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by

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ABSTRACT. - Relations between vector-valued measures and Hilbert space representations of quantum logics are studied. It is shown that a sum logic admits a faithful Hilbert space representation if and only if Segal product defined on bounded observables of the logic is distributive.

INTRODUCTION

Vector-valued measures on quantum logics have been studied by several authors, e.g. [7], [14], [16], [19], [12], [13]. In [16], there has been proved

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that if there exists a vector-valued measure $\xi$ on a quantum logic $L$ with values in a Hilbert space $H$, then there is a logic morphism $\Phi$ from $L$ into the logic $L(H)$ of all orthogonal projections on $H$ and a vector $v_\xi \in H$ such that $\xi(a) = \Phi(a) v_\xi$ for all $a \in L$. In [7], there has been proved that a sum logic with distributive Segal product admits a rich family of vector-valued measures. These two facts are used in the present paper to prove that a sum logic admits a faithful lattice $\sigma$-morphism into a Hilbert space logic $L(H)$ if and only if Segal product defined on bounded observables of the logic is distributive. In analogy with representations of C*-algebras we call a morphisms from a quantum logic $L$ into a Hilbert space logic $L(H)$ a representation of $L$ in $H$. We show that any representation of a sum logic can be extended to a representation of observables by self-adjoint operators which preserves sums and Segal products of bounded observables. We also show that the existence of joint distribution of type 1 for a finite set of bounded observables on a sum logic with distributive Segal product implies the existence of joint distribution of type 2 for these observables in a given state $m$ on $L$, and the latter joint distribution is identical with the joint distribution of type 1.

1. BASIC FACTS ABOUT LOGICS

A (quantum) logic $L$ is a partially ordered set with 0 and 1 and with orthocomplementation $': L \to L$ such that

(i) $(a')' = a$,
(ii) $a \leq b \Rightarrow b' \leq a'$,
(iii) $a \lor a' = 1$, $a \land a' = 0$,
(iv) $a \leq b' \Rightarrow a \lor b$ exists in $L$,
(v) $a \leq b \Rightarrow b = a \lor (a' \land b)$ (orthomodularity).

Elements $a, b \in L$ are orthogonal (written $a \perp b$) if $a \leq b'$. A logic $L$ is a $\sigma$-logic if $\lor a_i$ exists in $L$ for every sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of $L$.

A measure on $L$ is a map $m: L \to [0, \infty)$ such that $m(0) = 0$ and $m(a \lor b) = m(a) + m(b)$ for any $a, b \in L$, $a \perp b$. A measure $m$ on $L$ is $\sigma$-additive (or a $\sigma$-measure) if $m(\lor a_i) = \sum_{i \in N} m(a_i)$ for any sequence $(a_i)_{i \in N}$ of pairwise orthogonal elements of $L$ such that $\lor a_i$ exists in $L$.

A measure $m$ on $L$ is faithful if $m(a) = 0$ implies $a = 0$, and a measure $m$ is a state if $m(1) = 1$.

Let $L_1, L_2$ be logics. A map $\Phi: L_1 \to L_2$ is a morphism if

(i) $\Phi(1) = 1$, 

(ii) \(a, b \in L_1, a \perp b \Rightarrow \Phi (a) \perp \Phi (b)\) and \(\Phi (a \lor b) = \Phi (a) \lor \Phi (b)\). A morphism \(\Phi : L_1 \to L_2\) is a \(\sigma\)-morphism if \(\Phi (\bigvee a_i) = \bigvee \Phi (a_i)\) for any sequence \((a_i)_{i \in \mathbb{N}}\) of mutually orthogonal elements of \(L_1\) such that \(\bigvee a_i\) exists. A morphism \(\Phi : L_1 \to L_2\) is a lattice morphism if \(\Phi (a \lor b) = \Phi (a) \lor \Phi (b)\) for every \(a, b \in L_1\) such that \(a \lor b (a \land b)\) exists in \(L_1\).

A subset \(A\) of a logic \(L\) is a compatible subset if there is a Boolean subalgebra \(B\) of \(L\) such that \(A \subseteq B\). A two-element set \(\{a, b\}\) is compatible (or \(a\) and \(b\) are compatible, written \(a \leftrightarrow b\)) iff there are \(a_1, b_1, c \in L\) pairwise orthogonal and such that \(a = a_1 \lor c, b = b_1 \lor c\). (We note that \(a_1 = a \land b', b_1 = a' \land b, c = a \land b\). If \(L\) is a lattice then a subset \(A\) of \(L\) is compatible iff \(a \leftrightarrow b\) for every \(a, b \in A\).

Let \(L\) be a \(\sigma\)-logic. An observable on \(L\) is a \(\sigma\)-morphism \(x : B (\mathbb{R}) \to L\), where \(B (\mathbb{R})\) is the \(\sigma\)-algebra of Borel subsets of the real line \(\mathbb{R}\). If \(x\) is an observable and \(f : \mathbb{R} \to \mathbb{R}\) is a Borel function, then \(x \cdot f^{-1}\) is also an observable. If \(x\) is an observable and \(m\) is a \(\sigma\)-additive state on \(L\), then \(m_x : B (\mathbb{R}) \to \{0, 1\}, m_x (E) = m (x (E))\) is a probability measure on \(B (\mathbb{R})\), which is called the probability distribution of the observable \(x\) in the state \(m\). The expectation of \(x\) in \(m\) is then given by

\[
m (x) = \int_{\mathbb{R}} m_x (dt),
\]

if the integral exists. An observable \(x\) is bounded if there is a compact subset \(E \subseteq \mathbb{R}\) such that \(x (E) = 1\). If \(x\) is bounded then \(m (x)\) is finite for every state \(m\) on \(L\).

