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Jump processes and diffusions in relativistic stochastic mechanics

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ABSTRACT. — There are two stochastic descriptions of relativistic quantum spinless particles. In the first one, suggested by Feynman's path integral approach to Klein-Gordon propagator, the main ingredients are diffusions in Minkowski space parametrized by some kind of proper time. The second description, on the contrary, is based upon jump Markov processes giving the space position of the particle in the sense of Newton and Wigner with the time as parameter. In this paper we bridge the gap between these different probabilistic scenarios by constructing the space jump processes by means of space-time diffusions. We give some applications to the nonrelativistic limit of the hamiltonian semigroup

$$t \mapsto \exp - \frac{t}{\hbar} \mathrm{H}$$

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where

$$H = \sqrt{-c^2 \hbar^2 \Delta + \frac{M^2 c^4}{\hbar^2}} - M c^2 + V(.).$$

RÉSUMÉ. — Il y a deux descriptions stochastiques de la mécanique quantique relativiste pour une particule sans spin. Dans la première, suggéré par la théorie de Feynman, les ingrédients principaux sont des diffusions dans l'espace de Minkowski paramétrisé par une sorte de temps propre. La deuxième description, par contre, emploie des processus de Markov à sauts donnant la position de la particule dans l'espace au sens de Newton et Wigner avec le temps comme paramètre. Notre papier va relier les deux prescriptions probabilistes car nous construisons les processus à sauts dans l'espace par les diffusions dans l'espace-temps. Nous donnons aussi des applications à l'étude de la limite non relativiste du semi-group hamiltonien

$$t \mapsto \exp - \frac{t}{\hbar} H$$

ou

$$H = \sqrt{-c^2 \hbar^2 \Delta + \frac{M^2 c^4}{\hbar^2}} - M c^2 + V(.).$$

1. INTRODUCTION

The task of extending Nelson's stochastic mechanics ([1] to [4]) to the relativistic realm has not been yet fully performed, nevertheless there are partial results both for Klein-Gordon and Dirac equations ([5] to [10]). In the case of Klein-Gordon theory two different approaches exist which we briefly discuss here since they are the starting point of our paper. For simplicity we bound ourselves to free evolution where a single particle picture of (positive frequency) Klein-Gordon wave functions is physically meaningful. The first probabilistic scenario ([5], [6], [8], [9]) rests upon diffusions $s \mapsto x_s^\mu$, $\mu = 0, 1, \ldots, D$ in $(1+D)$-dimensional space-time. If $s \mapsto x_s^\mu$ obeys classical relativistic mechanics revisited à la Nelson ([1], [5], [8]), one can reconstruct from it a wave function

$\psi(s, x^0, \ldots, x^D) = \psi(s, x)$ which satisfies the "Schrödinger equation in Minkowski space":

$$i\hbar \frac{\partial \psi}{\partial s} = \frac{\hbar^2}{2M} \Box \psi = \frac{\hbar^2}{2M} \partial_\mu \partial^\mu \psi \tag{1}$$

whose stationary solutions

$$\psi(s, x^0, \ldots, x^D) = \psi(s, x) = \varphi(x) \exp i \frac{Mc^2}{2\hbar} s$$

obey Klein-Gordon equation $\Box \varphi + \frac{Mc^2}{\hbar} \varphi = 0$. The relativistic covariance of such description is tricky because space-time is lacking of a natural positive definite metric $\eta_{\mu\nu}(x)$ as it is required in the Fokker-Planck equation

$$\frac{\partial \rho}{\partial s} = \frac{\hbar}{2M} \eta_{\mu\nu} \partial^\mu \partial^\nu \rho - \partial_\mu (b^\mu \rho)$$

for the probability density $\rho(s, x^0, \ldots, x^D) = \rho(s, x)$ of $s \mapsto x_s^\mu$.

In order to overcome this difficulty, people proposed two solutions. In the first one [6] a positive definite metric $\eta_{\mu\nu}(x)$ is introduced in Minkowski space by means of suitable fields of Lorentz transformations (a sort of gauges fields) while the second approach [8] forsakes the Markov property by choosing $s \mapsto x_s^\mu$ inside the larger class of Bernstein's stochastic processes. By doing that, as shown in [8], the obviously covariant equation

$$\frac{\partial \rho}{\partial s} = -\frac{\hbar}{2M} \Box \rho - \partial_\mu (v^\mu \rho) \tag{2}$$

for $\rho(s, x)$ naturally emerges. The situation is especially simple when

$$\psi(s, x) = \varphi(x) \exp i \frac{Mc^2}{2\hbar} s$$

describes a spinless relativistic particle of mass M at rest namely,

$$\varphi(x) = (2\pi)^{-D/2} \sqrt{\frac{\hbar}{Mc}} \exp - i \frac{Mc}{\hbar} x^0$$

(ground state for a free particle). The corresponding diffusion, when it is chosen to start from the origin of Minkowski space, is, in both versions, $s \mapsto x_s^\mu$ with

$$x_s^0 = cs + \sqrt{\frac{\hbar}{M}} w_s^0, \qquad x_s^\alpha = \sqrt{\frac{\hbar}{M}} w_s^\alpha \, \alpha = 1, \ldots, D \tag{3}$$

Here $s \mapsto (w_s^0, \ldots, w_s^D)$ is a $(1 + D)$-dimensional Wiener process. In absence of quantum fluctuations $(\hbar = 0)\, s \mapsto x_s^\mu$ gives precisely the worldline of a particle at rest while, when $\hbar > 0$, the space-time path $s \mapsto x_s^\mu$ wanders wildly over Minkowski space by crossing infinitely often each spacelike hypersurface $x^0 = ct$ coherently with Feynman's ideas.

