(S\cdot f^{\pm}_{,\vartheta})_{,\rho}d\vartheta+(S\cdot f^{\pm}_{,\varphi})_{,\rho}d\varphi$$

$$=\frac{1}{2\pi}((\ln S)'''h\pm 2(\ln S)''h_{,\tau})\,d\rho$$

$$+\frac{1}{2\pi}((\ln S)''dh\pm(\ln S)'dh_{,\tau}-dh_{,\tau\tau})$$

$$=\frac{1}{2\pi S^2}((\ln S)'Z_\rho\mp Z_{\rho,\tau})(\tau\pm\rho,x)\,d\rho$$

$$+\frac{1}{2\pi(S^2)'}((\ln S)''Z_\omega\pm(\ln S)'Z_{\omega,\tau}-Z_{\omega,\tau\tau})(\tau\pm\rho,x), \quad (1.8)$$

where Z_ω means $Z_\vartheta d\vartheta+Z_\varphi d\varphi$.

If RW has nonpositive curvature, these are the desired fields of the magnetic dipole. The "comoving" limit of both advanced and retarded fields does not differ from solution (0.4).

If RW is the Einstein Universe and if $h=h_0(\tau)\cos\vartheta$, the fields $*\mathrm{B}$ and $(-*\mathrm{A}_{s,\tau})$ coincide with the electric and magnetic fields, respectively, which Kohler [4] derived for an electric dipole on this spacetime. But for two reasons, this solution is not complete: the minor reason is that the singularity $(-\mathrm{Z}^\flat.\boldsymbol{\delta}_0)$ in the electric field is missing, and the major reason is that the Green functions (9.1) are not global in case of the Einstein Universe. To correct this, $\mathrm{A}_s^\pm(\tau,x)$ and $\mathrm{B}^\pm(\tau,x)$ must be replaced by

$$\begin{aligned} &\mathrm{A}_s^\pm(\tau,x)+\sum_{m=1}^{\infty}(\mathrm{A}_s^\pm(\tau\pm 2m\pi,x)-\mathrm{A}_s^\mp(\tau\pm 2m\pi,x)) \\ &\mathrm{B}^\pm(\tau,x)+\sum_{m=1}^{\infty}(\mathrm{B}^\pm(\tau\pm 2m\pi,x)-\mathrm{B}^\mp(\tau\pm 2m\pi,x)), \end{aligned} \tag{1.9}$$

respectively. With $\mathrm{S}=\sin\rho$ inserted, these two sums are the advanced $(+)$ and retarded $(-)$ 1-form potential and magnetic 2-form for the time dependent magnetic dipole on Robertson-Walker spacetimes with positive curvature; the fields vanish at the point antipodal to the location of the dipole. The advanced and retarded Lorentz forces provoked at the point (τ,x) by a permanent dipole on such a spacetime are equally strong, if both $\Omega^2(\tau')$ and the vibration of the dipole are symmetric with respect to τ.

Neither the expressions (1.9) nor the solution of Kohler can be applied to the comoving case discussed in the previous section: while (1.9) turns into pure nonsense unless $\Omega(\tau)=0$ for finite τ, Kohlers solution turns into the static field of a static double dipole occupying antipodal points on S^3.

By now, the patient reader should be convinced that there exists a method of treating ordinary electrodynamics which can be generalized to Robertson-Walker spacetimes, but that the existence of eigenmodes and the global Green functions essentially depend on the spatial topology.

ACKNOWLEDGEMENTS

I would like to thank Prof. Dr. G. Süssmann, Prof. Dr. M. Schottenloher and the group around Prof. Dr. J. Ehlers as well as himself for encouragement, guidance through the literature and, last but not least, many useful discussions.

Note added in proof: Since there have been questions about it, $\mathscr{I}$ would like to remark that the decoupling procedure described in section VII also holds for the Schwarzschild spacetime.

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(*Manuscript received October* 12, 1989.)