A set \(\{x_i | i \in I\}\) of observables on \(L\) is compatible if \(\bigcup x_i (B (\mathbb{R}))\) is a compatible set. We note that the range \(x (B (\mathbb{R}))\) of an observable \(x\) is a Boolean sub-\(\sigma\)-algebra of \(L\). If \(x_1, x_2, \ldots, x_n\) are compatible observables on \(L\), then there is an observable \(u\) and Borel functions \(f_1, \ldots, f_n\) such that \(x_i = u \cdot f_i^{-1}, i \leq n\).

Let \(L\) be a \(\sigma\)-logic and let \(S\) be a set of \(\sigma\)-additive states on \(L\). We shall say that \(S\) is strongly ordering if for any \(a, b \in L, a \nless b\) there is \(m \in S\) such that \(m (a) = 1, m (b) \neq 1\).

Let \(L\) be a \(\sigma\)-logic and let the set \(\mathcal{S} (L)\) of all \(\sigma\)-additive states on \(L\) be strongly ordering. We shall say that \(L\) is a sum logic if for any two bounded observables \(x, y\) on \(L\) there is a unique bounded observable \(z\) such that \(m (x) + m (y) = m (z)\) for every \(m \in \mathcal{S} (L)\). The observable \(z\) is called the sum of \(x\) and \(y\) and we write \(z = x + y\). If \(x\) and \(y\) are bounded observables on a sum logic, then

\[
x \circ y = \frac{1}{4} [(x + y)^2 - (x - y)^2]
\]
defines Segal product of \( x \) and \( y \). Segal product is distributive if for any bounded observables \( x, y, z \) we have \((x+y)\ast z = x\ast y + x\ast z\).

We note that a sum logic is always a lattice and \([q_a + q_b, 2] = a \land b\) for any \( a, b \in L \), where \( q_d \) denotes the (unique) observable such that \( q_d, 1 = d, q_d, 0 = d' \).

For more details about quantum logics see ([2], [11], [21]). Sum logics have been introduced and studied in [11], where they are called logics with Uniqueness and Existence properties. Due to Christensen-Yeadon-Paszkiewicz-Matveichuk theorem ([5], [22], [20], [18]), there is a rich class of \( W^* \)-algebras whose projection logics are sum logics with distributive Segal product.

2. VECTOR-VALUED MEASURES ON QUANTUM LOGICS

Let \( H \) be a Hilbert space (real or complex). An \( H \)-valued measure on a logic \( L \) is a map \( \xi : L \to H \) such that \( a, b \in L, a \perp b \Rightarrow (\xi (a), \xi (b)) = 0 \) and \( \xi (a \vee b) = \xi (a) + \xi (b) \). An \( H \)-valued measure \( \xi \) on \( L \) is \( \sigma \)-additive if for any sequence \((a_i)_{i \in N}\) of pairwise orthogonal elements of \( L \) such that \( \bigvee a_i \) exists in \( L \) we have \( \xi (\bigvee a_i) = \sum \xi (a_i) \), where the series on the right converges in norm in \( H \). If \( \xi : L \to H \) is an \( H \)-valued measure, then the map \( a \to \|\xi (a)\|^2 \) is a measure on \( L \) which is \( \sigma \)-additive iff \( \xi \) is \( \sigma \)-additive. We shall say that \( \xi \) is an \( H \)-valued state if \( a \to \|\xi (a)\|^2 \) is a state. The problem of existence of \( H \)-valued (\( \sigma \)-) states on a logic is not trivial in general, since the existence of such state entails the existence of a state on \( L \). It is well-known that there are logics with no states, and hence no \( H \)-valued states [10]. The quotient algebra \( B (R)/I \) of the Borel algebra \( B (R) \) with respect to the \( \sigma \)-ideal \( I \) of all subsets of the first category is an example of a logic with no \( \sigma \)-states [3], hence no \( H \)-valued \( \sigma \)-states. But there exist finitely additive states on \( B (R)/I \), and to every finitely additive state \( m \), the function \( K_m (a, b) = m (a \land b), a, b \in B (R)/I \) is positive definite, which implies the existence of an \( H \)-valued state \( \xi \) such that \( m (a) = \|\xi (a)\|^2, a \in B (R)/I \) (see [17], [7]).