The second probabilistic scenario about Klein-Gordon equation [10] gives up explicit relativistic covariance as it uses Markov processes $t \mapsto \xi_t$ in space with the time t as parameter. When a positive frequency solution $\varphi(t, \mathbf{x})$ of Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi - \frac{M^2 c^2}{\hbar^2} = 0$$

is given, one can construct a jump Markov process $t \mapsto \xi_t$ such that

$$\mathbb{P}rob(\xi_t \in B) = \int_B |((-\Delta + M^2 c^2/\hbar^2)^{1/4} \varphi)(t, \mathbf{x})|^2 \, d\mathbf{x}^D \tag{4}$$

at every time t and for each Borel subset $B \subseteq \mathbb{R}^D$, the right hand side of this equality just being the quantum mechanical probability of finding, at time t, the particle localized inside B in the sense of Newton and Wigner [11]. The jump character of stochastic processes involved is at variance with the Nelson's theory, indeed it is a relativistic feature which disappears in the nonrelativistic limit. If

$$\varphi(t, \mathbf{x}) = (2\pi)^{-D/2} \sqrt{\frac{\hbar}{Mc}} \exp - i \frac{Mc^2}{\hbar} t$$

the corresponding jump Markov process $t \mapsto \xi_t^c$ has independent and stationary increments and, when it is chosen to start from the origin of $\mathbb{R}^D$, its characteristic function is given by

$$\mathbb{E}(\exp i\,\mathbf{p}\,.\,\xi_t^c) = \exp t\, L(\mathbf{p}) = \exp t\left(\frac{Mc^2}{\hbar} - \sqrt{c^2\,\|\mathbf{p}\|^2 + \frac{M^2 c^4}{\hbar^2}}\right) \tag{5}$$

The stochastic process $t \mapsto \xi_t^c$ admits, as generator, the pseudodifferential operator

$$\mathbb{L} = \frac{Mc^2}{\hbar}\mathbb{I} - \sqrt{-c^2\Delta + \frac{M^2 c^4}{\hbar^2}\mathbb{I}}$$

which is related to the relativistic free quantum hamiltonian

$$H_0 = \sqrt{-c^2\hbar^2\Delta + M^2 c^4\,\mathbb{I}}$$

by $H_0 = -\hbar\,\mathbb{L} + Mc^2\,\mathbb{I}$. The fact that $\mathbb{L}$ is the generator or infinitesimal operator of a time homogeneous Markovian family, has been exploited by T. Ichinose and H. Tamura [12] in constructing a Feynman-Kac formula for quantum hamiltonians of the form $H = H_0 - Mc^2\,\mathbb{I} + V(.)$ where $V(.) : \mathbb{R}^D \mapsto \mathbb{R}$ is some well behaving potential. In Ichinose and Tamura formula the ground state process $t \mapsto \xi_t^c$ plays the same role as the Wiener process $t \mapsto \mathbf{w}_t$ in the usual Feynman-Kac formula.

From the standpoint of relativistic stochastic mechanics, the ground state processes (3) and (5), albeit not associated to a normalizable wave function, are of the utmost importance. In fact, all processes describing other quantum states can be constructed on the probability space of (3) in the first scenario and of (5) in the second by solving suitable stochastic differential equations. Therefore we concentrate ourselves upon the task of finding useful relations between the two stochastic processes (3) and (5). We are, in fact, able to construct, in a natural way, (5) from (3) and this provides the core of our paper.

In the nonrelativistic limit it is known that the path space probability measure of $t \mapsto \xi_t^c$, which lives on cadlag paths, is weakly convergent to the Wiener measure and this result gives some control [13], by probabilistic techniques, on the nonrelativistic limit for the hamiltonian semigroups $t \mapsto \exp -\frac{t}{\hbar} \mathrm{H}$ where $\mathrm{H} = \mathrm{H}_0 - \mathrm{M} c^2 \mathbb{I} + \mathrm{V}(.)$. We obtain a stronger result as we construct both $t \mapsto \xi_t^c$ and the Wiener process $t \mapsto \mathbf{w}_t$ on the same probability space and it is easy to prove that $t \mapsto \xi_t^c$ converges in probability to $t \mapsto \mathbf{w}_t$, as $c \uparrow +\infty$, uniformly in $0 \leqq t \leqq \mathrm{T}$ for all $\mathrm{T} > 0$. This fact allows us to get a more transparent probabilistic control of the nonrelativistic limit for the hamiltonian semigroup $t \mapsto \exp -\frac{t}{\hbar} \mathrm{H}$. We derive the ground state jump process $t \mapsto \xi_t^c$ from the space-time diffusion

$$s \mapsto x_s^\mu = \left(cs + \sqrt{\frac{\hbar}{\mathrm{M}}} w_s^0, \ldots, \sqrt{\frac{\hbar}{\mathrm{M}}} w_s^{\mathrm{D}} \right)$$

in a very simple way. Let $\tau_c(t)$ be the least of s at which $s \mapsto x_s^\mu$ hits the "space at time t" namely the spacelike hypersurface

$$\Sigma_t = \{ (x^0, \ldots, x^{\mathrm{D}}) \in \mathbb{R}^{1+\mathrm{D}} : x^0 = ct \},$$

then we claim that $\xi_t^c = \sqrt{\frac{\hbar}{\mathrm{M}}} \mathbf{w}_{\tau_c(t)}$, in other words the space random variable ξ_t^c represents the intersection of the path $s \mapsto x_s^\mu$ with the spacelike hypersurface Σ_t at the first hitting time s.

