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Variational principles for a relativistic stochastic mechanics

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ABSTRACT. — We propose an extension to the relativistic case of the stochastic variational principles both of Lagrangian and Eulerian type. The action we use is the mean classical action evaluated on the paths of relativistic covariant diffusions. The resulting equations of motion are the relativistic stochastic Lagrange equations.

RÉSUMÉ. — Nous proposons une extension au cas relativiste des principes variationnels de type lagrangien et eulérien. Nous utilisons comme action l'action moyenne classique évaluée sur les trajectoires des diffusions covariantes relativistes. Les équations du mouvement obtenues sont les équations de Lagrange stochastiques relativistes.

1. INTRODUCTION

The problem of formulation of the stochastic mechanics in a relativistic framework has not at the moment a unique and complete answer ([1]-[9]), however some satisfactory results have been obtained for the description of a relativistic spinless particle in a classical electromagnetic field. A kind of approach utilizes stochastic diffusion processes in the four dimensional space-time, which are labelled by an invariant parameter (proper time). This approach tries to implement in a stochastic framework the Feynman's picture [10] of relativistic quantum mechanics in which the virtual paths moving backward in time are interpreted as antiparticles moving forward.

The first attempt in this direction is due to Guerra-Ruggiero ([1], [2]), who consider random processes which are characterized by suitable conditions on conditional expectations and which do not have the usual Markov property. Dohrn-Guerra ([3], [4]) proposed a different way to overcome the difficulty of constructing non trivial relativistic Markov diffusions. They use Markov processes on $\mathbf{R}^4$ but allow the Brownian metric to be different from the kinetic one, provided that two metrics verify a compatibility condition. Then the Klein-Gordon equation turns out from an Eulerian variational principle. The covariance property is recovered considering a class of processes connected with generalized frames of reference.

Recently a new class of processes has been introduced [5], which take values on the Minkowski space and are labelled by an invariant time, as solutions of stochastic equations with boundary conditions both at initial and final time. These processes belong to a Bernstein [11] class of processes and do not have in general the usual Markov property. The notion of transport derivative naturally associated with this kind of processes is used to give the stochastic relativistic Lagrange equations, which are known to be equivalent to the Klein-Gordon equation.

A different and independent approach to the relativistic stochastic mechanics of a spinless particle has been used by De Angelis [8]. He is able to reproduce the kinematics of the system utilizing a 3d Markov jump process labelled by the physical time. In [9] it is shown that this jump process can be recovered from the relativistic process introduced in [5] by a suitable elimination of the invariant time.

In this paper we formulate variational principles of Lagrangian and Eulerian type using the processes introduced in [5]. We construct the stochastic action starting from the classical one in the covariant four-dimensional formulation in terms of the proper time. Then the mean classical action evaluated on the stochastic paths, in analogy with the classical stochastic case [12], gives the stochastic action in terms of a Lagrangian, which is manifestly covariant.

Moreover the Lagrangian principle, when suitable boundary conditions on the variation are chosen, gives rise to rotational solutions which can be seen as a generalization to the relativistic case of the Morato solutions [13].

2. THE RELATIVISTIC PROCESS

Work on a flat Minkowski space-time M^4 endowed with the usual Lorentz metric $(1, -1, -1, -1)$. We consider the stochastic process on M^4, introduced in [5], defined in the interval $[0, T]$ by means of the following stochastic integral equations

$$q^0(s) = q'^0 + \int_0^s ds'\, b^0(q(s'), s') + w^0(s)$$
$$\mathbf{q}(s) = \mathbf{q}' - \int_s^T ds'\, \mathbf{b}(q(s'), s') + \mathbf{w}(T-s) \qquad (2.1)$$

where $\mathbf{w} = (w^1, w^2, w^3)$. w^0 and w^j for $j = 1, 2, 3$ are four independent brownian motions starting from 0 at time 0. $b = (b^0, \mathbf{b})$ is a quadrivector drift. $q' = (q'^0, \mathbf{q}')$ is a random variable given independently of $w(s) = (w^j)_{j=0,1,2,3}$, $s \in [0, T]$.

(2.1) can be considered as a particular type of stochastic integral equation with mixed boundary conditions. One can give sense to (2.1) constructing a solution using a method of successive approximations [14]. Then under the usual regularity conditions on b, it is possible to show that there exists a unique solution for the fixed point problem (2.1).