Another example of a logic possessing no \( \sigma \)-additive \( H \)-valued state, is the logic \( E (V) \) of all splitting subspaces of an incomplete inner-product space \( V \) of \( \aleph_0 \)-orthogonal dimension (we recall that a subspace \( M \subset V \) is splitting if \( M + M^\perp = V \), since \( V \) is complete iff \( E (V) \) possesses at least one \( \sigma \)-state (see [9]). On the other hand, there exist many finitely additive \( H \)-valued states. Indeed, let \( H = \overline{V} \), where \( \overline{V} \) is the completion of \( V \) and for any \( x \in V, \|x\| = 1 \), define an \( H \)-valued mapping \( \xi_x : E (V) \to H \) via \( \xi_x (M) = x_M, M \in E (V) \), where \( x = x_M + x_M^\perp, x_M, x_M^\perp \in M, x_M^\perp \in M^\perp \). Then \( \xi_x \) is an \( H \)-valued state on \( L \).
In [13], an example of a finite logic is constructed, which possesses ordinary states, but does not have any H-valued state in any Hilbert space H.

The following principal criterion has been proved in [7]. Here we present its more compact form.

**THEOREM 2.1.** Let $L$ be a (σ-) logic and $m$ be a (σ-) measure on $L$. Then there is a Hilbert space $H$ and an $H$-valued (σ-) measure $ξ$ on $L$ such that $\| ξ (a) \|^2 = m (a)$, $a \in L$, if and only if there is a map $K_m : L \times L \to C$ (or $R$) such that

(i) $K_m (a, b) = m (a \land b)$ if $a \leftrightarrow b$,

(ii) $\sum_{i, j \leq n} \alpha_i \alpha_j K_m (a_i, a_j) \geq 0$ for all $\alpha_i \in C$ (or $\alpha_i \in R$), $a_i \in L$, $i \leq n$, $K_m (a, b) = K_m (b, a)$ in the real case.

**Proof.** If $ξ$ exists, we put $K_m (a, b) = (ξ (a), ξ (b))$. If $K$ with properties (i) and (ii) is given, the proof follows by a well-known theorem (see e.g. [17], p. 489) using the same ideas as in [7].

We note that if an $H$-valued state on a logic $L$ exists, then there exists an $H$-valued state in an infinite dimensional, real Hilbert space. Indeed, due to (i), (ii) in the above theorem, there is a probability space $(Ω, σ, P)$ and a Gaussian process $(ξ \in L)$ such that $K (a, b) = (ξ (a), ξ (b))$ is the covariance function, and $H = L^2 (Ω, σ, P)$ is that. Moreover, if $K$ is a covariance function, then the real part of $K$ is also a covariance function (see [17]), hence we may choose a real $H$.

Two $H$-valued measures $ξ$, $η$ on $L$ are said to be biorthogonal if for every $a, b \in L$, $a \perp b$ we have $(ξ (a), η (b)) = 0$. Following statement is straightforward.

**LEMMA 2.2.** Let $ξ$, $η$ be $H$-valued measures on $L$. Following conditions are equivalent:

(i) $ξ$, $η$ are biorthogonal,

(ii) for every $α, β \in C$ (or $α, β \in R$ if $H$ is real) the map $a \mapsto αξ (a) + βη (a)$ from $L$ into $H$ is an $H$-valued measure.

A family $\mathcal{N}$ of $H$-valued measures on $L$ is said to be biorthogonal if every two measures $ξ$, $η \in \mathcal{N}$ are biorthogonal. A biorthogonal family $\mathcal{N}$ is a maximal biorthogonal family if every $H$-valued measure on $L$, which is biorthogonal to every member of $\mathcal{N}$, necessarily belongs to $\mathcal{N}$. By Lemma 2.4, every maximal biorthogonal family is a linear space over $C$ (or over $R$). Clearly, every biorthogonal family is contained in a maximal one.

Following theorem shows that the family of all $H$-valued measures on $L$ (and also every maximal biorthogonal family) is sequentially closed.
THEOREM 2.3 (Nikodym theorem). — Let L be a σ-logic and let $\xi_n$, $n \in \mathbb{N}$ be H-valued σ-measures on L. If for any $a \in L$ there is $\xi(a) = \lim \xi_n(a)$ (i.e. $\| \xi_n(a) - \xi(a) \| \to 0$), then $\xi$ is an H-valued measure on L.

Proof. — Let $\xi(a) = \lim \xi_n(a)$. If $a \perp b$, then

$$\xi(a \vee b) = \lim \xi_n(a \vee b) = \lim \xi_n(a) + \lim \xi_n(b) = \xi(a) + \xi(b).$$

We claim to show $\xi(a) = \sum_{i \in \mathbb{N}} \xi(a_i)$ if $a_i \perp a_j$, $i \neq j$, and $a = \bigvee a_i$. The functions $m(b) = \| \xi(b) \|^2$, $m_n(b) = \| \xi_n(b) \|^2$, $b \in L$, are additive and σ-additive measures on L. Moreover, for any $b \in L$,

$$|m(b) - m_n(b)| = \| \xi_n(b) \|^2 - \| \xi(b) \|^2 = \| \xi_n(b) - \xi(b) \|^2 \leq |\xi_n(b) - \xi(b)|. K \to 0,$$

where

$$K = \sup \{ m_n(1), m(1) \mid n \in \mathbb{N} \} < \infty.$$

Hence, by [6], $m(a) = \sum_{i \in \mathbb{N}} m(a_i)$. Therefore,

$$\| \xi - \sum_{i \leq n} \xi(a_i) \|^2 = \| \xi \wedge (\bigvee a_i') \|^2 = m(a \wedge (\bigvee a_i')) = m(a) - \sum_{i \leq n} m(a_i) \to 0,$$

hence $\xi(a) = \sum_{i \in \mathbb{N}} \xi(a_i)$.