The plan of the paper is the following: in the second section we show that such construction provides the right result, namely that $t \mapsto \sqrt{\frac{\hbar}{\mathrm{M}}} \mathbf{w}_{\tau_c(t)}$ is a stochastic process with independent and time homogeneous increments whose characteristic function is $\exp t \mathrm{L}(\mathbf{p})$. In the third section we study the limiting behavior of Markov time $\tau_c(t)$ when $c \uparrow +\infty$ and we show that $\tau_c(t)$ converges in probability to t uniformly on each compact interval. It follows that $\xi_t^c = \sqrt{\frac{\hbar}{M}}\, \mathbf{w}_{\tau_c(t)}$ converges in probability to $\sqrt{\frac{\hbar}{M}}\, \mathbf{w}_t$ uniformly in $0 \leqq t \leqq T$ for all $T > 0$. In the fourth section, we exploit this result by studying, via probabilistic techniques, the nonrelativistic limit of the hamiltonian semigroup $t \mapsto \exp - \frac{t}{\hbar} H$ where $H = H_0 - M c^2 \mathbb{I} + V(.)$, a typical result will be that $t \mapsto \exp - \frac{t}{\hbar} H$ converges strongly to

$$t \mapsto \exp - \frac{t}{\hbar}\left(- \frac{\hbar^2}{2M} \Delta + V(.) \right)$$

in each $L^p(\mathbb{R}^D)$, $1 \leqq p < +\infty$, uniformly in $0 \leqq t \leqq T$, when $V(.)$ is continous and bounded below. Finally, in the conclusions, we describe some problems which are worth studying along this line.

2. CONSTRUCTION OF $t \mapsto \xi_t^c$ THROUGH BROWNIAN MOTION

We begin this section by studying some elementary properties of the Markov time

$$\tau_c(t) = \inf \left\{ s \geqq 0 : cs + \sqrt{\frac{\hbar}{M}}\, w_s^0 = ct \right\}$$

where $t \geqq 0$. Let $s \mapsto w_s$ be a one-dimensional Wiener process and, for $\alpha \geqq 0$, $\beta \geqq 0$, define

$$\tau(\alpha, \beta) = \inf \{ s \geqq 0 : w_s = \alpha - \beta s \} \tag{6}$$

It is well known [14] that $\alpha \mapsto \tau(\alpha, 0)$ is a Lévy process, namely a nonnegative and nondecreasing stochastic process with independent and time homogeneous increments, moreover $\tau(\alpha, 0) < +\infty$ a. s. and, for all nonnegative γ, $\mathbb{E}(\exp - \gamma \tau(\alpha, 0)) = \exp - \alpha \sqrt{2\gamma}$.

LEMMA 1. – *For all* $\beta \geqq 0$, *the stochastic process* $\alpha \mapsto \tau(\alpha, \beta)$ *is a Lévy process, moreover* $\tau(\alpha, \beta) < +\infty$ *a. s. and, for all nonnegative* γ, $\mathbb{E}(\exp - \gamma \tau(\alpha, \beta)) = \exp \alpha (\beta - \sqrt{\beta^2 + 2\gamma})$.

Proof. – The first part of Lemma can be exactly proven as in the known particular case $\beta = 0$ while the easy inequality $\tau(\alpha, 0) \geqq \tau(\alpha, \beta)$ assures that a. s. $\tau(\alpha, \beta) < +\infty$. Therefore the only thing we must demonstrate is the formula

$$\mathbb{E}(\exp - \gamma \tau(\alpha, \beta)) = \exp \alpha (\beta - \sqrt{\beta^2 + 2\gamma}).$$

By Dynkin-Hunt theorem, the stochastic process $s \mapsto \tilde{w}_s = w_{s+\tau(\alpha,\beta)} - w_{\tau(\alpha,\beta)}$ is a new Wiener process independent from $\tau(\alpha, \beta)$. Let X be any nonnegative random variable independent from $s \mapsto \tilde{w}_s$ and $\tilde{\tau}(X) = \inf\{s \geqq 0 : \tilde{w}_s = X\}$. It is easy to see, from $\tau(\alpha, 0) \geqq \tau(\alpha, \beta)$, that $\tau(\alpha, 0) = \tau(\alpha, \beta) + \tilde{\tau}(\beta\tau(\alpha, \beta))$. From this observation it follows that

$$\exp - \alpha \sqrt{2\tilde{\gamma}} = \mathbb{E}(\exp - \tilde{\gamma}\tau(\alpha, 0)) = \mathbb{E}(\exp - (\tilde{\gamma} + \beta\sqrt{2\tilde{\gamma}})\,\tau(\alpha, \beta))$$

because

$$\mathbb{E}(\exp - \tilde{\gamma}\tilde{\tau}(X) \mid X) = \exp - X\sqrt{2\tilde{\gamma}}.$$

By choosing $\sqrt{2\tilde{\gamma}} = -(\beta - \sqrt{\beta^2 + 2\gamma})$ we obtain the result.

If we observe that

$$\tau_c(t) = \inf\left\{ s \geqq 0 : w_s^0 = \sqrt{\frac{Mc^2}{\hbar}}\,t - \sqrt{\frac{Mc^2}{\hbar}}\,s \right\}$$

we conclude that $t \mapsto \tau_c(t)$ is a Lévy process with

$$\mathbb{E}\left(\exp - \frac{\hbar \|\mathbf{p}\|^2}{2M} \tau_c(t) \right) = \exp t \left(\frac{Mc^2}{\hbar} - \sqrt{c^2 \|\mathbf{p}\|^2 + \frac{M^2 c^4}{\hbar^2}} \right)$$

If $s \mapsto \mathbf{w}_s = (w_s^1, \ldots, w_s^D)$ is a D-dimensional Wiener process independent from $\tau_c(t)$ it follows that $t \mapsto \xi_t^c = \sqrt{\frac{\hbar}{M}}\,\mathbf{w}_{\tau_c(t)}$ is a stochastic process with independent and time homogeneous increments starting from the origin of $\mathbb{R}^D$ as $\tau_c(0) = 0$ and the only thing we must still show is that $t \mapsto \xi_t^c$ has the right characteristic function.

