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The theory of Kaluza-Klein-Jordan-Thiry revisited

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ABSTRACT. — The field equations of gravity coupled to electromagnetism and to the U(1) Jordan-Thiry scalar field are reviewed and given both in the usual 4-dimensional space of General Relativity, and in the 5-dimensional formalism of Kaluza and Klein. The relation between the five and the 4-dimensional geodesics is studied as well as the general description of matter sources. In particular, we show that any attempt to describe the motion of usual test particles (or matter fields) in the 5-dimensional space leads to experimental inconsistencies. On the contrary, if matter is described as usual in the 4-dimensional space, the theory leads to the same *P*arametrized-*P*ost-*N*ewtonian parameters as General Relativity for the Schwarzschild solution, although the scalar field is coupled to gravity.

RÉSUMÉ. — Les équations de la gravitation couplée à l'électromagnétisme et au champ scalaire U(1) de Jordan-Thiry sont réexposées et écrites dans l'espace de dimension 4 de la Relativité Générale, ainsi que dans le

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formalisme pentadimensionnel de Kaluza et Klein. Nous étudions la relation entre les géodésiques 4- et 5-dimensionnelles, et la description générale des sources matérielles. En particulier, nous montrons que décrire le mouvement des particules usuelles (ou les champs matériels) dans l'espace de dimension 5, conduit à des incompatibilités expérimentales. En revanche, si la manière est décrite comme d'habitude dans l'espace de dimension 4, la théorie conduit aux mêmes paramètres Post-Newtoniens que la Relativité Générale pour la solution de Schwarzschild, malgré la présence du champ scalaire couplé à la gravitation.

INTRODUCTION

As is well known, gravity and electromagnetism can be unified in the 5-dimensional formalism of Kaluza [1] and Klein [2]. However, unless a rather unphysical hypothesis is made, the theory also incorporates a scalar field which, roughly speaking, describes how the size of the 5th dimension (the radius of a small circle) changes from place to place. A 4-dimensional description of the theory incorporates a set of 15 equations (10 modified Einstein equations, 4 modified Maxwell equations and a 15th equation for the scalar field). Jordan [3] and Thiry [4] were the first to consider the scalar field (that we shall call σ) not as a nuisance, but as an interesting prediction of the theory, which should be tested experimentally. In the present decade, the old idea of Kaluza has been taken up and many attempts to find a unified theory have been made along these lines. The low-energy behaviour of all these Kaluza-Klein-like theories (and even of superstring) should be somehow tightly related to a theory of general relativity with a $U(1)$ scalar field since, after all, gravity and electromagnetic forces govern our 4-dimensional world. Despite an enormous amount of literature on the subject of Kaluza-Klein-like theories in recent years (*see* for instance [5] to [9]), it is not easy to find an account of the original theory of Thiry (*see* however [3], [4], [10], [11], [12]). This partly motivates the present paper.

In order to be experimentally tested, a theory of gravity should incorporate both a description of geometry (in the present case gravity, electromagnetism and a scalar field), and a description of matter (equations of motion for test particles, description of matter sources...). Thiry [4] made a specific choice for the expression of the 5-dimensional energy-momentum tensor and imposed that test particles should follow geodesics of this 5-dimensional space-time. Some experimental consequences of this choice have been studied in the past, and most physicists remember that it leads to difficulties. The literature often quotes the calculation [13] which shows that this theory is incompatible with the measurement of Mercury perihelion shift. However, this calculation is incorrect, although we agree with the conclusion. Moreover, this is not the worst experimental contradiction, and we point out two others, related with the fact that test particles are supposed to follow the geodesics of the 5-dimensional manifold, in this *original* theory. A space-like geodesic in the 5-dimensional space-time may appear space-like, time-like or even null for the 4-dimensional observer (this depends upon the local value of the scalar field); it can also describe charged particles of arbitrary mass (including imaginary). On the contrary, time-like (or null) geodesics in the 5-dimensional space-time always appear as time-like (or null) in our 4-dimensional space-time but they cannot describe charged particles of mass smaller than 10^{20} GeV. For this reason the papers dealing with the experimental consequences of this theory ([10], [13] to [17]) made the hypothesis that test particles are tachyons in the 5-dimensional manifold (although this was not always recognized explicitly) and that the scalar field σ is small enough – but cannot take arbitrary negative values – in order for this test particle to appear as a usual time-like particle like the electron. Nowadays, we would not like to use tachyonic particles in a Unified Quantum Field Theory anyway, but we show that this choice leads to very serious experimental inconsistencies already at the classical level. This is another motivation for our study.

The experimental predictions of the theory depend of course upon our description of matter fields, and the choice made by Thiry (and considered in the literature) is not the only one. For example, a fluid which appears to be perfect in the 4-dimensional space can be described in several ways in the 5-dimensional space. It is also possible to describe matter as usual, in the 4-dimensional space, like in Einstein-Maxwell theory. With this choice, we find that the static and spherically symmetrical solution of the field equations in the vacuum – *i.e.* the generalized Schwarzschild solution – is almost not modified by the presence of the scalar field coupled to gravity and electromagnetism: the post-Newtonian parameters are the same as in General Relativity. This discussion of the possible choices to describe matter fields, which imply different analysis of the experimental consequences, is a third motivation for our study.

Our paper is organized as follows: In the first part we study gravity, electromagnetism and the scalar field σ in empty space; we also give the generalized Schwarzschild solution (assuming no electromagnetism), and we see how this is modified when we add an electromagnetic field or a cosmological constant. In the second part, we study matter: action of geometry on matter – *i.e.* geodesics – and action of matter on geometry, in particular the different possibilites for writing field equations in the presence of matter. We then discuss several experimental difficulties of the original Jordan-Thiry theory.

N.B.: The sign conventions and notations that we use are given in Appendix B.

1. GRAVITATION AND ELECTROMAGNETISM

1.1. The 5-dimensional manifold

1.1.1. *Kaluza's idea*

Special Relativity predicts that the electric and magnetic fields are components of the same tensor $F_{\mu\nu}$. Trying to imitate this unification of two fields, Kaluza's idea [1] was to describe the gravitational field $g_{\mu\nu}$ and the electric potential A_μ as components of the same tensor. Gravitation and electromagnetism would then be given by a single theory. The most natural way of implementing this idea is to construct a second-order symmetrical tensor γ_{mn}, the components of which include $g_{\mu\nu}$ and A_μ. Since $g_{\mu\nu}$ is symmetrical with 10 independent components and A_μ is a 4-vector, γ_{mn} must have at least 14 independent components. It can be interpreted as a metric tensor in 5 dimensions and will have therefore 15 independent components. The extra degree of freedom is interpreted as a scalar field in Jordan-Thiry theory. This theory will be a natural generalization of General Relativity, and the connection will be the Levi-Civita one (hence compatible with the metric and torsionless).