We note that if $\xi_n$, $n \in \mathbb{N}$ belong to a maximal orthogonal family $\mathcal{N}$, then $\xi$ also belongs to $\mathcal{N}$. Indeed, if $a \perp b$, then for any $\eta \in \mathcal{N}$, $(\xi(a), \eta(b)) = \lim (\xi_n(a), \eta(b)) = 0$.

Following theorem has been proved in [16] for lattice logics and complex Hilbert spaces, but the method of the proof can be applied to logics which are not necessarily lattices and real Hilbert spaces as well.

THEOREM 2.4. — Let L be a logic and let H be a Hilbert space (real or complex). Let $\mathcal{N}$ be a maximal biorthogonal family of H-valued measures on L. For every $a \in L$ put $\mathcal{N}(a) = \{ \xi(a) \mid \xi \in \mathcal{N} \}$. Then following statements hold.

(i) For every $a \in L$, $\mathcal{N}(a)$ is a closed linear subspace of H.

(ii) For every $a$, $b \in L$, $a \perp b$, we have $\mathcal{N}(a) \perp \mathcal{N}(b)$ and $\mathcal{N}(a \vee b) = \mathcal{N}(a) \vee \mathcal{N}(b)$, i.e. $\Phi(a \vee b) = \Phi(a) + \Phi(b)$, where $\Phi(a)$ denotes the projection on $\mathcal{N}(a)$. If, in addition, all the measures in $\mathcal{N}$ are σ-additive, then for every sequence $(a_i)_{i \in \mathbb{N}}$ of mutually orthogonal elements of L such that $\bigvee a_i$ exists in L we have $\Phi(\bigvee a_i) = \sum_{i \in \mathbb{N}} \Phi(a_i)$, where the sum converges in the strong operator topology on H.
(iv) For every $\xi \in \mathcal{N}$ there is a vector $v_\xi \in H$ such that $\xi(a) = \Phi(a) v_\xi$, $a \in L$.

**Corollary 2.5.** Let $\xi$ be an $H$-valued measure on $L$. Then there is a closed subspace $H_0$ of $H$, a morphism $\Phi$ from $L$ into $L(H_0)$ and a vector $v_\xi \in H_0$ such that $\xi(a) = \Phi(a) v_\xi$ for every $a \in L$. If, in addition, $L$ is a $\sigma$-logic and $\xi$ is $\sigma$-additive, then $\Phi$ is a $\sigma$-morphism.

**Proof.** The measure $\xi$ is contained in at least one maximal biorthogonal family of $H$-valued measures on $L$, so that we can apply Theorem 2.4. We put $H_0 = \mathcal{N}(1)$. Then $H_0$ is a closed subspace of $H$. From $\Phi(a) \mathcal{N}(1) = \Phi(a) (\mathcal{N}(a) \lor \mathcal{N}(a')) = \Phi(a) \mathcal{N}(a)$ we get $\mathcal{N}(a) \subset \mathcal{N}(1) = H_0$ for every $a \in L$, and $\Phi(a) + \Phi(a') = \Phi(1)$, $\Phi(a) \cap \Phi(a')$ imply that $\Phi(a') = \Phi(a') \lor \Phi(1)$. Hence $\Phi$ is a morphism from $L$ into $L(H_0)$. For every $a \in L$, $\Phi(a) \xi(1) = \Phi(a) (\xi(a) + \xi(a')) = \xi(a) = \Phi(a) v_\xi$, hence we may put $v_\xi = \xi(1) e H_0$.

Following example shows that the morphism $\Phi$ need not be a lattice morphism. Let $L = MO(3)$ be "Chinese lantern" (Fig.). Let $H = \mathbb{R}^3$ and let \( \{x, y, z\} \) be an orthogonal base in $H$. For every $t = (\alpha, \beta, \gamma) \in \mathbb{R}^3$ define $\xi_t(a) = \alpha x, \xi_t(b) = \beta y, \xi_t(c) = \gamma z, \xi_t(a') = \beta y + \gamma z, \xi_t(b') = \alpha x + \gamma z, \xi_t(c') = \alpha x + \beta y + \gamma z$. It is easy to check that $\mathcal{N} = \{\xi_t | t \in \mathbb{R}^3\}$ is a biorthogonal family of $H$-valued measures on $L$. Now let $\xi$ be an $H$-valued measure on $L$ which is biorthogonal to all members of $\mathcal{N}$. Then $\xi(a) \bot \xi_t(a')$ for all $t \in \mathbb{R}^3$ implies that $\xi(a) = \alpha_1 x$ for some $\alpha_1 \in \mathbb{R}$. Similarly, $\xi(b) = \beta_1 y$, $\xi(c) = \gamma_1 z$ for some $\beta_1, \gamma_1 \in \mathbb{R}$. Further, $\xi(a') \bot \xi_t(a)$ for every $t$ implies that $\xi(a') = \beta_2 y + \gamma_2 z$, and analogically $\xi(b') = \alpha_2 x + \gamma_3 z$, $\xi(c') = \alpha_3 x + \beta_3 y$ for some $\alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3 \in \mathbb{R}$. From $\xi(1) = \xi(a) + \xi(a') = \xi(b) + \xi(b') = \xi(c) + \xi(c')$ we obtain $\alpha_1 x + \beta_2 y + \gamma_2 z = \alpha_2 x + \beta_1 y + \gamma_3 z = \alpha_3 x + \beta_3 y + \gamma_1 z$, and independence of $x, y, z$ entails $\alpha_1 = \alpha_2 = \alpha_3, \beta_1 = \beta_2 = \beta_3, \gamma_1 = \gamma_2 = \gamma_3$. Hence $\xi \in \mathcal{N}$, i.e. $\mathcal{N}$ is a maximal biorthogonal family. Now $\mathcal{N}(a) = [x]$,
\( \mathcal{N} (b) = [y], \mathcal{N} (c) = [z], \mathcal{N} (a') = [y, z], \mathcal{N} (b') = [x, z], \mathcal{N} (c') = [x, y], \mathcal{N} (1) = H, \) where \([u, v, \ldots]\) denotes the linear subspace of \( H \) generated by the vectors \( u, v, \ldots \) in \( H \). But \( \mathcal{N} (a \lor b) = \mathcal{N} (1) = \mathcal{N} (a) \lor \mathcal{N} (b) = [x, y] \).