THEOREM 1. – *If* $\xi_t^c = \sqrt{\frac{\hbar}{M}}\,\mathbf{w}_{\tau_c(t)}$ *then* $t \mapsto \xi_t^c$ *is a stochastic process with independent and time homogeneous increment whose characteristic function is given by*

$$\mathbb{E}(\exp i\,\mathbf{p}.\xi_t^c) = \exp t \left(\frac{Mc^2}{\hbar} - \sqrt{c^2 \|\mathbf{P}\|^2 + \frac{M^2 c^4}{\hbar^2}} \right).$$

Proof. – We already know that $t \mapsto \xi_t^c$ has independent and time homogeneous increments. Furthermore, since

$$\mathbb{E}\left(\exp i\,\mathbf{p}.\sqrt{\frac{\hbar}{M}}\,\mathbf{w}_{\tau_c(t)} \mid \tau_c(t) \right) = \exp - \frac{\hbar \|\mathbf{p}\|^2}{2M} \tau_c(t)$$

it follows that

$$\mathbb{E}\left(\exp i\,\mathbf{p}.\sqrt{\frac{\hbar}{M}}\,\mathbf{w}_{\tau_c(t)} \right) = \mathbb{E}\left(\exp - \frac{\hbar \|\mathbf{p}\|^2}{2M} \tau_c(t) \right)$$

and it is sufficient to apply Lemma 1.

This theorem is a straightforward generalisation of the well known construction of the D-dimensional Cauchy process $t \mapsto \mathbf{c}_t$, with generator $-\sqrt{-\Delta}$ as $\mathbf{c}_t = \mathbf{w}_{\tau(t)}$ where $\tau(t) = \inf\{s \geqq 0 : w_s^0 = t\}$ and it provides the link between the space-time diffusion

$$s \mapsto \left(cs + \sqrt{\frac{\hbar}{M}} w_s^0, \sqrt{\frac{\hbar}{M}} w_s^1, \ldots, \sqrt{\frac{\hbar}{M}} w_s^D \right)$$

and the jump Markov process $t \mapsto \xi_t^c$ which we described in the introduction.

In Nelson's stochastic mechanics, the diffusion associated to the ground state wave function $\psi(t, \mathbf{x}) = \text{Const.}$ of a free nonrelativistic particle with mass M is $t \mapsto \sqrt{\frac{\hbar}{M}} \mathbf{w}_t$, therefore we expect that the jump process $t \mapsto \xi_t^c$ converges to $t \mapsto \sqrt{\frac{\hbar}{M}} \mathbf{w}_t$ when $c \uparrow +\infty$. In the next section we study the nonrelativistic limit of $t \mapsto \xi_t^c$ and we show, indeed, that $\lim_{c \to +\infty} \xi_t^c = \sqrt{\frac{\hbar}{M}} \mathbf{w}_t$ in probability.

3. THE NONRELATIVISTIC LIMIT OF $t \mapsto \xi_t^c$

We begin this section by studying the behaviour of

$$\tau_c(t) = \inf\left\{ s \geqq 0 : cs + \sqrt{\frac{\hbar}{M}} w_s^0 = ct \right\}$$

when $c \uparrow +\infty$, in the following $(\Omega, \mathscr{F}, \mathbb{P}(.))$ will be the probability space underlying the $(1+D)$-dimensional Brownian motion

$$s \mapsto (w_s^0, w_s^1, \ldots, w_s^D).$$

LEMMA 2. — *For all* $t \geqq 0$ *and* $\varepsilon > 0$

$$\lim_{c \to +\infty} \mathbb{P}(|\tau_c(t) - t| > \varepsilon) = 0$$

Proof. — From

$$\mathbb{E}(\exp - \gamma \tau_c(t)) = \exp t \left(\frac{M c^2}{\hbar} - \sqrt{\frac{2 \gamma M c^2}{\hbar} + \frac{M^2 c^4}{\hbar^2}} \right)$$

it follows that $\mathbb{E}(\tau_c(t)) = t$ and $\mathbb{E}(|\tau_c(t) - t|^2) = \frac{\hbar}{M c^2}$ and the result is an obvious consequence of Chebyshev's inequality.

Now we can prove a better result.

LEMMA 3. — *For all* $T>0$ *and* $\varepsilon>0$

$$\lim_{c \to +\infty} \mathbb{P}(\sup_{0 \leqq t \leqq T} |\tau_c(t)-t| > \varepsilon)=0.$$

Proof. — (i) Because $t \mapsto \tau_c(t)$ is nondecreasing and

$$\sqrt{\frac{M c^2}{\hbar}}(t-\tau_c(t)) = w^0_{\tau_c(t)},$$

$$\mathbb{P}(\sup_{0 \leqq t \leqq T} |\tau_c(t)-t| > \varepsilon) \leqq \mathbb{P}\left(\sup_{0 \leqq s \leqq \tau_c(T)} |w^0_s| > \sqrt{\frac{M c^2}{\hbar}}\varepsilon\right)$$

(ii) but, for arbitrary $\delta>0$,

$$\begin{aligned}\mathbb{P}&\left(\sup_{0 \leqq s \leqq \tau_c(T)} |w^0_s| > \sqrt{\frac{M c^2}{\hbar}}\varepsilon\right)\\ &= \mathbb{P}\left(\sup_{0 \leqq s \leqq \tau_c(T)} |w^0_s| > \sqrt{\frac{M c^2}{\hbar}}\varepsilon \,\&\, \tau_c(T) > T+\delta\right)\\ &+ \mathbb{P}\left(\sup_{0 \leqq s \leqq \tau_c(T)} |w^0_s| > \sqrt{\frac{M c^2}{\hbar}}\varepsilon \,\&\, \tau_c(T) \leqq T+\delta\right)\\ &\leqq \mathbb{P}(\tau_c(T) > T+\delta) + \mathbb{P}\left(\sup_{0 \leqq s \leqq T+\delta} |w^0_s| > \sqrt{\frac{M c^2}{\hbar}}\varepsilon\right)\end{aligned}$$

and the result follows from Lemma 2 and

$$\lim_{c \to +\infty} \mathbb{P}\left(\sup_{0 \leqq s \leqq T+\delta} |w^0_s| > \sqrt{\frac{M c^2}{\hbar}}\varepsilon\right)=0.$$