1.1.2. *U(1)-invariance*

Even if one tries to describe the laws of physics in an extra-dimensional space, it remains to explain why our Universe appears to be only 4-dimensional. Kaluza's idea was to suppose *a priori* that in some coordinates, the metric γ_{mn} does not depend on x^5. (Klein [2] suggested that the metric was a periodical function of x^5.)

Today we express this idea as follows: the universe is locally homeomorphic to a "tube", product of a circle by the usual 4-dimensional space-time – that we will call M –. Moreover, the 5-dimensional metric is invariant with respect to the group U(1) acting on the "internal space" S^1. In yet another language, the 5-dimensional universe is a U(1) principal bundle endowed with a U(1) invariant metric.

In this paper, we do not use explicitly the fact that the internal space is compact. However, if we suppose that it is a circle, we may set $dx^5 = \mathcal{R}\, d\theta$

and use (for instance) the Dirac equation to show that the charge e is quantized and related to the radius $\mathcal{R}$ of this circle by :

$$\mathcal{R}=\frac{2\sqrt{G}}{e}\ (\approx 10^{-33}\,m)$$

1.2. Dimensional reduction

1.2.1. *Metric tensor*

The 5-dimensional line element:

$$d\sigma^2=\gamma_{\mu\nu}\,dx^\mu\,dx^\nu+2\,\gamma_{\mu 5}\,dx^\mu\,dx^5+\gamma_{55}\,(dx^5)^2$$

can be written:

$$d\sigma^2=\left(\gamma_{\mu\nu}-\frac{\gamma_{\mu 5}\,\gamma_{\nu 5}}{\gamma_{55}}\right)dx^\mu\,dx^\nu+\gamma_{55}\left(dx^5+\frac{\gamma_{\mu 5}}{\gamma_{55}}\,dx^\mu\right)^2$$

We will often consider adapted frames, in which the 5th unitary vector is given by the Pfaff form: $\omega^{\underline{5}}=\sqrt{\gamma_{55}}\left(dx^5+\frac{\gamma_{\mu 5}}{\gamma_{55}}\,dx^\mu\right)$. [Underlined indexes are associated with the adapted frame.] Locally, the 4-dimensional space orthogonal to this vector will be interpreted as the usual space-time. Let us call:

$$g_{\mu\nu}=\gamma_{\mu\nu}-\frac{\gamma_{\mu 5}\,\gamma_{\nu 5}}{\gamma_{55}},\tag{1 a}$$

$$\kappa\,A_\mu=\frac{\gamma_{\mu 5}}{\gamma_{55}},\tag{1 b}$$

and:

$$e^{2\sigma}=\gamma_{55}\tag{1 c}$$

[We choose the signature of the metric to be $(-,+,+,+,+)$, therefore $\gamma_{55}>0$. A signature $(-,+,+,+,-)$ would lead to a negative energy for the electromagnetic field]. The quantities $g_{\mu\nu}$, A_μ and σ are independent of x^5, and κ is a constant which will be related to G. As we shall see, A_μ behaves like the electromagnetic potential, and σ behaves as a scalar field in M.

The 5-dimensional line element can now be written in terms of 4-dimensional quantities:

$$d\sigma^2=g_{\mu\nu}\,dx^\mu\,dx^\nu+e^{2\sigma}\,(dx^5+\kappa\,A_\mu\,dx^\mu)^2\tag{2}$$

1.2.2. *Einstein tensor*

Ricci tensor in an adapted frame:

By writing that the torsion of the 5-dimensional manifold vanishes, one can derive formulae giving the 5-Ricci tensor $\mathbb{R}_{mn}$ in terms of the 4-tensor $R_{\underline{\mu\nu}}$, the 4-vector A_μ and the 4-saclar σ (*see* for instance [11], [12] or [18]):

$$\mathbb{R}_{\underline{\mu\nu}} = R_{\underline{\mu\nu}} - e^{-\sigma}\nabla_{\underline{\mu}}(\partial_{\underline{\nu}} e^{\sigma}) - \frac{\kappa^2 e^{2\sigma}}{2} F_{\underline{\mu\rho}} F_{\underline{\nu}}{}^{\underline{\rho}} \tag{3a}$$

$$\mathbb{R}_{\underline{\mu 5}} = \frac{\kappa}{2} e^{-2\sigma}\nabla_{\underline{\rho}}(e^{3\sigma} F_{\underline{\mu}}{}^{\underline{\rho}}) \tag{3b}$$

$$\mathbb{R}_{\underline{55}} = -e^{-\sigma}\nabla_{\underline{\rho}}(\partial^{\underline{\rho}} e^{\sigma}) + \frac{\kappa^2 e^{2\sigma}}{4} F_{\underline{\mu\nu}} F^{\underline{\mu\nu}} \tag{3c}$$

where ∇_μ is the 4-dimensional covariant differentiation. Hence, the 5-curvature scalar reads:

$$\mathbb{R} = R - 2e^{-\sigma}\Box e^{\sigma} - \frac{\kappa^2}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} \tag{3d}$$

Einstein tensor:

In an adapted frame, $\mathbb{E}_{mn}$ is easily deduced from $\mathbb{R}_{mn}$, and one can then derive the corresponding formulae in a coordinate frame. The 5-Einstein tensor $\mathbb{E}_{mn}$ is related to the 4-dimensional one $E_{\mu\nu}$ by:

$$\begin{aligned} \mathbb{E}_{\mu\nu} = E_{\mu\nu} + \frac{\kappa^2 e^{2\sigma}}{2}\left(\frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - F_{\mu\rho} F_\nu{}^\rho\right) \\ - e^{-\sigma}(\nabla_\mu(\partial_\nu e^{\sigma}) - g_{\mu\nu}\nabla_\rho \partial^\rho e^{\sigma}) \\ + \kappa(A_\mu \mathbb{E}_{\nu 5} + A_\nu \mathbb{E}_{\mu 5}) - \kappa^2 A_\mu A_\nu \mathbb{E}_{55} \end{aligned} \tag{3e}$$

$$\mathbb{E}_{\mu 5} = \frac{\kappa}{2} e^{-\sigma}\nabla_\rho(e^{3\sigma} F_\mu{}^\rho) + \kappa A_\mu \mathbb{E}_{55} \tag{3f}$$

$$\mathbb{E}_{55} = \frac{e^{2\sigma}}{2}\left(\frac{3\kappa^2 e^{2\sigma}}{4} F_{\mu\nu} F^{\mu\nu} - R\right) \tag{3g}$$