### 3. REPRESENTATIONS OF QUANTUM LOGICS

In analogy with representations of \( \mathbb{C} \)-algebras, we shall call every (\( \sigma \)-) morphism from a (\( \sigma \)-) logic into a Hilbert space logic \( L (H) \) a (\( \sigma \)-) representation of \( L \) in \( H \). We shall say that a representation \( \Phi \) of \( L \) in \( H \) is faithful if \( \Phi (a) = 0 \) implies \( a = 0 \). If \( L \) is a lattice logic and \( \Phi \) is a faithful lattice representation, then \( \Phi \) is one-to-one.

Let \( \Phi_i \) be representations of \( L \) in \( H_i, i \in I \). Put \( H = \bigoplus_{i \in I} H_i, \Phi = \bigoplus_{i \in I} \Phi_i, \Phi (a) = (\Phi_i (a))_{i \in I} \). Then clearly, \( \Phi \) is a representation of \( L \) in \( H \), which is the direct sum of the representations \( \Phi_i, i \in I \).

Let \( m \) be a measure on \( L \). If there is a representation \( \Phi \) of \( L \) in a Hilbert space \( H \) such that \( m(a) = \| \Phi (a) v \|^2, a \in L, \) for a vector \( v \in H \), we shall call \( \Phi \) the representation associated with the measure \( m \). Clearly, a representation \( \Phi \) associated with a measure \( m \) is a \( \sigma \)-representation iff \( m \) is \( \sigma \)-additive, and if \( m \) is faithful, then also \( \Phi \) is faithful.

**Lemma 3.1.** — Let \( \Phi \) be a representation of a logic \( L \) in a Hilbert space \( H \). Then

(i) if \( a, b \in L \), \( a \leftrightarrow b \Rightarrow \Phi (a \land b) = \Phi (a) \Phi (b) = \Phi (b) \Phi (a) = \Phi (a) \land \Phi (b) \). In particular, if \( B \) is a Boolean subalgebra of \( L \), then \( \Phi (B) \) is a Boolean subalgebra of \( L (H) \).

(ii) for every state \( m \) on \( L (H) \) there is a state \( m^k \) on \( L \) such that \( m^k (a) = m (\Phi (a)), a \in L \).

If \( \Phi \) is a \( \sigma \)-representation of a \( \sigma \)-logic \( L \), then

(iii) to every observable \( x \) on \( L \) there is a self-adjoint operator \( \Phi (x) \) on \( H \) with the spectral measure \( E \mapsto \Phi (x) (E) = \Phi (x (E)), E \in B (R) \). If \( x \) is an observable and \( f : R \to R \) is a Borel function, then \( \Phi (x \cdot f^{-1}) = \Phi (x) \cdot f^{-1} = f (\Phi (x)) \). In particular, if \( x \) and \( y \) are compatible observable on \( L \), then \( \Phi (x) \) and \( \Phi (y) \) commute and \( \Phi (x + y) = \Phi (x) + \Phi (y), \Phi (x \cdot y) = \Phi (x) \cdot \Phi (y) \), where the operations + and \( \cdot \) are defined by the functional calculus for compatible observables on \( L \), resp. on \( L (H) \).

Proof of the lemma is standard and we omit it.