LEMMA 4. — *Let* $s \mapsto \mathbf{w}_s$ *be a* D-*dimensional Wiener process, then, for all* $T>0$ *and* $\varepsilon>0$

$$\lim_{\delta \to 0} \mathbb{P}(\sup_{0 \leqq t \leqq T} \sup_{s:|s-t| \leqq \delta} \|\mathbf{w}_s-\mathbf{w}_t\| > \varepsilon)=0.$$

Proof. — Let δ_n be any sequence of positive numbers $\leqq 1$ which converges to 0 as $n \to \infty$ and $B_{n,\varepsilon} = \{\omega \in \Omega : \sup_{0 \leqq t \leqq T} \sup_{s:|s-t| \leqq \delta_n} \|\mathbf{w}_s-\mathbf{w}_t\| > \varepsilon\}$. Since $s \mapsto \mathbf{w}_s(w)$ is uniformly continous on compact intervals (and also globally a.s.), $\chi_{B_{n,\varepsilon}}(w)$ will be 0 for n large enough. The result follows, therefore, from Lebesque's theorem on dominated pointwise convergence.

We can discuss now the non relativistic behaviour of $\xi^c_t = \sqrt{\frac{\hbar}{M}}\mathbf{w}_{\tau_c(t)}$.

THEOREM 2. – *For all* $T>0$ *and* $\varepsilon>0$

$$\lim_{c\to+\infty} \mathbb{P}\left(\sup_{0\leqq t\leqq T} \left\| \xi_t^c - \sqrt{\frac{\hbar}{M}} \mathbf{w}_t \right\| > \varepsilon \right) = 0.$$

Proof. – (i) For $\delta>0$ let $A_{c,\delta}=\{\omega\in\Omega: \sup_{0\leqq t\leqq T} |\tau_c(t)-t|\leqq\delta\}$, we know, by Lemma 3, that $\lim_{c\to+\infty} \mathbb{P}(CA_{c,\delta})=0$, (ii) on the other hand, for each $\delta>0$,

$$\mathbb{P}\left(\sup_{0\leqq t\leqq T} \left\| \sqrt{\frac{\hbar}{M}} \mathbf{w}_{\tau_c(t)} - \sqrt{\frac{\hbar}{M}} \mathbf{w}_t \right\| > \varepsilon \right) \leqq \mathbb{P}\left(\sup_{0\leqq t\leqq T} \sup_{s:|s-t|\leqq\delta} \|\mathbf{w}_s - \mathbf{w}_t\| > \sqrt{\frac{\hbar}{M}}\varepsilon \right) + \mathbb{P}(CA_{c,\delta})$$

and the result follows by Lemma 4.

The Theorem 2 is our main result about the nonrelativistic limit of the ground state jump process $t\mapsto\xi_t^c$. As an application, we will study the nonrelativistic limit of the semigroup $t\mapsto P_t^c$ where

$$(P_t^c\psi)(\mathbf{x}) = \mathbb{E}\left(\psi(\mathbf{x}+\xi_t^c) \exp -\frac{1}{\hbar}\int_0^t V(\mathbf{x}+\xi_s^c)\,ds \right)$$

4. NONRELATIVISTIC LIMIT OF RELATIVISTIC FEYNMAN-KAC FORMULAS

In this section we want to apply the results previously obtained to the nonrelativistic limit of the hamiltonian semigroup $t\mapsto \exp -\frac{t}{\hbar} H$ where

$$H = \sqrt{-c^2\hbar^2\Delta + M^2c^4\mathbb{1}} - Mc^2\mathbb{1} + V(.)$$

for a potential $V(.)$ which, for simplicity, we suppose continous and bounded below. Let $H_{nr} = -\frac{\hbar^2}{2M}\Delta + V(.)$ then we want to show, by using probabilistic techniques, that

$$\lim_{c\to+\infty} \exp -\frac{t}{\hbar} H = \exp -\frac{t}{\hbar} H_{nr}$$

where this convergence is studied in each $L^p(\mathbb{R}^D)$, $1\leqq p\leqq+\infty$.

By bearing in mind Ichinose-Tamura [12] and Feynman-Kac formulas, we define the semigroups $t \mapsto \mathrm{P}_t^c$ and $t \mapsto \mathrm{P}_t^\infty$ by:

$$(\mathrm{P}_t^c \psi)(\mathbf{x}) = \mathbb{E}\left(\psi(\mathbf{x}+\xi_t^c) \exp -\frac{1}{\hbar}\int_0^t \mathrm{V}(\mathbf{x}+\xi_s^c)\,ds\right) \tag{10}$$

$$(\mathrm{P}_t^\infty \psi)(\mathbf{x}) = \mathbb{E}\left(\psi\left(\mathbf{x}+\sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t\right) \exp -\frac{1}{\hbar}\int_0^t \mathrm{V}\left(\mathbf{x}+\sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_s\right)ds\right) \tag{11}$$

and we are interested in the convergence of the first semigroup to the second when $c \uparrow +\infty$. In order to do that, we first consider the free case $\mathrm{V}(.)=0$.