1.3. 5-dimensional Einstein's equations for empty space

1.3.1. *Natural generalization of pure gravity*

Like in General Relativity, one can write the lagrangian density:

$$L = \frac{1}{16\pi \mathbb{G}} \mathbb{R}\sqrt{-\gamma} \tag{4}$$

where $\gamma = \det(\gamma_{mn})$ and $\mathbb{G}$ is a constant (the gravitational constant in the 5-dimensional space). It leads to the generalized Einstein's equations:

$$\mathbb{E}_{mn} = 0 \quad \Leftrightarrow \quad \mathbb{R}_{mn} = 0 \tag{5}$$

Kaluza wrote these equations in [1], but he wanted to eliminate the prediction of the unknown massless scalar σ. Therefore, he supposed later that σ was a constant parameter of the theory. This assumption does not lead to inconsistencies if only the equations of motion are considered (Kaluza's purpose was mainly to find a space where charged particles move on geodesics; *cf.* § 2.1.1). But when the field equations (5) are written, σ must actually be taken as a field (*cf.* § 1.4.3.). Jordan [3] and Thiry [4] were the first to consider this scalar not as a nuisance, but as an interesting prediction modifying Einstein-Maxwell theory, which should be tested experimentally.

1.3.2. *A first attempt of physical interpretation*

Using formula (3 *d*), the lagrangian (4) reads:

$$L=\frac{\sqrt{-g}}{16\pi\mathbb{G}e^{-\sigma}}\left(R-\frac{\kappa^2}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}-2e^{-\sigma}\Box e^{\sigma}\right) \tag{6}$$

(where the third term involvind $\Box e^{\sigma}$ can be removed since it is a total divergence). This lagrangian describes gravity coupled to electromagnetism and a scalar field, but the first term shows that the 4-dimensional gravitational constant must be defined as: $G=\mathbb{G}e^{-\sigma}$ —up to a constant multiplicative factor—. Therefore, in this original version of Jordan-Thiry theory, G appears to be *variable* if σ is not a constant. This can be cured by a conformal transformation.

1.3.3. *Conformal transformation*

- Field redefinitions allow us to rewrite the *same* theory in a different way. Let us redefine the metric tensor by a conformal transformation ([10], [14], [19], [20]):

$$g^*_{\mu\nu}=e^{\tau}g_{\mu\nu}$$

where τ is a 4-scalar, depending on the four coordinates (x^1, x^2, x^3, x^4). $g^*_{\mu\nu}$ is always a tensor intrinsically defined on M, so it can also be considered as the usual metric of General Relativity.

- One can define new tensors corresponding to this conformal metric. The equations: $g^*_{\mu\nu}=e^{\tau}g_{\mu\nu}$ and: $g^{*\mu\nu}=e^{-\tau}g^{\mu\nu}$ yield:

$$\Gamma^{*\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}+\frac{1}{2}\left(\delta^{\lambda}_{\mu}\partial_{\nu}\tau+\delta^{\lambda}_{\nu}\partial_{\mu}\tau-g_{\mu\nu}g^{\lambda\rho}\partial_{\rho}\tau\right) \tag{7a}$$

then:

$$R^*=e^{-\tau}\left(R-3\nabla_{\lambda}(g^{\lambda\rho}\partial_{\rho}\tau)-\frac{3}{2}g^{\lambda\rho}\partial_{\lambda}\tau\,\partial_{\rho}\tau\right) \tag{7b}$$

and:

$$E^*_{\mu\nu} = E_{\mu\nu} - \nabla_\mu(\partial_\nu\tau) + \frac{1}{2}\partial_\mu\tau.\partial_\nu\tau + g_{\mu\nu}\nabla_\lambda(g^{\lambda\rho}\partial_\rho\tau) + \frac{1}{4}g_{\mu\nu}g^{\lambda\rho}\partial_\lambda\tau\,\partial_\rho\tau \qquad (7\,c)$$

The electromagnetic tensor is still:

$$F^*_{\mu\nu} = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

But:

$$F^{*\ \nu}_{\mu} = e^{-\tau} F_\mu^{\ \nu} \qquad \text{and} \qquad F^{*\,\mu\nu} = e^{-2\tau} F^{\mu\nu} \qquad (7\,d)$$

• Of course, in order to write the "old" tensors in terms of the "new" ones, we need only to change the sign of τ. For instance:

$$R = e^{\tau}\left(R^* + 3\,\Box^*\tau - \frac{3}{2}\partial_\rho\tau\,\partial^{*\rho}\tau\right) \qquad (7\,b')$$

where the stars (*) mean that the conformal metric must be used to raise and lower the indices (this * has of course no relation with Hodge duality !). In particular:

$$\partial^{*\rho}\tau = g^{*\lambda\rho}\partial_\lambda\tau$$

and:

$$\Box^*\tau = \nabla^*_\mu\partial^{*\mu}\tau = \frac{1}{\sqrt{-g^*}}\partial_\mu(\sqrt{-g^*}\,g^{*\mu\nu}\partial_\nu\tau) = \partial_\mu(g^{*\mu\nu}\partial_\nu\tau) + \Gamma^{*\ \mu}_{\mu\lambda}g^{*\lambda\nu}\partial_\nu\tau$$

• The lagrangian (6) of the theory can now be written in terms of the conformal metric:

$$L = \frac{\sqrt{-g^*}}{16\pi\mathbb{G}}\left\{e^{\sigma-\tau}\left[R^* + \frac{3}{2}\partial_\rho\tau\,\partial^{*\rho}(\tau - 2\sigma)\right]\right.$$
$$\left. - \frac{\kappa^2}{4}e^{3\sigma}F_{\mu\nu}F^{*\mu\nu} + \nabla^*_\rho[e^{\sigma-\tau}\partial^{*\rho}(3\tau - 2\sigma)]\right\} \qquad (8)$$

(where the last term can be removed since it is a total divergence). Notice that the electromagnetic term is a conformal invariant (*i.e.* the factor $e^{3\sigma}$ is not modified by the redefinition of the metric). The 4-dimensional gravitational constant reads now: $G = \mathbb{G}\,e^{\tau-\sigma}$ and can be imposed to be the constant $\mathbb{G}$ by setting $\tau = \sigma$.

(Notice that this conformal transformation is nothing else than a convenient change of variables, but it does not change anything to the physics.)

• *Conformal transformation for* σ*:*

A general study of extra-dimensional Kaluza-Klein theories has been developed in [18]. For more than 4+1 dimensions, a conformal transformation on $g_{\mu\nu}$ is not enough to obtain a constant value for G. The metric of the "internal" space must also be conformally transformed.