**Theorem 3.2.** — Let \( L \) be a \( \sigma \)-logic, \( \mathcal{S} (L) \) be the set of all \( \sigma \)-additive states on \( L \) and let \( (L, \mathcal{S} (L)) \) be a sum logic. Then there is a faithful lattice \( \sigma \)-representation of \( L \) in a (real) Hilbert space \( H \) iff Segal product on bounded observables on \( L \) is distributive.
Proof. — Let \((L, \mathcal{P}(L))\) be a sum logic with distributive Segal product. Let \(m \in \mathcal{P}(L)\) and put \(K_m(a, b) = m(q_a \circ q_b)\), \(a, b \in L\). Due to distributivity of Segal product, \(K_m(a, b)\) satisfies the conditions of Theorem 2.1. Indeed, let \((\alpha_i)_{1 \leq n} \subset \mathbb{R}\). We have

\[
\sum_{i, j \leq n} \alpha_i \alpha_j K_m(a_i, a_j) = \sum_{i, j \leq n} \alpha_i \alpha_j m(q_{a_i} \circ q_{a_j}) = m((\sum_{i \leq n} \alpha_i q_{a_i})^2) \geq 0 \quad \text{(see [7])}.
\]

Therefore, there is a (real) Hilbert space \(H_m\) and a \(\sigma\)-additive \(H_m\)-valued state \(\xi_m : L \rightarrow H_m\) such that \(K_m(a, b) = (\xi_m(a), \xi_m(b))\) for any \(a, b \in L\). By Theorem 2.4, there is a \(\sigma\)-morphism \(\Phi_m : L \rightarrow L(H^0_m), H_m \in L(H_m)\), and a vector \(v_m \in H_m^0\) such that \(m(a) = \|\Phi(a)v_m\|^2\) for every \(a \in E\). Without any loss of generality we may assume that \(H_m = H_m^0\). Now construct the direct sum \(\Phi = \oplus \{\Phi_m \mid m \in \mathcal{P}(L)\}\). Since \(\mathcal{P}(L)\) is strong, \(\Phi\) is a faithful representation of \(L\) in \(H = \oplus H_m\). By Lemma 3.1, to every observable \(x\) on \(L\), there corresponds a s. a. operator \(\Phi(x)\) on \(H\). Let \(x, y\) be bounded observables on \(L\). Then \(\Phi(x), \Phi(y)\) are bounded. Let \(v \in H, ||v|| = 1\), and let \(s_v\) be the corresponding state on \(L(H)\). From

\[
s_v(\Phi(x + y)(E)) = s_v^L((x + y)(E)), \quad E \in B(\mathbb{R}),
\]

we obtain

\[
s_v(\Phi(x + y)) = s_v^L(x + y) = s_v^L(x) + s_v^L(y) = s_v(\Phi(x)) + s_v(\Phi(y)),
\]

as \(s_v^L \in \mathcal{P}(L)\). Therefore \(\Phi(x + y)v, v = ((\Phi(x) + \Phi(y))v, v)\) for every \(v \in H\), and hence \(\Phi(x + y) = \Phi(x) + \Phi(y)\). As \(\Phi(x, f^{-1}) = \Phi(x, f^{-1})\), we have \(\Phi(x^2) = \Phi(x)^2\). This entails that \(\Phi\) preserves sums and Segal products of bounded observables. Let \(a, b \in L\), then \((q_a + q_b) \{2\} = a \wedge b\) implies \(\Phi((q_a + q_b) \{2\}) = \Phi(a \wedge b)\). On the other hand, \(q_a = q_a^2\) and \(\Phi(q_a) = (\Phi(q_a))^2\) imply that \(\Phi(q_a)\) is a projection, and we have \(\Phi(q_a) \{1\} = \Phi(q_a) \{1\} = \Phi(a)\). Therefore

\[
\Phi((q_a + q_b) \{2\}) = \Phi(q_a + q_b) \{2\} = (\Phi(q_a) + \Phi(q_b)) \{2\} = (\Phi(a) \wedge \Phi(b)).
\]

Hence \(\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)\). \(i.e.\ \Phi\) is a lattice morphism.

Now suppose that \((L, \mathcal{P}(L))\) is a sum logic which admits a faithful \(\sigma\)-representation in a Hilbert space \(H\). Similarly as in the first part of this proof, we show that \(\Phi(x + y) = \Phi(x) + \Phi(y)\) and \(\Phi(x^2) = \Phi(x)^2\) for any bounded observables \(x, y\) on \(L\), so that \(\Phi\) preserves sums and Segal products, and \(\Phi\) preserves lattice operations. Let \(x, y, z\) be bounded observables on \(L\). We have

\[
\Phi[(x + y) \circ z - (x \circ z + y \circ z)] = (\Phi(x) + \Phi(y)) \circ \Phi(z)
\]

\[
- (\Phi(x) \circ \Phi(z) + \Phi(y) \circ \Phi(z)) = 0,
\]

and as \(\Phi\) is faithful, this entails that \((x + y) \circ z = x \circ z + y \circ z\).

**Corollary 3.3.** — Let \((L, \mathcal{P}(L))\) be a sum logic. Then every \(\sigma\)-representation of \(L\) in a Hilbert space \(H\) is a lattice representation. In addition, the
extension of the representation to bounded observables on \( L \) preserves sums and Segal products.

**Proof.** It follows immediately from the proof of Theorem 3.2.

**Theorem 3.4.** Let \((L, \mathcal{S}(L))\) be a sum logic and let \( H \) be a Hilbert space. Let \( m \in \mathcal{S}(L) \) and let there be an \( H \)-valued state \( \xi \) on \( L \) such that \( m(a) = \| \xi(a) \|^2, a \in L \). Then \( \text{Re} \, K_m(a, b) = m(q_a \circ q_b), a, b \in L \).