This process belongs to the class of Bernstein processes [11] in the sense that

$$E\{f(q(t))/\mathscr{P}_s, \mathscr{F}_{s'}\} E\{f(q(t))/N_s, N_{s'}\} \qquad (2.2)$$

for $s \leqq t \leqq s'$, where $\mathscr{P}_s (\mathscr{F}_{s'})$ is the σ-algebra generated by $q(u)$, for $0 \leqq u \leqq s$ $(s' \leqq u \leqq T)$ and N_s, $N_{s'}$ are the present σ-algebras generated by $q(s)$, $q(s')$ respectively.

Furthermore the process q has a peculiar property. Let us introduce the following σ-algebras. Let X_s, Y_s denote the σ-algebras generated by $q^0(s)$ and $\mathbf{q}(s)$ respectively and by $X_{<s}$, $Y_{>s}$ the ones generated by $q^0(s')$, $0 \leqq s' \leqq s$ and by $\mathbf{q}(s')$, $s \leqq s' \leqq T$. Since the process $q(t)$ for times $s \leqq t \leqq s'$ can be reconstructed from (2.1) by the only knowledge of $q^0(s)$ and $\mathbf{q}(s')$ then it is easily seen that the solution of (2.1) verify

$$E\{f(q(t))/X_{>s}, Y_{<s'}\} = E\{f(q(t))/X_s, Y_{s'}\} \qquad (2.3)$$

for $s \leqq t \leqq s'$.

Therefore we can introduce a probability transition density $p(sx \mid qt \mid s'\mathbf{y})$ depending on 3 times such that for any smooth function f

$$E\{f(q(t))/q^0(s) = x, \mathbf{q}(s') = \mathbf{y}\} = \int dq\, f(q)\, p(sx \mid qt \mid s'\mathbf{y}) \qquad (2.4)$$

This probability transition density satisfies a differential equation in q, t. The following derivation of this equation relies mainly on the property

(2.3). Consider

$$\frac{d}{dt}\mathrm{E}\{f(q(t))/q^0(s)=x,\ \mathbf{q}(s')=\mathbf{y}\}$$
$$= \lim_{h\to 0} h^{-1}\,\mathrm{E}\{f(q(t+h))-f(q(t))/q^0(s)=x,\ \mathbf{q}(s')=\mathbf{y}\}$$
$$= \lim_{h\to 0} h^{-1}\,\mathrm{E}\{\mathrm{E}\{f(q(t+h)) -f(q(t))/\mathrm{X}_t,\ \mathrm{Y}_{t+h}\}/q^0(s)=x,\ \mathbf{q}(s')=\mathbf{y}\} \quad (2.5)$$

The second equality follows from (2.3). The limit

$$\lim_{h\to 0} h^{-1}\,\mathrm{E}\{f(q(t+h))-f(q(t))/\mathrm{X}_t,\ \mathrm{Y}_{t+h}\}=(\mathscr{D}f)(q(t),t) \quad (2.6)$$

which has been considered in [5] is the natural definition of the mean derivative associate with the processes we are considering. The explicit expression of $\mathscr{D}$ is

$$(\mathscr{D}f)(q(t))=(b^\alpha\partial_\alpha f)(q(t))+\frac{1}{2}(\partial_\alpha\partial^\alpha f)(q(t)) \quad (2.7)$$

where $\partial_\alpha\partial^\alpha$ is the d'alambertian operator $\partial_0\partial_0-\partial_i\partial_i$.
Moreover we have

$$\frac{d}{dt}\mathrm{E}\{f(q(t))/q^0(s)=x,\ \mathbf{q}(s')=\mathbf{y}\}=\int dq\,\partial_t p(sx\,|\,qt\,|\,s'\,\mathbf{y})\,f(q)$$
$$=\int dq\,f(q)\left[-\partial_\alpha(b^\alpha p(sx\,|\,qt\,|\,s'\,\mathbf{y})) +\frac{1}{2}\partial_\alpha\partial^\alpha p(sx\,|\,qt\,|\,s'\,\mathbf{y})\right] \quad (2.8)$$

where the second equality turns out from (2.5), (2.7).