In Jordan-Thiry theory, this additional rescaling is not necessary. It corresponds to a redefinition of σ: $\sigma^* = \frac{3}{2}\sigma$. We have kept the non-transformed σ in our formulae, because they would have been a little more complicated with σ^*. But higher dimensional Kaluza-Klein theorists must exercise care when using the present paper, in which only the necessary conformal transformation has been made.

1.3.4. *Field equations for empty space*

• If the conformal metric $g^*_{\mu\nu} = e^\sigma g_{\mu\nu}$ is used, the lagrangian (8) of the theory reads:

$$L = \frac{\sqrt{-g^*}}{16\pi G}\left\{R^* - \frac{3}{2}\partial_\rho\sigma\,\partial^{*\rho}\sigma - \frac{\kappa^2}{4}e^{3\sigma}F_{\mu\nu}F^{*\mu\nu}\right\}\ (+\ \text{total divergence}) \quad (8')$$

(In the 5-dimensional *empty* space, there are not matter and no electric sources, but the electromagnetic field is not necessarily vanishing.) One can choose the origin of σ so that Einstein-Maxwell theory is recovered for $\sigma = 0$. In this case, one must set:

$$\kappa^2 = 4G$$

so that the electromagnetic lagrangian is the usual one.

• One can derive the field equations directly from this lagrangian, or rewrite the 5-dimensional ones (5) by using the dimensional-reduction formulae (4) and the conformal transformation (7). They read:

$$E^*_{\mu\nu} = 8\pi G\,(T^{*\,\text{e.m.}}_{\mu\nu} + \tau^*_{\mu\nu}) \quad (9a)$$

$$\nabla^*_\nu H^{*\mu\nu} = 0 \quad (9b)$$

$$\Box^*\sigma = G H^*_{\mu\nu} F^{*\mu\nu} \quad (9c)$$

where:

$$\tau^*_{\mu\nu} = \frac{3}{16\pi G}\left(\partial_\mu\sigma\,\partial_\nu\sigma - \frac{1}{2}g^*_{\mu\nu}\partial_\rho\sigma\,\partial^{*\rho}\sigma\right) \quad (9d)$$

is the stress-energy tensor for the scalar field [one usually finds a factor $\frac{1}{4\pi}$ in front of the parenthesis, for the stress-energy tensor of a scalar field, but here σ is dimensionless, and the extra factor $\frac{3}{4G}$ is understandable$\Bigg]$,

$$T^{*\,\text{e.m.}}_{\mu\nu} = \frac{1}{4\pi}\left(\frac{1}{2}\left[H^{*\,\rho}_{\mu} F_{\nu\rho} + F^{*\,\rho}_{\mu} H_{\nu\rho}\right] - \frac{1}{4} g_{\mu\nu} H_{\rho\sigma} F^{*\,\rho\sigma}\right) \tag{9e}$$

is the (modified) Maxwell tensor, and

$$H_{\mu\nu} = e^{3\sigma} F_{\mu\nu} \tag{9f}$$

Notice that Maxwell's equations (9 *b*) and the Maxwell tensor (9 *e*) are not the usual ones in the vacuum. They are modified by σ, as if there were an electric permittivity $\varepsilon = e^{3\sigma}$ and a magnetic permeability $\mu = e^{-3\sigma}$ (so that $\varepsilon\mu = 1$ as usual in the vacuum). $F_{\mu\nu}$ is the tensor of electric field **E** and magnetic induction **B**, whereas $H_{\mu\nu}$ (9*f*) can be interpreted as the tensor of electric displacement **D** and magnetic field **H** in the vacuum. Hence, Jordan-Thiry theory modifies Einstein's and Maxwell's equations not only by considering the stress-energy tensor of the scalar field σ in (9 *a*), but also by making the vacuum behave like a material body. The 15 field equations (9 *a*), (9 *b*), (9 *c*) are completely coupled, $g_{\mu\nu}$, A_μ and σ appearing in each of them.

• *Bianchi's identities:*

Bianchi's identities can be added to the field equations (9) of the theory. Actually, because of the definition of the Riemann-Christoffel tensor $R^*_{\mu\nu\rho\sigma}$ in terms of $\Gamma^{*\,\lambda}_{\mu\nu}$, they are still valid in Jordan-Thiry theory:

$$R^*_{\mu\nu\rho\sigma;\,\lambda} + R^*_{\mu\nu\lambda\rho;\,\sigma} + R^*_{\mu\nu\sigma\lambda;\,\rho} = 0 \tag{10a}$$

[where the semicolon (;) means covariant differentiation, using the *conformal* metric $g^*_{\mu\nu}$].

Similarly, the first group of Maxwell's equations follows from the definition of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$F^*_{\mu\nu;\,\lambda} + F^*_{\lambda\mu;\,\nu} + F^*_{\nu\lambda;\,\mu} = 0 \tag{10b}$$

1.3.5. *Cosmological constant*

If the cosmological constant Λ does not vanish, the lagrangian is a simple generalization of the General Relativity one:

$$L_{(\Lambda\neq 0)} = L_{(\Lambda=0)} + \frac{\Lambda}{8\pi G}\sqrt{-\gamma} = L_{(\Lambda=0)} + \frac{\Lambda}{8\pi G} e^{-\sigma}\sqrt{-g^*}$$

and the field equations read:

$$\mathbb{E}_{mn} + \Lambda\,\gamma_{mn} = 0$$

In terms of 4-dimensional fields, they read:

$$E^*_{\mu\nu} = 8\pi G(T^{*\,\text{e.m.}}_{\mu\nu} + \tau^*_{\mu\nu}) - \Lambda e^{-\sigma} g^*_{\mu\nu} \tag{11 a}$$

$$\nabla^*_\nu H^{*\mu\nu} = 0 \tag{11 b}$$

$$\Box^* \sigma = G H^*_{\mu\nu} F^{*\mu\nu} - \frac{2\Lambda}{3} e^{-\sigma} \tag{11 c}$$

• *Warning:*

Since most of the other sections use only the conformal metric, we will no longer use the *-notation for the metric, but stars should be understood everywhere. For example, the notations given in Appendix B and the generalized Schwarzschild solution of next section correspond to the *conformal* metric $g^*_{\mu\nu}$.

1.4. Generalized Schwarzschild solution

1.4.1. *Useful form for the* 4-*dimensional line element*

In this section, we derive the solution of the field equations for empty space, near a star. We assume that the fields are static and spherically symmetrical, and we first study the case of a vanishing electromagnetic field. The electromagnetic corrections will be considered in section 1.4.4.