LEMMA 5. — *If* $\psi(.) : \mathbb{R}^\mathrm{D} \to \mathbb{C}$ *is continous and bounded then, for all* $\mathrm{T}>0$ *and all compact* $\mathbf{K} \subset \mathbb{R}^\mathrm{D}$

$$\lim_{c \to +\infty} \mathbb{E}\left(\left|\psi(\mathbf{x}+\xi_t^c) - \psi\left(\mathbf{x}+\sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t\right)\right|\right) = 0$$

uniformly w.r.t. $\mathbf{x} \in \mathbf{K}$ *and* $0 \leqq t \leqq \mathrm{T}$

Proof. — For $a>0$, $0<b\leqq 1$ and $\mathrm{T}>0$ let

$$\mathrm{B}_{a,b,\mathrm{T}} = \left\{\omega \in \Omega : \max_{0\leqq t\leqq \mathrm{T}} \|\mathbf{w}_t\| \leqq a \;\&\; \sup_{0\leqq t\leqq \mathrm{T}} \left\|\xi_t^c - \sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t\right\| \leqq b\right\}$$

Because

$$\begin{aligned}\mathbb{E}\left(\left|\psi(\mathbf{x}+\xi_t^c) - \psi\left(\mathbf{x}+\sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t\right)\right|\right) \\ \leqq 2\|\psi\|_\infty \mathbb{P}\left(\max_{0\leqq t\leqq \mathrm{T}} \|\mathbf{w}_t\| > a\right) \\ + 2\|\psi\|_\infty \mathbb{P}\left(\sup_{0\leqq t\leqq \mathrm{T}} \left\|\xi_t^c - \sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t\right\| > b\right) \\ + \mathbb{E}\left(\left|\psi(\mathbf{x}+\xi_t^c) - \psi\left(\mathbf{x}+\sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t\right)\right| \chi_{\mathrm{B}_{a,b,\mathrm{T}}}(\omega)\right)\end{aligned}$$

given any $\varepsilon>0$ we can first choose $a>0$ such that

$$2\|\psi\|_\infty \mathbb{P}\left(\max_{0\leqq t\leqq \mathrm{T}} \|\mathbf{w}_t\| > a\right) < \frac{\varepsilon}{3}$$

Under the conditions $\mathbf{x} \in \mathbf{K}$ and $\omega \in \mathrm{B}_{a,b,\mathrm{T}}$, both points $\mathbf{x}+\xi_t^c$ and $\mathbf{x}+\sqrt{\frac{\hbar}{\mathrm{M}}}\mathbf{w}_t$ belong to some fixed compact $\tilde{\mathbf{K}}(\mathbf{K}, a) \subset \mathbb{R}^\mathrm{D}$ for each $0\leqq t\leqq \mathrm{T}$. Therefore, by the uniform continuity of $\psi(.)$ inside $\tilde{\mathbf{K}}$ we can choose

$0<b\leqq 1$ such that

$$\mathbb{E}\left(\left|\psi(\mathbf{x}+\xi_t^c)-\psi\left(\mathbf{x}+\sqrt{\frac{\hbar}{M}}\mathbf{w}_t\right)\right|\chi_{B_{a,b,T}}(\omega)\right)\leqq\frac{\varepsilon}{3}$$

for all $\mathbf{x}\in\mathbf{K}$ and all $0\leqq t\leqq T$ and, finally, Theorem 2 provides the result.

It is easy to see, by the same technique, that

$$\lim_{c\to+\infty}\mathbb{E}\left(\left|\psi(\mathbf{x}+\xi_t^c)-\psi\left(\mathbf{x}+\sqrt{\frac{\hbar}{M}}\mathbf{w}_t\right)\right|\right)=0$$

uniformly w.r.t. $\mathbf{x}\in\mathbb{R}^D$ and $0\leqq t\leqq T$ if $\psi(.)$ vanishes to the infinity, therefore

$$\lim_{c\to+\infty}\|P_t^c\psi-P_t^\infty\psi\|_\infty=0$$

when $V(.)=0$ and $\psi(.)$ is vanishing to the infinity. Let $\psi_\xi(.)$ be the translate of $\psi(.)$ by $\xi\in\mathbb{R}^D$ namely $\psi_\xi(\mathbf{x})=\psi(\mathbf{x}-\xi)$.

LEMMA 6. – *Let* $\psi(.)\in L^p(\mathbb{R}^D)$ *with* $1\leqq p<+\infty$, *then, for all* $T>0$

$$\lim_{c\to+\infty}\mathbb{E}(\|\psi(.)-\psi_{\xi_t^c-\sqrt{(\hbar/M)}\mathbf{w}_t}(.)\|_p^p)=0$$

uniformly in $0\leqq t\leqq T$.

Proof. – Let

$$B_{b,T}=\left\{\omega\in\Omega:\ \sup_{0\leqq t\leqq T}\left\|\xi_t^c-\sqrt{\frac{\hbar}{M}}\mathbf{w}_t\right\|\leqq b\right\}.$$

As

$$\begin{aligned}\mathbb{E}(\|\psi(.)-\psi_{\xi_t^c-\sqrt{\hbar/M)}\mathbf{w}_t}(.)\|_p^p)&\\ \leqq 2^p\|\psi\|_p^p\,\mathbb{P}\left(\sup_{0\leqq t\leqq T}\left\|\xi_t^c-\sqrt{\frac{\hbar}{M}}\mathbf{w}_t\right\|>b\right)&\\ +\mathbb{E}(\|\psi(.)-\psi_{\xi_t^c-\sqrt{(\hbar/M)}\mathbf{w}_t}(.)\|_p^p\chi_{B_{b,T}}(\omega))&\end{aligned}$$

we can choose $b>0$ such that the last term be arbitrarily small for all $0\leqq t\leqq T$ and then we exploit again the Theorem 2.

In the free case $V(.)=0$, by Fubini's theorem, convexity of $x\mapsto x^p$ and translation invariance of Lebesque's measure

$$\|P_t^c\psi-P_t^\infty\psi\|_p^p\leqq\mathbb{E}(\|\psi(.)-\psi_{\xi_t^c-\sqrt{\hbar/M)}\mathbf{w}_t}(.)\|_p^p)$$

therefore we just proved that

$$\lim_{c\to+\infty}\|P_t^c\psi-P_t^\infty\psi\|_p=0$$

uniformly for $0\leqq t\leqq T$ when $V(.)=0$.

In order to extend such results to the case of a nonvanishing potential we prove first the following Lemma.