Finally, since f is arbitrary, we obtain the equation for p

$$\partial_t p(sx\,|\,qt\,|\,s'\,\mathbf{y})+\partial_\alpha(b^\alpha p(sx\,|\,qt\,|\,s'\,\mathbf{y}))-\frac{1}{2}\partial_\alpha\partial^\alpha p(sx\,|\,qt\,|\,s'\,\mathbf{y})=0 \quad (2.9)$$

The boundary conditions on p are

$$\lim_{t\to s}\int dq^0\,p(sx\,|\,qt\,|\,s'\,\mathbf{y})=\delta(\mathbf{y}-\mathbf{q})$$
$$\lim_{t\to s'}\int d\mathbf{q}\,p(sx\,|\,qt\,|\,s'\,\mathbf{y})=\delta(x-q^0) \quad (2.10)$$

Also the density $\rho(q,t)$ satisfies an equation of the same kind as one can see from the relation

$$\rho(q,t)=\int dx\,d\mathbf{y}\,p(sx\,|\,qt\,|\,s'\,\mathbf{y})\,\rho(sx,s'\,\mathbf{y}) \quad (2.11)$$

$\rho(sx, s'\mathbf{y})$ in (2.11) is the two times initial probability density. In other words $\rho(sx, s'\mathbf{y})$ is the joint probability that the $3d$ backward component of the process is in $\mathbf{y}$ at time s' and the $1d$ forward component is in x at time s. Equations (2.9) and the analogous one for the density exhibit the relativistic covariance property. This is the reason why we call the process defined by (2.1) a relativistic process. From the point of view of the process, the stochastic equations (2.1) assume a different form under a change of reference frame. Henceforth the class of all processes which are obtained from (2.1) by a Lorentz boost has to be considered. It turns out that the probability transition densities of all the processes of this covariant class will satisfy the equation (2.9), provided that the boundary condition (2.10) is changed according to the Lorentz transformation (*see* [5]). We can conclude that the relativistic process is characterized by the covariant equation (2.9). Therefore we can work in the reference frame in which the stochastic equations assume the form (2.1) withouth any lack of generality. As a remark we note that the process we have introduced is markovian when the two point initial density $\rho(sx, s'\mathbf{y})$ is a product $\rho(sx)\rho(s'\mathbf{y})$ and the space component of the drift depends only of $\mathbf{q}(s)$ as well as the time component only of $q^0(s)$. In fact in this case one can separate the evolution of the 3-dimensional part of the process from the 1-dimensional one which are independently markovian. It can be shown that in the general case the process is not markovian. As a simple example one can consider the case of zero drift but an initial condition $\rho(sx, s'\mathbf{y})$ which does not factorize. What happens is that the correlation between $q^0(0)$ and $\mathbf{q}(\mathrm{T})$, given by $\rho(sx, s'\mathbf{y})$, does not allow the future to be independent of the past if the present is known. The main advantage in our opinion of the introduction of the relativistic process is the manifestly relativistic covariant form of the equation for the density and for the probability transition. In [5] it is shown that the operator $\mathscr{D}$ (and the time reversal counterpart $\mathscr{D}^*$) naturally associated to this kind of process allows to construct a relativistic stochastic Newton equation, which is equivalent to the Klein-Gordon equation. In the following we will show that the theory can be founded on a variational principe whose main ingredient is this kind of relativistic process.

3. PATHWISE VARIATIONAL PRINCIPLE

The relativistic action for a classical particle in a given external electromagnetic field, with vector potential A can be written explicitly as a functional of the paths $q(s)$ in the Minkowski space-time labelled by a proper time parameter s

$$\mathbb{A} = -\frac{1}{2}m\int_{s0}^{s_1} ds\frac{dq^\alpha}{ds}\frac{dq_\alpha}{ds} - \frac{e}{c}\int_{s0}^{s_1} ds\,\mathrm{A}_\alpha\frac{dq^\alpha}{ds}. \tag{3.1}$$

Since we promote the configuration variables q to stochastic processes evolving in the time s under the equations (2.1) $\left(\text{we introduce a diffusion coefficient } \sigma = \sqrt{\frac{\hbar}{m}}\right)$ we need to give meaning to (3.1). To this end we consider, following Nelson [12], the mean value of the classical action on the stochastic trajectories of the relativistic process.