The line element can be written:

$$ds^2 = -e^\nu dt^2 + e^{-\nu} dr^2 + e^{\lambda-\nu} d\Omega^2 \qquad [\text{where: } d\Omega^2 = d\theta^2 + \sin^2\theta . d\varphi^2] \tag{12}$$

(neither the standard nor the isotropic forms are used, for simplicity of the following calculations; this is the choice made in [13]). Let us recall that the *conformal* metric is used in this section, though no stars (*) are written, in order to simplify the notations.

1.4.2. *Affine connection and Ricci tensor*

The affine connection is given by: $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\nu g_{\mu\rho} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})$.

If a prime means differentiation with respect to r, the nonvanishing components are:

$$-\Gamma^r_{rr} = e^{-2\nu} \Gamma^r_{tt} = \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{\nu'}{2}$$

$$-e^{-\lambda} \Gamma^r_{\theta\theta} = \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\lambda' - \nu'}{2} \qquad (\text{and } \Gamma^r_{\varphi\varphi} = \sin^2\theta . \Gamma^r_{\theta\theta})$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta . \cos\theta; \qquad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \cot\theta$$

The Ricci tensor is given by: $R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\lambda\rho} - \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\rho\nu}$.

Rotational and time-reversal invariances of the metric lead to:

$$R_{\varphi\varphi}=\sin^2\theta\,.\,R_{\theta\theta}, \qquad \text{and} \qquad R_{\mu\nu}=0 \text{ for } \mu\neq\nu$$

The other components are:

$$R_{rr}=\frac{\nu''-2\lambda''-\nu'^2-\lambda'^2+\lambda'\nu'}{2}$$
$$R_{\theta\theta}=1+\frac{\nu''-\lambda''-\lambda'^2+\lambda'\nu'}{2}\,e^{\lambda}$$
$$R_{tt}=\frac{\nu''+\lambda'\nu'}{2}\,e^{2\nu}$$

1.4.3. *Field equations and solution*

• If the electromagnetic field vanishes, the field equations for empty space read:

$$E_{\mu\nu}=8\pi G\tau_{\mu\nu} \quad\Rightarrow\quad R_{\mu\nu}=8\pi G\left(\tau_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\tau_{\rho}{}^{\rho}\right)$$
$$0=0$$
$$\nabla_{\mu}\partial^{\mu}\sigma=0 \quad\Rightarrow\quad \sigma''+\lambda'\sigma'=0$$

The only nonvanishing component of the Ricci tensor is then: $R_{rr}=\frac{3}{2}\sigma'^2$.

The field equations read then:

$$2\lambda''+\lambda'^2+\nu'^2+3\sigma'^2=0$$
$$(e^{\lambda})''=2$$
$$(\nu' e^{\lambda})'=0$$
$$(\sigma' e^{\lambda})'=0$$

• They lead to the solution:

$$e^{\lambda}=r^2-ar \tag{13a}$$
$$\nu=\frac{b}{a}\ln\left(1-\frac{a}{r}\right) \qquad (\Leftrightarrow \nu' e^{\lambda}=b) \tag{13b}$$
$$\sigma=\frac{d}{a}\ln\left(1-\frac{a}{r}\right) \qquad (\Leftrightarrow \sigma' e^{\lambda}=d) \tag{13c}$$

where a, b and d are constants verifying:

$$a^2=b^2+3d^2 \tag{13d}$$

The corresponding line element reads:

$$ds^2=-\left(1-\frac{a}{r}\right)^{b/a}dt^2+\left(1-\frac{a}{r}\right)^{-(b/a)}dr^2+r^2\left(1-\frac{a}{r}\right)^{1-(b/a)}d\Omega^2 \tag{14}$$

The solution given here is chosen to approach the Minkowski metric for $r \to \infty$. It is unique, with the restriction that one can redefine r and t by affine transformations. [By changing the origin for r and the notations, one can not recover the solution given in [13] or [21], because the value for σ (13 *c*) differs from ours by a factor 2.]

This metric reduces to Schwarzschild solution when $d=0$, *i.e.* for σ constant, as expected.

1.4.4. *Electromagnetic corrections*

• The field equations are given by (8). If the electromagnetic field is not symmetrical, a general solution is not easy to write, and not very useful for our purpose. If one wants to compute only the order of magnitude of electromagnetic corrections to the generalized Schwarzschild solution (13), one can consider a spherically symmetrical electromagnetic field as an approximation.

Using a metric (12), the equations for R_{tt} and σ read:

$$(\nu' e^{\lambda})' = G e^{\lambda-\nu+3\sigma}(F_{\mu\nu}F^{\mu\nu} - 4 F_{t\mu}F^{t\mu}) = 2G e^{\lambda-\nu+3\sigma}(\mathbf{E}^2+\mathbf{B}^2)$$
$$(\sigma' e^{\lambda})' = G e^{\lambda-\nu+3\sigma} F_{\mu\nu}F^{\mu\nu} = 2G e^{\lambda-\nu+3\sigma}(\mathbf{B}^2-\mathbf{E}^2)$$

where $\mathbf{E}^2 = -F_{t\mu}F^{t\mu}$ and $\mathbf{B}^2 = \frac{1}{2} F_{\mu\nu}F^{\mu\nu} + \mathbf{E}^2$.

One can then compute ν' and σ' at a point r_0:

$$\nu' e^{\lambda} = 2G \int_0^{r_0} e^{\lambda-\nu+3\sigma}(\mathbf{E}^2+\mathbf{B}^2)\,dr + K_\nu \tag{15a}$$

$$\sigma' e^{\lambda} = 2G \int_0^{r_0} e^{\lambda-\nu+3\sigma}(\mathbf{B}^2-\mathbf{E}^2)\,dr + K_\sigma \tag{15b}$$

where K_ν and K_σ are constants depending on the matter distribution of the star (*cf.* § 2.2.2).

Notice that spherical symmetry and Gauss's theorem allowed us to integrate from $r=0$ to $r=r_0$, and not on all space.

• *Another approximation:*

If the electromagnetic field is negligible for $r>R$ (where R is some chosen value), equations (15) prove that $\nu' e^{\lambda}$ and $\sigma' e^{\lambda}$ are almost constant for $r>R$. Hence, the generalized Schwarzschild solution (13) is valid, with:

$$b = 2G \int_0^{R} e^{\lambda-\nu+3\sigma}(\mathbf{E}^2+\mathbf{B}^2)\,dr + K_\nu \tag{16a}$$

$$d = 2G \int_0^{R} e^{\lambda-\nu+3\sigma}(\mathbf{B}^2-\mathbf{E}^2)\,dr + K_\sigma \tag{16b}$$

Then, K_ν and K_σ are the values of b and d for a vanishing electromagnetic field.