**Proof.** By Corollary 2.5 and Corollary 3.3, to any bounded observables \( x, y \) on \( L \) there correspond bounded self-adjoint operators \( \Phi(x), \Phi(y) \) on \( H \) and \( \Phi(x \circ y) = \Phi(x) \circ \Phi(y), \Phi(x + y) = \Phi(x) + \Phi(y) \). We have

\[
K_m(a, b) = (\xi(a), \xi(b)) = (\Phi(a) v_{\xi_a}, \Phi(b) v_{\xi_b}) = (\Phi(b) \Phi(a) v_{\xi_a}, v_{\xi_b}) = (\Phi(q_a) \Phi(q_b) v_{\xi_a}, v_{\xi_b}) = K_m(b, a).
\]

On the other hand,

\[
m(q_a \circ q_b) = (\Phi(q_a) \circ \Phi(q_b) v_{\xi_a}, v_{\xi_b}) = (\frac{1}{2} (\Phi(q_a) \Phi(q_b) + \Phi(q_b) \Phi(q_a)) v_{\xi_a}, v_{\xi_b}) = \text{Re} \, K_m(a, b).
\]

We note that the result obtained by Hamhalter in [12] that every \( \sigma \)-additive state on a projection logic \( L(\Omega) \) of a \( \text{W}^* \)-algebra \( \Omega \) without any type \( I_2 \) direct summand on a Hilbert space \( H \) with \( \dim H = \infty \) can be represented by an \( H \)-valued state (with values in the same Hilbert space \( H \)), follows directly from our criterion in Theorem 2.1 for \( K_m(P, Q) = m(PQ) = \text{Tr}(TPQ), m \in \mathcal{S}(L), P, Q \in L(\Omega) \), where \( T = \sum_i c_i (\ldots, e_i) e_i \). It suffices to put \( \xi(P) = \sum_i c_i^{1/2} P e_i \). Nevertheless, the method of proof he used is very interesting.

**Theorem 3.5.** Let \((L, \mathcal{S}(L))\) be a sum logic and let \( m \in \mathcal{S}(L) \).

(i) There is an \( H \)-valued state \( \xi \) on \( L \) such that \( m(a) = \| \xi(a) \|^2, a \in L \), iff \( m(((x + y) \circ z - (x \circ z + y \circ z))^2) = 0 \) for any three bounded observables \( x, y, z \) on \( L \).

(ii) If \( s \in \mathcal{S}(L), s \leq m \) [in the sense that \( m(a) = 0 \Rightarrow s(a) = 0, a \in L \) and if there is an \( H \)-valued state \( \eta \) such that \( m(a) = \| \eta(a) \|^2, a \in L \), then there is an \( H \)-valued state \( \xi \) on \( L \) such that \( s(a) = \| \eta(a) \|^2, a \in L \).

**Proof.** (i) Let \( m(a) = \| \xi(a) \|^2, a \in L \), and let \( \xi(a) = \Phi(a) v_{\xi_a} \). Then

\[
m(((x + y) \circ z - (x \circ z + y \circ z))^2)
= \| (\Phi(x) + \Phi(y)) \circ (\Phi(z) - (\Phi(x) \circ \Phi(z) + \Phi(y) \circ \Phi(z))) v_{\xi_a} \|^2 = 0.
\]

On the other hand, if the above condition is satisfied, Schwarz inequality implies that \( m(((x + y) \circ z - (x \circ z + y \circ z))^2) = 0 \) for any \( x, y, z \), and it is easy to check that \( K_m(a, b) = m(q_a \circ q_b) \) satisfies conditions (i)-(ii) of Theorem 2.1, and hence \( m \) is representable by an \( H \)-valued state \( \xi \).

(ii) follows directly from (i).
Our next remark concerns some relations between $H$-valued states and representations of $C^*$-algebras.

Let $(L, \mathcal{P}(L))$ be a sum logic such that $L$ is a projection logic of a $C^*$-algebra $\mathcal{U}$ (e.g. $L$ is a projection logic of a $W^*$-algebra of operators acting on a complex separable Hilbert space $H$, and a unit cyclic vector $v_L$ for $\pi_L$ such that $s(A) = (\pi(a) v_L, v_L)$ $(A \in \mathcal{U})$ (see [15], p. 278, [4], p. 64). If $a \in L$, $\pi(a)$ is a projection on $H_s$ and hence $s(a) = \|\pi(a) v_L\|^2$. If we put $\eta(a) = \pi(a) v_L$, then it is easy to check that $a \rightarrow \pi(a) v_L$, $a \in L$, is an $H_s$-valued $\sigma$-additive state on $L$.

We note that, using the results in [1], similar results may be obtained for suitable types of JB-algebras.