The Lagrangian is obtained considering a conditional expectation suitable for the relativistic process, that is the one suggested by the property (2.3). Consider the limit

$$\lim_{h\to 0} h^{-2}\,\mathrm{E}\left\{\frac{1}{2}m\,dq^\alpha dq_\alpha - \frac{e}{c}h\,\mathrm{A}_\alpha dq^\alpha / q^0(s) = x,\ \mathbf{q}(s+h) = \mathbf{y}\right\} \tag{3.2}$$

The limit (3.2) can be computed taking into account the expressions for the increment dq up to the order $h^{3/2}$ and using the property of independence of the brownian increments for different components (*see* for example [5]).

The result is

$$\frac{1}{2}m[b^\alpha b_\alpha + \sigma^2\partial_\alpha b^\alpha] - \frac{e}{c}\left[\mathrm{A}_\alpha b^\alpha + \frac{1}{2}\sigma^2\partial_\alpha \mathrm{A}\right] + \lim_{h\to 0} h^{-1}2\sigma^2. \tag{3.3}$$

The divergent term is independent of the specific path and can be dropped in the variation. Therefore we introduce the Lagrangian

$$\mathscr{L}(q, s) = \frac{1}{2}m[b^\alpha b_\alpha + \sigma^2\partial_\alpha b^\alpha] - \frac{e}{c}\left[\mathrm{A}_\alpha b^\alpha + \frac{1}{2}\sigma^2\partial_\alpha \mathrm{A}^\alpha\right] \tag{3.4}$$

We can also define the time reversal counterpart of $\mathscr{L}$ as

$$\mathscr{L}^*(q, s) = \frac{1}{2}m[b^{*\alpha} b^*_\alpha + \sigma^2\partial_\alpha b^{*\alpha}] - \frac{e}{c}\left[\mathrm{A}_\alpha b^{*\alpha} + \frac{1}{2}\sigma^2\partial_\alpha \mathrm{A}^\alpha\right] \tag{3.5}$$

where $b^* = (b^0 + \sigma^2\partial^0 \ln\rho, \mathbf{b} + \sigma^2\nabla\ln\rho)$. The Lagrangian reduces to the classical one in the limit $\sigma \to 0$.

The action associated to $\mathscr{L}$ is

$$\mathscr{A} = \int_{s0}^{s_1} ds\,\mathrm{E}\{\mathscr{L}(q, s)\} \tag{3.6}$$

Since the proper time s is not observable we will only be interested to Lagrangians independent of s, but for sake of completeness we formulate the variational principle in the general case.

We introduce the following class of variations inspired by [13] (*see* also [15]). Given a drift b we consider a varied drift b'. Let us call $q(s)$ and $q'(s)$ two paths under b and b' respectively. We consider only variations $q'-q$ such that q' and q have the same brownian realization $w^0(s')$, $0 \leqq s' \leqq s$ and $\mathbf{w}(s')$, $T-s \leqq s' \leqq T$. In other words we require that

$$d[q'-q](s) = [b'(q'(s), s) - b(q(s), s)]\, ds \tag{3.7}$$

Introducing a small variational parameter ε and putting $\varepsilon\delta b^\alpha = b'^\alpha - b^\alpha$ we have for the infinitesimal variations δq the equations

$$d\delta q(s) = \partial_\alpha b\, \delta q^\alpha\, ds + \delta b \tag{3.8}$$

The processes δq have differentiable paths but $\delta q(s)$ depend on the entire realization $w^0(s')$, $0 \leqq s' \leqq s$ and $\mathbf{w}(s')$, $T-s \leqq s' \leqq T$.

The variation of the action (3.6) under the variations δq is

$$\begin{aligned}\delta\mathscr{A} &= \int_{s_0}^{s_1} ds\, \mathrm{E}\{\delta\mathscr{L}(q, s; \mathrm{A})\} \\ &= \int_{s_0}^{s_1} ds\, \mathrm{E}\Big\{\Big[-mv^\alpha - \frac{e}{c}\mathrm{A}_\alpha\Big]\frac{d}{ds}\delta q^\alpha - mu^\alpha\,\partial_\beta b_\alpha - \frac{e}{c}\partial_\beta \mathrm{A}_\alpha\, \delta q^\beta b^\alpha \\ &\qquad + \frac{1}{2}\sigma^2\,\partial_\beta\Big[-m\,\partial_\alpha b^\alpha - \frac{e}{c}\partial_\alpha \mathrm{A}^\alpha\Big]\delta q^\beta\Big\}\end{aligned} \tag{3.9}$$