In this approximation, ν and σ are still given by equations (13 *b*) and (13 *c*), for $r > \mathrm{R}$:

$$\nu = \frac{b}{a} \ln \left(1 - \frac{a}{r}\right) \approx -\frac{b}{r} \tag{17 a}$$

$$\sigma = \frac{d}{a} \ln \left(1 - \frac{a}{r}\right) \approx -\frac{d}{r} \tag{17 b}$$

where

$$a^2 = b^2 + 3\,d^2 \tag{17 c}$$

These equations are useful in order to evaluate the order of magnitude of electromagnetic corrections to ν and σ, but let us recall that two approximations have been done:

– The electromagnetic field is supposed to be spherically symmetrical [this is admittedly not realistic, but will allow us to get a rough estimate].

– It is supposed to be negligible when r is greater than a given value R.

1.4.5. *Modifications induced by a cosmological term*

When the cosmological constant does not vanish, field equations are given by equations (11). One can compute the corrections to the generalized Schwarzschild solution (13) at first order in Λ. The field equations read:

$$\begin{gathered} 2\lambda'' + \lambda'^{\,2} + \nu'^{\,2} + 3\,\sigma'^{\,2} \approx -4\,\Lambda \\ (e^{\lambda})'' \approx 2 - 4\,r^2\,\Lambda \\ (\nu'\,e^{\lambda})' \approx -2\,\Lambda\,r^2 \\ (\sigma'\,e^{\lambda})' \approx -\frac{2}{3}\,\Lambda\,r^2 \end{gathered}$$

(where a flat metric is used for the terms involving Λ). They yield:

$$e^{\lambda} = r^2 - ar - \frac{\Lambda\,r^4}{3} \tag{18 a}$$

$$\nu'\,e^{\lambda} \approx b - \frac{2\,\Lambda\,r^3}{3} \quad \Rightarrow \quad \nu' \approx \frac{b}{r^2 - ar} - \frac{2\,\Lambda\,r}{3}$$

$$\sigma'\,e^{\lambda} \approx d - \frac{2\,\Lambda\,r^3}{9} \quad \Rightarrow \quad \nu' \approx \frac{d}{r^2 - ar} - \frac{2\,\Lambda\,r}{9}$$

Then:

$$\nu \approx \frac{b}{a} \ln \left(1 - \frac{a}{r}\right) - \frac{\Lambda\,r^2}{3} \tag{18 b}$$

$$\sigma \approx \frac{d}{a} \ln \left(1 - \frac{a}{r}\right) - \frac{\Lambda\,r^2}{9} \tag{18 c}$$

And, neglecting Λ:

$$a^2 \approx b^2 + 3\,d^2 \tag{18 d}$$

This is an *approximate* solution, where b and d are assumed to be negligible with respect to r, and large with respect to $\sqrt{\Lambda}\,.\,r^2$.

2. MATTER

In order to test the experimental predictions of the theory, one needs to describe matter fields and write the equations of motion of usual test particles. Two different attitudes are possible.

The first (and historical) one is to consider that we live in the 5-dimensional space, then that the equations of motion are given by its geodesics. Since the signature of this space is hyperbolic, geodesics are of three possible types. The aim of this section is to show that none of them can be used to describe usual particles. More generally, we will see that any attempt to describe usual matter in the 5-dimensional space leads to experimental difficulties.

The second possible attitute is to consider the 5-dimensional space only as a method to unify gravitation and electromagnetism, but not matter. Consequently, there is no real reason to incorporate matter artificially in the extra-dimensional space. Therefore, particle trajectories will be described as in Einstein-Maxwell theory, *i.e.* by 4-dimensional geodesics with a Lorentz force. With this choice, all the experimental problems that we will discuss in this section obviously disappear. [We will also see in Appendix A that all the predictions related to the post-Newtonian parameters of Schwarzschild solution are the same as in General Relativity, hence consistent with experiment.]

2.1. Action of geometry on matter

We devote this section to the 4-dimensional description of the 5-dimensional geodesics.

2.1.1. *Geodesics of the 5-dimensional manifold*

- *Equations in the 5-dimensional space:*

. The geodesics are written:

$$\mathbb{U}^n \nabla_n \mathbb{U}^m = 0$$

where $\mathbb{U}^m$ is the unitary 5-velocity, verifying $\mathbb{U}_m \mathbb{U}^m = \operatorname{sgn}(d\sigma^2)$. It remains only to express the Christoffel symbols Γ^l_{mn} in terms of $g_{\mu\nu}$, A_μ and σ.

The equation for the fifth coordinate can be written:

$$\mathbb{U}^n \partial_n \mathbb{U}_5 - \mathbb{U}^n \Gamma^l_{5n} \mathbb{U}_l = 0 \quad \Leftrightarrow \quad \mathbb{U}^n \partial_n \mathbb{U}_5 - \frac{1}{2} \mathbb{U}^n \mathbb{U}^p (\partial_n \gamma_{5p} - \partial_p \gamma_{5n}) = 0$$

The second term vanishes, and one finds that $\mathbb{U}_5$ is a constant. We shall denote it as:

$$q = \mathbb{U}_5 = \text{Const.} \tag{19}$$

$\left[q \text{ will be related to } \frac{e}{m}, \text{ but it is not a charge!} \right]$

By writing the other equations in terms of the 4-dimensional metric $g_{\mu\nu}$, and the fields $F_{\mu\nu}$ and σ, one finds:

$$\frac{d\mathbb{U}^\mu}{|d\sigma|} + \Gamma^\mu_{\lambda\nu} \mathbb{U}^\lambda \mathbb{U}^\nu = \kappa q F^\mu{}_\nu \mathbb{U}^\nu + e^{-2\sigma} q^2 \partial^\mu \sigma$$

(where $\Gamma^\mu_{\lambda\nu}$ are the 4-dimensional Christoffel symbols, corresponding to the metric $g_{\mu\nu}$, before conformal rescaling). In a more general theory where the group U(1) is replaced by a group $\mathcal{G}$ or a homogeneous space $\mathcal{G}/\mathcal{H}$, this equation can be generalized and this is discussed in [18].