On the other hand, let $\xi : L \rightarrow H$ be a $\sigma$-additive $H$-valued state on $L$ such that $s(a) = \|\xi(a)\|^2$ for some state $s \in \mathcal{P}(L)$. Then there is a lattice $\sigma$-morphism $\Phi_\xi : L \rightarrow L(H)$ such that $\xi(a) = \Phi_\xi(a) v_L$, $a \in L$, for some unit vector $v_L \in H$ and $\Phi_\xi$ can be extended to a linear and Segal product preserving map from bounded observables on $L$ (i.e. Hermitian elements of $\mathcal{U}$) into the algebra $\mathcal{B}(H)$ of bounded operators on $H$. This map $\Phi_\xi$ can be in a natural way extended to a linear Jordan morphism $\Phi_\xi$ from $\mathcal{U}$ into $\mathcal{B}(H)$ (see also [16]). By [4], 3.1.2, p. 17), every Jordan morphism is a combination of a morphism and antimorphism. In case that $\Phi_\xi$ is a morphism, we put $H_\xi = \{\Phi(A) v_L | A \in \mathcal{U}\}^-$ (where $M^-$ means the closure of $M$, $M \subset L$, in $H$). Then since $H_\xi$ is invariant, the map $A \rightarrow P_\xi \Phi_\xi(A)$, $A \in \mathcal{U}$, where $P_\xi$ is the projection in $H$ onto $H_\xi$, is a cyclic representation of $\mathcal{U}$ in $H_\xi$ such that

$$s(A) = (P_\xi \Phi_\xi(A) v_L, v_L) = (\Phi_\xi(A) v_L, v_L),$$

$$\xi(a) = P_\xi \Phi_\xi(a) v_L = \Phi_\xi(a) v_L.$$ 

By [15], Prop. 4.5.3, this cyclic representation $(H_\xi, P_\xi \Phi_\xi, v_L)$ is isomorphic to the cyclic representation $(H_s, \pi_s, v_s)$ produced from $s$ by the GNS construction, in the sense that there is an isomorphism from $H_s$ onto $H_\xi$ such that $v_L = U v_s$, $P_\xi \Phi_\xi(A) = U \pi_s(A) U^*(A \in \mathcal{U})$.

Our next results concerns joint distributions of observables. Recall that observables $x_1, x_2, \ldots, x_n$ on a logic $L$ have a joint distribution of type 1 in a state $m$ if there is a probability measure $\mu_1$ on $B(R^n)$ such that

$$\mu_1(\{E \times E \times \ldots \times E\}) = m(x_1(E_1) \wedge \ldots \wedge x_n(E_n))$$

for any $E_1, \ldots, E_n \in B(R)$, and bounded observables $x_1, \ldots, x_n$ on a sum logic have a joint distribution of type 2 in a state $m$ if there is a measure $\mu_2$ on $B(R^n)$ such that

$$\mu_2(\{(t_1, \ldots, t_n) \in R^n | \sum_{i \leq n} \alpha_i t_i \in E\}) = m((\sum_{i \leq n} \alpha_i x_i)(E))$$

for any \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) and \( E \in \mathcal{B} (\mathbb{R}) \) (see [8]).

**Theorem 3.7.** — Let \((L, \mathcal{P} (L))\) be a sum logic. Let \( m \in \mathcal{P} (L) \) and let there be an \( H \)-valued state \( \xi : L \to H \) such that \( m (a) = \| \xi (a) \| ^2, a \in L \). If for a given set \( x_1, \ldots, x_n \) of bounded observables joint distribution of type 1 in the state \( m \) exists, then there exists also joint distribution of type 2, and the two joint distributions are identical.

**Proof.** — Let us define so-called commutator \( c \) of \( x_1, \ldots, x_n \) by

\[
c = \bigwedge_{E_1 \ldots E_n} \bigvee_{i_1 \ldots i_n=0} \bigwedge_{j=0} x_j (E_j)^{i_j}
\]

where \( a^1 = a, a^0 = a', a \in L \). It is known that \( c \) exists and that \( x_1, \ldots, x_n \) have a type 1 joint distribution in \( m \) iff \( m (c) = 1 \) (see [8]). But \( m (c) = 1 \) implies \( \| \xi (c) \| ^2 = \| \Phi (c) v_k \| ^2 = 1 \), hence \( \Phi (c) v_k = v_k \). Since \( \Phi : L \to L (H) \) is a lattice \( \sigma \)-morphism, we obtain that \( \Phi (c \xi) \) is the commutator of \( \Phi (x_1), \ldots, \Phi (x_n) \), and hence the latter observables have a type 1 joint distribution in the vector state \( m v_k \) on \( L (H) \) corresponding to \( v_k \). Now by [8], type 2 joint distribution of \( \Phi (x_1), \ldots, \Phi (x_n) \) in \( m v_k \) exists and is equal to the type 1 joint distribution. Let \( \alpha_1, \ldots, \alpha_n \in \mathbb{R}, E \in \mathcal{B} (\mathbb{R}) \). Then

\[
m (x_1 (E_1) \wedge \ldots \wedge x_n (E_n)) = \Phi (x_1 (E_1) \wedge \ldots \wedge x_n (E_n)) v_k v_k = (\Phi (x_1 (E_1)) \wedge \ldots \wedge \Phi (x_n (E_n)) v_k v_k = m v_k \Phi (x_1 (E_1)) \wedge \ldots \wedge \Phi (x_n (E_n)),
\]

\[
m \left( \bigotimes_{i \leq n} x_i (E) \right) = \Phi \left( \bigotimes_{i \leq n} x_i (E) \right) v_k v_k = m v_k \left( \bigotimes_{i \leq n} \Phi (x_i) (E) v_k v_k \right).
\]

The latter equalities show that there is a type 2 joint distribution of \( x_1, \ldots, x_n \) in \( m \), which equals to the type 1 joint distribution.

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