where we have set $v+u=b$, $v-u=b^*$ and we have taken into account that

$$\int_{s_0}^{s_1} ds\, \mathrm{E}\Big\{\Big[u^\alpha\,\delta b_\alpha + \frac{1}{2}\sigma^2\,\partial_\alpha\,\delta b^\alpha\Big]\Big\} = 0. \tag{3.10}$$

Let us now examine the integral in (3.9) which contains the temporal derivative of δq. Denoting $p_\alpha = mv_\alpha + \frac{e}{c}\mathrm{A}_\alpha$ we have

$$\begin{aligned}&\int_{s_0}^{s_1} ds\, \mathrm{E}\Big\{\Big[-mv_\alpha - \frac{e}{c}\mathrm{A}_\alpha\Big]\frac{d}{ds}\delta q^\alpha\Big\} \\ &\qquad = -\lim_{h\to 0} h^{-1}\int_{s_0}^{s_1} ds\, \mathrm{E}\{p_\alpha[\delta q^\alpha(s+h) - \delta q^\alpha(s)]\} \\ &\qquad = \lim_{h\to 0} h^{-1}\int_{s_0}^{s_1} ds\, \mathrm{E}\{[p_\alpha(s+h) - p_\alpha(s)]\,\delta q^\alpha(s)\} + b.c.\end{aligned} \tag{3.11}$$

where $b.c. = -\mathrm{E}\{(p_\alpha\,\delta q^\alpha)(s_1)(p_\alpha\,\delta q^\alpha)(s_0)\}$. Finally, taking into account the properties of the variations δq, we get

$$(3.11)\quad = \int_{s_0}^{s_1} ds\,\mathrm{E}\{-\mathscr{D}\mathbf{p}(s)\,\delta\mathbf{q}(s) + \mathscr{D}p_0(s)\,\delta q^0(s)\}$$
$$+ \lim_{h\to 0} h^{-1}\{\mathrm{E}\{-[\mathbf{w}(\mathrm{T}-s)-\mathbf{w}(\mathrm{T}-s-h)]\,\delta\mathbf{q}(s)\}$$
$$+\mathrm{E}\{[w_0(s+h)-w_0(s)]\,\delta q^0(s)\}\} + b.c.$$
$$= \int_{s_0}^{s_1} ds\,\mathrm{E}\{\mathscr{D}p_\alpha\,\delta q^\alpha(s)\} + b.c. \quad (3.12)$$

The second line gives zero because $\delta q(s)$ depends on $w^0(s')$, $0 \leqq s' \leqq s$ and $\mathbf{w}(s')$, $\mathrm{T}-s \leqq s' \leqq \mathrm{T}$ while the increments $\mathbf{w}(\mathrm{T}-s)-\mathbf{w}(\mathrm{T}-s-h)$ and $w_0(s+h)-w_0(s)$ are independent of it.

For the action we have

$$\delta\mathscr{A} = \int_{s_0}^{s_1} ds\,\mathrm{E}\left\{\delta q^\beta\left[\mathscr{D}p_\beta - mu^\alpha\,\partial_\beta b_\alpha - \frac{e}{c}\,\partial_\beta\,\mathrm{A}_\alpha b^\alpha\right.\right.$$
$$\left.\left. + \frac{1}{2}\,\sigma^2\,\partial_\beta\left[-m\,\partial_\alpha b^\alpha - \frac{e}{c}\,\partial_\alpha \mathrm{A}^\alpha\right]\right]\right\} + b.c. \quad (3.13)$$

We choose the following initial condition on the variations [16]

$$\mathrm{E}\{\delta q^\alpha(s_0)/q(s_0) = q\} = 0. \quad (3.14)$$

The boundary term at the final time is controlled adding to the action a Lagrangian multiplier $\mathrm{E}\{\mathrm{S}_1(q(s_1))\}$. Under the variation this term becomes

$$\mathrm{E}\{\partial_\beta\,\mathrm{S}_1(q(s_1))\,\delta q^\beta(s_1)\} \quad (3.15)$$

which gives together with the boundary term $\mathrm{E}\{(p_\beta\,\delta q^\alpha)(s_1)\}$ the following term in $\delta\mathscr{A}$

$$\mathrm{E}\{\partial_\beta\,\mathrm{S}(q(s_1)) + p_\beta(q(s_1), s_1)\} \quad (3.16)$$