. In order for the gravitational constant G to be a constant, one must use the conformal metric $g^*_{\mu\nu}$. One finds:

$$\frac{d\mathbb{U}^\mu}{|d\sigma|} + \Gamma^{*\mu}_{\lambda\nu} \mathbb{U}^\lambda \mathbb{U}^\nu = \kappa q e^\sigma F^{*\mu}{}_\nu \mathbb{U}^\nu + \mathbb{U}^\mu \mathbb{U}^\nu \partial_\nu \sigma + \left(e^{-\sigma} q^2 + \frac{1}{2} \left| \frac{ds^*}{d\sigma} \right|^2 \right) \partial^{*\mu} \sigma \tag{20}$$

This equation looks like the Einstein-Maxwell one:

$$\frac{d^2 x^\mu}{|ds^*|^2} + \Gamma^{*\mu}_{\lambda\nu} \frac{dx^\lambda}{|ds^*|} \cdot \frac{dx^\nu}{|ds^*|} = \frac{e}{m} F^{*\mu}{}_\nu \frac{dx^\nu}{|ds^*|}$$

but two differences must be pointed out:

– The scalar field σ appears on the right-hand side of equation (20).

– The differentiations are made with respect to the 5-line element $|d\sigma|$ in (20), but not to the proper time $|ds^*|$.

. To relate the constant q to $\frac{e}{m}$, one must write equation (20) in the 4-dimensional space, *i.e.* by expressing the 5-dimensional line element $|d\sigma|$ in terms of the proper time $|ds^*|$. Lorentz force can be recovered by comparing with Einstein-Maxwell theory, or by writing the newtonian limit of the equations of motion.

● *Unitary 5-velocity:*

. To write equation (20) in terms of the 4-unitary velocity $u^{*\mu}$, one must relate the unitary 5-velocity $\mathbb{U}^m$ to it.

When we derived the expression of $g_{\mu\nu}$ in terms of γ_{mn} in section 1.2.1, we found:

$$d\sigma^2 = ds^2 + \gamma_{55}\left(dx^5 + \frac{\gamma_{\mu 5}}{\gamma_{55}} dx^\mu\right)^2.$$

Since the world line of a physical particle is always time-like, the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ is known to be negative, but we do not know *a priori* the sign of $d\sigma^2$:

$$\begin{aligned} \mp |d\sigma^2| = -|ds^2| + \frac{1}{\gamma_{55}}(\gamma_{55} dx^5 + \gamma_{\mu\nu} dx^\mu)^2 \\ \Leftrightarrow \quad |d\sigma^2|(e^{-2\sigma}[\gamma_{55}\mathbb{U}^5 + \gamma_{\mu 5}\mathbb{U}^\mu]^2 \pm 1) = |ds^2| \\ \Leftrightarrow \quad \frac{|ds|}{|d\sigma|} = \sqrt{e^{-2\sigma} q^2 \pm 1} \end{aligned}$$

From now on, the upper sign will always correspond to $d\sigma^2 < 0$, and the lower one to $d\sigma^2 > 0$.

If the conformal line element $|ds^*|$ is used, one finds:

$$\frac{|ds^*|}{|d\sigma|} = e^{\sigma/2}\sqrt{e^{-2\sigma} q^2 \pm 1} \tag{21}$$

. One can then derive the formulae giving $\mathbb{U}^m$ in terms of $u^{*\mu}$ and q:

$$\mathbb{U}^\mu = \frac{dx^\mu}{|d\sigma|} = \frac{dx^\mu}{|ds^*|}\frac{|ds^*|}{|d\sigma|} \quad \Rightarrow \quad \mathbb{U}^\mu = e^{\sigma/2}\sqrt{e^{-2\sigma} q^2 \pm 1}\, u^{*\mu} \tag{22 a}$$

Let us recall that we found (19):

$$\mathbb{U}_5 = q \tag{22 b}$$

Then:

$$\begin{aligned} \gamma_{55}\mathbb{U}^5 + \gamma_{\mu 5}\mathbb{U}^\mu = \mathbb{U}_5 = q \\ \Rightarrow \quad \mathbb{U}^5 = e^{-2\sigma} q - 2\sqrt{e^\sigma \mathrm{G}}\sqrt{e^{-2\sigma} q^2 \pm 1}\, \mathrm{A}_\mu u^{*\mu} \end{aligned} \tag{22 c}$$

One can also compute:

$$\mathbb{U}_\mu = \gamma_{\mu 5}\mathbb{U}^5 + \gamma_{\mu\nu}\mathbb{U}^\nu = e^{-(\sigma/2)}\sqrt{e^{-2\sigma} q^2 \pm 1}\, u^*_\mu + 2\sqrt{\mathrm{G}}\, q\, \mathrm{A}_\mu \tag{22 d}$$

• *Equations written in the 4-dimensional space:*

Using formulae (22), the equations of motion (20) read:

$$\begin{aligned} u^{*\nu}\nabla^*_\nu u^{*\mu} = \kappa q e^\sigma \frac{|d\sigma|}{|ds|} \mathrm{F}^{*\mu}{}_\nu u^{*\nu} \\ + \left(\frac{1}{2} + q^2 e^{-\sigma}\frac{|d\sigma|^2}{|ds^*|^2}\right).(\partial^{*\mu}\sigma + u^{*\mu}u^{*\nu}\partial_\nu\sigma) \end{aligned}$$

To find Lorentz's force, one must then define:

$$\frac{e}{m} = \kappa q e^{\sigma} \frac{|d\sigma|}{|ds^*|} \tag{23 a}$$

Or, if one uses: $\kappa = 2\sqrt{G}$ (*cf.* § 1.3.4), and (21):

$$\frac{e}{m} = 2q\sqrt{\frac{e^{\sigma}\,G}{e^{-2\sigma}q^2 \pm 1}} \tag{23 b}$$

$\left(\text{One finds: } \frac{e}{m} = \kappa q \sqrt{\frac{e^{\tau}}{e^{-2\sigma}q^2 \pm 1}}\right.$ for a general conformal rescaling; $\left. cf.\ \S\ 1.3.3.\right)$ Then, $\frac{e}{m}$ is not a constant. We shall devote section 2.1.3 to this problem.

The equations of motion read then, in the 4-dimensional space:

$$u^{*\nu}\nabla^*_\nu u^{*\mu} = \frac{e}{m} F^{*\mu}{}_\nu u^{*\nu} + \left(\frac{1}{2} + \frac{e^2}{4Gm^2} e^{-3\sigma}\right).(\partial^{*\mu}\sigma + u^{*\mu}u^{*\nu}\partial_\nu\sigma) \tag{24}$$

Notice that this equation gives the usual geodesics of General Relativity, when σ is a constant. The presence of σ is then the only modification to General Relativity.