LEMMA 7. — *Let* $V(.):\mathbb{R}^D \mapsto \mathbb{R}$ *be continous and bounded below, then, for all compact* $\mathbf{K} \subset \mathbb{R}^D$ *and for all* $T>0$

(i)

$$\lim_{c \to +\infty} \mathbb{E}\left(\sup_{\mathbf{x} \in \mathbf{K}} \left| \exp - \frac{1}{\hbar}\int_0^t V(\mathbf{x}+\xi_s^c)\, ds - \exp - \frac{1}{\hbar}\int_0^t V\Big(\mathbf{x}+\sqrt{\frac{\hbar}{M}}\mathbf{w}_s\Big)\, ds \right| \right) = 0$$

moreover, for all $1 \leqq p < +\infty$

$$\text{(ii)}\quad \lim_{c \to +\infty} \mathbb{E}\left(\sup_{\mathbf{x} \in \mathbf{K}} \left| \exp - \frac{1}{\hbar}\int_0^t V(\mathbf{x}+\xi_s^c - \xi_t^c)\, ds - \exp - \frac{1}{\hbar}\int_0^t V\Big(\mathbf{x}+\sqrt{\frac{\hbar}{M}}\mathbf{w}_s - \xi_t^c\Big)\, ds \right|^p \right) = 0$$

uniformly in $0 \leqq t \leqq T$.

Proof. — We prove only (i) as the proof of (ii) is exactly similar. Without any loss of generality we can suppose $V(.) \geqq 0$.

Let $B_{a,b,T}$ as in Lemma 5 and

$$\alpha_t(\mathbf{x}) = \frac{1}{\hbar}\int_0^t V(\mathbf{x}+\xi_s^c)\, ds,\quad \beta_t(\mathbf{x}) = \frac{1}{\hbar}\int_0^t V\left(\mathbf{x}+\sqrt{\frac{\hbar}{M}}\mathbf{w}_s\right) ds$$

then

$$\mathbb{E}(\sup_{\mathbf{x} \in \mathbf{K}} |\exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x})|)$$

$$\leqq 2\,\mathbb{P}(\max_{0 \leqq t \leqq T} \|\mathbf{w}_t\| > a) + 2\,\mathbb{P}\left(\sup_{0 \leqq t \leqq T} \left\| \xi_t^c - \sqrt{\frac{\hbar}{M}}\mathbf{w}_t \right\| > b \right) + \mathbb{E}(\chi_{B_{a,b,T}}(\omega) \sup_{\mathbf{x} \in \mathbf{K}} |\exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x})|)$$

Given any $\varepsilon > 0$ we first choose $a > 0$ such that $2\,\mathbb{P}(\max_{0 \leqq t \leqq T} \|\mathbf{w}_t\| > a) < \varepsilon/3$.

Under the condition $\omega \in B_{a,b,T}$,

$$|\alpha_t(\mathbf{x}) - \beta_t(\mathbf{x})| \leqq \frac{1}{\hbar}\int_0^T \left| V(\mathbf{x}+\xi_s^c) - V\left(\mathbf{x}+\sqrt{\frac{\hbar}{M}}\mathbf{w}_s\right) \right| ds$$

can be made uniformly small in $\mathbf{x} \in \mathbf{K}$ and $0 \leqq t \leqq T$ when $b \downarrow 0$. Then, since

$$|\exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x})| \leqq |\alpha_t(\mathbf{x}) - \beta_t(\mathbf{x})|$$

we can choose $0 < b \leqq 1$ such that

$$\mathbb{E}(\chi_{B_{a,b,T}}(\omega) \sup_{\mathbf{x} \in \mathbf{K}} |\exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x})|) \leqq \frac{\varepsilon}{3}$$

for all $0 \leqq t \leqq T$.
In this way, for each $\varepsilon > 0$, we can choose $0 < b \leqq 1$ such that

$$\mathbb{E}(|\exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x})|) \leqq \frac{2\varepsilon}{3} + 2\,\mathbb{P}\left(\sup_{0 \leqq t \leqq T} \left\| \xi_t^c - \sqrt{\frac{\hbar}{M}}\mathbf{w}_t \right\| > b\right)$$

for all $0 \leqq t \leqq T$ and the result is again a consequence of Theorem 2
Now we can state the first theorem of this section.

THEOREM 3. — *Let* $V(.)$ *continous and bounded below, then, if* $\psi(.)$ *is a continous and bounded function, for all* $T > 0$ *and for all compact* $\mathbf{K} \subset \mathbb{R}^D$

$$\lim_{c \to +\infty} (P_t^c \psi)(\mathbf{x}) = (P_t^\infty \psi)(\mathbf{x})$$

uniformly w.r.t. $\mathbf{x} \in \mathbf{K}$ *and* $0 \leqq t \leqq T$.

Proof. — Without any loss of generality we suppose $V(.) \geqq 0$.
Let

$$\alpha_t(\mathbf{x}) = \frac{1}{\hbar}\int_0^t V(\mathbf{x} + \xi_s^c)\,ds$$

and

$$\beta_t(\mathbf{x}) = \frac{1}{\hbar}\int_0^t V\left(\mathbf{x} + \sqrt{\frac{\hbar}{M}}\mathbf{w}_s\right)ds$$

as in Lemma 7. Because

$$|(P_t^c \psi)(\mathbf{x}) - (P_t^\infty)(\mathbf{x})| \leqq \mathbb{E}\left(\left|\psi(\mathbf{x} + \xi_t^c) - \psi\left(\mathbf{x} + \sqrt{\frac{\hbar}{M}}\mathbf{w}_t\right)\right|\right) + \|\psi\|_\infty\, \mathbb{E}(|\exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x})|)$$

the result is a consequence of Lemma 5 and the first part of Lemma 7.
It is not difficult to see that

$$\lim_{c \to +\infty} \|P_t^c \psi - P_t^\infty \psi\|_\infty = 0$$

uniformly in $0 \leqq t \leqq T$ when $\psi(.)$ vanishes to the infinity. In order to manage $\lim_{c \to +\infty} P_t^c \psi$ in $L^p(\mathbb{R}^D)$ with $1 \leqq p < +\infty$ we need one more Lemma.