It is a matter of algebra now to get the expression of $\delta\mathscr{A}$. The conclusion is that the variation of the action with the boundary conditions is zero if and only if the fields v, u satisfy the equation

$$m\,\partial_t v_\beta + mv^\alpha\,\partial_\alpha v_\beta - mu^\alpha\,\partial_\alpha u_\beta - \frac{1}{2}\,m\,\sigma^2\,\partial_\alpha\partial^\alpha u_\beta + \frac{e}{c}\,\partial_t \mathrm{A}_\beta$$
$$+ \frac{e}{c}\,v^\alpha\,[\partial_\alpha \mathrm{A}_\beta - \partial_\beta \mathrm{A}_\alpha] + u^\alpha[\partial_\alpha p_\beta - \partial_\beta p_\alpha] + \frac{1}{2}\,\sigma^2\,\partial^\alpha[\partial_\alpha p_\beta - \partial_\beta p_\alpha] = 0 \quad (3.17)$$

and the initial condition

$$\partial_\beta\,\mathrm{S}_1(q(s_1)) = -p_\beta(q(s_1), s_1) \quad (3.18)$$

It is possible to prove [15] that the gradient condition (3.18) at time zero implies that p is a gradient at any time. So there exists some function S (q, s) such that

$$\partial_\beta S(q, s) = -p_\beta(q, s) = -mv_\beta - \frac{e}{c} A_\beta \qquad (3.19)$$

As a consequence (3.17) becomes

$$m\,\partial_t v_\beta + mv^\alpha \partial_\alpha v_\beta - mu^\alpha \partial_\alpha u_\beta - \frac{1}{2} m\sigma^2 \partial_\alpha \partial^\alpha u_\beta = -\frac{e}{c}\partial_t A_\beta - \frac{e}{c} v^\alpha [\partial_\alpha A_\beta - \partial_\beta A_\alpha] \qquad (3.20)$$

Taking into account (3.19), (3.20) implies the following equation for S

$$\partial_s S = \frac{1}{2m}\left\{\left[\partial_\alpha S + \frac{e}{c} A_\alpha\right]\left[\partial^\alpha S + \frac{e}{c} A^\alpha\right] - \hbar^2 \left[\frac{1}{\rho^{1/2}} \partial_\alpha \partial^\alpha \rho^{1/2}\right]\right\} \qquad (3.21)$$

We can write the right hand of (3.21) as $-H$, where H has the meaning of the Hamiltonian of the system and represents the energy in the stationary case. It is easy to verify that (3.21) and the equation for the density imply that the wave function

$$\Psi(q, s) = \rho\,(q, s)^{1/2} \exp \frac{i}{\hbar} \{S(q, s)\} \qquad (3.22)$$

satisfies the equation

$$i\hbar\,\partial_s \Psi = \left\{\left[i\partial_\alpha - \frac{e}{c} A_\alpha\right]\left[i\partial^\alpha - \frac{e}{c} A^\alpha\right]\right\}\Psi \qquad (3.23)$$

In the physical case the invariant time s is not observable and only the stationary solutions of (3.21) are relevant. The solution corresponding to the ground state energy has the form

$$S(q, s) = S_1(q) - \frac{1}{2} mc^2 (s - s_0) \qquad (3.24)$$

where the spatial dependence of S is determined by (3.18). The corresponding wave function assumes the Feynmann's hansatz form

$$\Psi(q, s) = \rho^{1/2} \exp \frac{i}{\hbar}\left\{S_1(q) - \frac{1}{2} mc^2 (s - s_0)\right\} \equiv \varphi(q) \exp \frac{i}{\hbar}\left\{-\frac{1}{2} mc^2 (s - s_0)\right\} \qquad (3.25)$$

where φ results to be a solution of the Klein-Gordon equation.

Remark. — It is possible to fix different boundary conditions in the variational principle [16]. Let us consider the following ones

$$E\{\delta q^\alpha(s_0)/q(s_0)=q\}=0, \qquad E\{\delta q^\alpha(s_1)/q(s_1)=q\}=0. \qquad (3.26)$$

Under this different choice of the boundary conditions we can get from the variational principle equation (3.17) which admits also non gradient solution for the momentum p. This equation appears to be a relativistic generalization of the Morato equation [13].

4. EULERIAN VARIATIONAL PRINCIPLE

In this section we explore a hydrodynamical approach to the variational principle. In this case the action $\mathscr{A}$ is regarded as a functional of the drift field.

Introduce the stochastic Hamilton-Jacobi function $S(q, s)$ as a solution of the equation

$$\mathscr{D}S(q, s)=\mathscr{L}(q, s) \qquad (4.1)$$

with the final condition

$$S_1(q)=S(q, s_1) \qquad (4.2)$$

It is easily seen that this equation implies

$$E\{S(q(s_1), s_1)-S(q(s_0), s_0)\}=\int_{s_0}^{s_1} ds\, E\{\mathscr{L}(q, s)\}. \qquad (4.3)$$

Therefore we can apply the machinery of the Guerra-Morato [17] variational principle based on the variation of the drift field b. The boundary conditions fix the density at times s_0, s_1.

The result is that the Hamilton-Jacobi condition

$$\partial_\beta S(q(s), s)=-mv_\beta-\frac{e}{c}A_\beta \qquad (4.4)$$

is necessary and sufficient for the stationarity of the action. The Klein-Gordon equation is recovered from the programming equation (4.4) and from the density equation (2.1). D. Dohrn and F. Guerra (DG) formulated a variational principle [3] of the hydrodynamical type for the Klein-Gordon equation in the more general framework of stochastic processes on manifolds (*see* also [4] for an exhaustive discussion). The processes they consider are Markov processes on a manifold M with a Brownian metric η. The kinetic metric g can be different from η provided a compatibility condition is verified. The relativistic Lorentzian case is implemented choosing

$$\eta^{\alpha\beta} = 2\, u^{\alpha} u^{\beta} - g^{\alpha\beta} \tag{4.5}$$

where u is some time oriented field. If g is the Lorentzian metric and $u = (1, 0, 0, 0)$ then η is the usual Brownian metric in R^4.

The relation between the relativistic process we use and the usual Markov process on R^4, that DG use, can be better understood if we define

$$\tilde{b} = (b^0, \mathbf{b} + \sigma \nabla \ln \rho) \tag{4.6}$$

where b is the drift of the relativistic process. Then our covariant equation for the density assumes the more familiar form

$$\partial_t \rho(q, t) + \partial_i (\tilde{b}^i \rho(q, t)) - \frac{1}{2} \partial_i \partial_i \rho(q, t) = 0 \tag{4.7}$$

which can now be interpreted as a Fokker-Plank equation for a stochastic Markov process in R^4 of DG type. A solution $\rho(q, t)$ of (4.7) which corresponds to the initial condition $\rho(s_0 x, s_1 \mathbf{y})$ for the relativistic process is also a solution of the Markov process in R^4 with initial condition $\rho(q, s_0)$, provided that two initial conditions $\rho(s_0 x, s_1 \mathbf{y})$ and $\rho(q, s_0)$ are compatible. Therefore the one time probability density coincides for both processes but probability involving more times are in general different. Moreover it comes out from (4.6), as expected, that the velocity field $v = (1/2)(b + b^*)$ is the same in both cases. For what concerns the action, our Lagrangian and the one of DG are different but their mean values coincide. Finally we note that the covariance properties of the theory in the DG scheme are obtained by introducing a class of Brownian metrics labelled by the field u and considering the corresponding Markov processes. In our approach on the contrary the covariance emerges from the manifestly covariant structure of continuity equation, mean derivatives and all the other quantities which characterize the process.

As last remark we point out that if we write down the velocity fields in terms of the wave function the mean stochastic Lagrangian differs from the quantum relativistic one by a constant

$$E\{\mathscr{L} - m^2 c^2\} = \int dq\, [\partial_\alpha \Phi\, \partial^\alpha \Phi^* - m^2 c^2 \Phi\Phi^*] = \int dq\, L(\Phi, \Phi^*) \tag{4.8}$$

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