LEMMA 8. — *Let* $V(.)$ *continous and bounded below and* $\psi(.)$ *continous with compact support then, for all* $1 \leqq p < +\infty$ *and for all* $T > 0$

$$\lim_{c \to +\infty} \|P_t^c \psi - P_t^\infty \psi\|_p = 0$$

uniformly in $0 \leqq t \leqq T$.
Proof. — Without any loss of generality we suppose $V(.) \geqq 0$.

By exploiting the easy inequality

$$\| \mathrm{P}_t^c \psi - \mathrm{P}_t^\infty \psi \|_p^p \leqq 2^p \, \mathbb{E}(\| \psi(.) - \psi_{\xi_t^c - \sqrt{(\hbar/\mathrm{M})}\, \mathbf{w}_t}(.) \|_p^p)$$
$$+ 2^p \, \mathbb{E}\left(\int_{\mathbb{R}^\mathrm{D}} | \psi(\mathbf{x} + \xi_t^c) |^p \, | \exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x}) |^p \, d^\mathrm{D} \mathbf{x} \right),$$

we obtain the result from Lemma 6 and the second half of Lemma 7 because, by translation invariance of Lebesque's measure

$$\mathbb{E}\left(\int_{\mathbb{R}^\mathrm{D}} | \psi(\mathbf{x} + \xi_t^c) |^p \, | \exp - \alpha_t(\mathbf{x}) - \exp - \beta_t(\mathbf{x}) |^p \, d^\mathrm{D} \mathbf{x} \right)$$
$$\leqq \| \psi \|_p^p \, \mathbb{E}\left(\sup_{\mathbf{x} \in \mathbf{K}} \left| \exp - \frac{1}{\hbar} \int_0^t \mathrm{V}(\mathbf{x} + \xi_s^c - \xi_t^c) \, ds - \exp - \frac{1}{\hbar} \int_0^t \mathrm{V}(\mathbf{x} + \xi_s^c - \xi_t^c) \, ds \right|^p \right)$$

where $\mathbf{K}$ is the support of $\psi(.)$.

Finally we can state our last theorem.

THEOREM 4. – *Let* $\psi(.) \in \mathrm{L}^p(\mathbb{R}^\mathrm{D})$ *with* $1 \leqq p < +\infty$ *and* $\mathrm{V}(.)$ *continous and bounded below then, for all* $\mathrm{T} > 0$

$$\lim_{c \to +\infty} \| \mathrm{P}_t^c \psi - \mathrm{P}_t^\infty \psi \|_p = 0$$

uniformly in $0 \leqq t \leqq \mathrm{T}$.

Proof. – Given $\psi(.) \in \mathrm{L}^p(\mathbb{R}^\mathrm{D})$ we can find a sequence $\psi_n(.)$ of continous functions with compact support which converges to $\psi(.)$ in the L^p norm. Now

$$\| \mathrm{P}_t^c \psi - \mathrm{P}_t^\infty \psi \|_p \leqq \| \mathrm{P}_t^c \psi - \mathrm{P}_t^c \psi_n \|_p + \| \mathrm{P}_t^\infty \psi - \mathrm{P}_t^\infty \psi_n \|_p + \| \mathrm{P}_t^c \psi_n - \mathrm{P}_t^\infty \psi_n \|_p$$

Without any loss of generality we suppose $\mathrm{V}(.) \geqq 0$.

As $\| \mathrm{P}_t^c f \|_p \leqq \| f \|_p$ and $\| \mathrm{P}_t^\infty f \|_p \leqq \| f \|_p$, we have the inequality

$$\| \mathrm{P}_t^c \psi - \mathrm{P}_t^\infty \psi \|_p \leqq 2 \| \psi - \psi_n \|_p + \| \mathrm{P}_t^c \psi_n - \mathrm{P}_t^\infty \psi_n \|_p$$

and we can exploit the convergence of $\psi_n(.)$ to $\psi(.)$ and the Lemma 8.

We have therefore seen that the semigroup $t \mapsto \mathrm{P}_t^c$ has good properties of convergence to $t \mapsto \mathrm{P}_t^\infty$ for $c \uparrow +\infty$ at least when the potential $\mathrm{V}(.)$ is continous and bounded below. More general potential could be considered but this would be outside our pourposes because the main target of this paper is building a bridge between the jump markov process $t \mapsto \xi_t^c$ and the Brownian motion with just some examples of applications.

5. CONCLUSIONS AND OUTLOOK

We implemented, at least for ground state processes, our program of bridging the gap between the alternative stochastic descriptions of a spinless relativistic quantum particle at which we alluded in the introduction. As an additional bonus, we reached a deeper understanding of Feynman-Kac formula for relativistic hamiltonians and a better perception of the probabilistic mechanism behind the nonrelativistic limit of hamiltonian semigroups, we think that this subject deserves further developments. A next interesting problem is the study of semiclassical limit ($\hbar \downarrow 0$) of relativistic quantum mechanics in the probabilistic approach of G. Jona-Lasinio, F. Martinelli and E. Scoppola [15] through the theory of small random perturbation of dynamical systems. We feel that the relation we found between the jump Markov process $t \mapsto \xi_t^c$ and the Brownian motion will help such analysis too.

After the completion of this paper, we learned that the construction of the process $t \mapsto \xi_t^c$ through the Brownian motion is a particular case of a general construction first introduced by D. Bakry [16]. R. Carmona, W. C. Masters and B. Simon ([17], [18]) recognized the usefulness of Bakry's idea in their investigations on the asymptotic behavior of the eigenfunctions of relativistic Schrödinger operators via the relativistic Feynman-Kac formula (10) and we refer to them for the description of a large class of acceptable potentials.

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