2.1.2. *Lagrangian formalism*

- The geodesics are given by the extremals of the integral: $\int L_1\, dp$ where:

$$L_1 = -\frac{|d\sigma|}{dp}, \quad \text{and} \quad p \text{ is any parameter describing the curve.} \tag{25}$$

- We know that the geodesics can also be obtained by extremizing the integral: $\int L_2\, dp$ where:

$$L_2 = \frac{1}{2}\gamma_{mn}\dot{x}^m\dot{x}^n, \quad \text{and} \quad \dot{x}^m = \frac{dx^m}{dp}, \tag{26}$$

p being an *affine* parameter here [*i. e.* a curvilinear abscissa on the geodesic in the 5-dimensional space]. Since the metric does not depend on x^5, Noether's theorem ensures that there is a conserved quantity. It is given by the Euler-Lagrange equation:

$$\frac{d}{dp}\left(\frac{\partial L_2}{\partial \dot{x}^5}\right) = \frac{\partial L_2}{\partial x^5} = 0 \quad \Rightarrow \quad \frac{\partial L_2}{\partial \dot{x}^5} = \text{Cont.}$$

If p is taken to be the proper time: $dp = |d\sigma|$, one can then define:

$$q = \mathbb{U}_5 = \gamma_{55}\,\mathbb{U}^5 + \gamma_{\mu 5}\,\mathbb{U}^\mu = \text{Cont.}$$

• Provided that we define: $\dfrac{e}{m} = \kappa q e^{\sigma} \dfrac{|d\sigma|}{|ds^*|}$ and express the 5-unitary velocity $\mathbb{U}^m$ in terms of the 4-unitary velocity $u^{*\mu}$ and the constant q, the other equations:

$$\frac{d}{dp}\left(\frac{\partial L_2}{\partial \dot{x}^{\mu}}\right) = \frac{\partial L_2}{\partial x^{\mu}}$$

lead to the generalized geodesics equation (24) given in the previous paragraph (§ 2.1.1).

• One can write the lagrangian in a more usual form, looking like the one of General Relativity for a charged particle. Let us define L_3 by a Legendre transformation on L_1:

$$L_3 = L_1 - \left(\frac{\partial L_1}{\partial \dot{x}^5}\right)\dot{x}^5 \tag{27}$$

Since $\dfrac{\partial L_1}{\partial x^5} = 0$, we know that $\dfrac{\partial L_1}{\partial \dot{x}^5}$ is a constant. It can be related to q:

$$\mp {L_1}^2 = \gamma_{mn} \dot{x}^m \dot{x}^n \quad \Rightarrow \quad \mp L_1 \frac{\partial L_1}{\partial \dot{x}^5} = \gamma_{55} \dot{x}^5 + \gamma_{\mu 5} \dot{x}^{\mu} = q \frac{|d\sigma|}{dp}$$

Then:

$$\frac{\partial L_1}{\partial \dot{x}^5} = \pm q.$$

Since L_1 depends on $(x^1, \ldots, \dot{x}^1, \ldots, \dot{x}^4, \dot{x}^5)$, we know that L_3 depends on $(x^1, \ldots, x^4, \dot{x}^1, \ldots, \dot{x}^4, \pm q)$. But q is a constant, then it can be considered as a parameter labelling the action L_3, depending only on the four coordinates $x^1, \ldots, x^4$ and their derivatives. (This is the Maupertuis principle, and is discussed for instance in [11], p. 159.)

Since: $q \dfrac{|d\sigma|}{dp} = \gamma_{55} \dot{x}^5 + \gamma_{\mu 5} \dot{x}^{\mu}$, this lagrangian reads:

$$L_3 = -\frac{|d\sigma|}{dp} \mp q \dot{x}^5 = \mp\left([e^{-2\sigma} q^2 \pm 1] \frac{|d\sigma|}{dp} - 2\sqrt{G}\, q A_{\mu} \dot{x}^{\mu}\right)$$

$$\Rightarrow \quad L_3 = \mp e^{-(\sigma/2)} \sqrt{e^{-2\sigma} q^2 \pm 1} \left(\frac{|ds^*|}{dp} - \frac{e}{m} A_{\mu} \dot{x}^{\mu}\right), \tag{28}$$

where: $\dfrac{e}{m} = 2q \sqrt{\dfrac{e^{\sigma} G}{e^{-2\sigma} q^2 \pm 1}}$.

Remark. — Notice that p is *any* parameter describing the trajectory, and not an affine one. This is fortunate, because an affine parameter for the geodesics of the 5-dimensional space is not affine for the 4-geodesics!

• When σ is constant, e and m are constant, and the lagrangian reads:

$$K(-m\sqrt{-g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}+e\,A_{\mu}\dot{x}^{\mu}),\quad \text{where K is a constant.}$$

The lagrangian of Einstein-Maxwell theory is recovered. Then, in the limiting case corresponding to σ constant, the extremals of the integral are the usual trajectories of charged particles in General Relativity.

2.1.3. *The sign of* $d\sigma^2$ *and the variation of* $\frac{e}{m}$

• Formula (23 *b*) shows that in Jordan-Thiry theory, $\frac{e}{m}$ is not a constant if σ varies (and no conformal rescaling can make it constant). If e is defined as the charge of the particle, m must be considered as an *effective* mass, depending on σ. If m_0 is the mass of the particle when σ vanishes, one finds:

$$m=m_0\,e^{-(\sigma/2)}\sqrt{\frac{e^{-2\sigma}q^2\pm 1}{q^2\pm 1}} \tag{29}$$

• We already know that the upper sign corresponds to $d\sigma^2<0$ (time-like trajectories in 5 dimensions), and the lower one to $d\sigma^2>0$ (space-like trajectories, *i.e.* tachyons in 5 dimensions). When the geodesics of the 5-dimensional manifold are analysed from the point of view of the effective 4-dimensional space-time, they appear as curves (generally not geodesics) whose space or time-like character depends on the local value of the scalar field σ. We have the following three cases:

. $d\sigma^2>0$ then $\dfrac{e^2}{4Gm^2}=\dfrac{q^2 e^{\sigma}}{q^2 e^{-2\sigma}-1}$, and m^2 is given by curve (a) below.

Notice that if $\sigma<\ln|q|$, then $m^2>0$ and this branch could describe usual particles since m^2 can be small if $(\ln|q|-\sigma)$ is small. This is indeed the choice made by Thiry [4], but we will see in the next paragraphs that it leads to experimental contradictions. Notice that if σ becomes larger than $\ln|q|$, m^2 becomes negative ([4]) and the corresponding trajectory becomes tachyonic in our 4-dimensional space-time.

. $d\sigma^2<0$ then $\dfrac{e^2}{4Gm^2}=\dfrac{q^2 e^{\sigma}}{q^2 e^{-2\sigma}+1}$, and m^2 is given by curve (b) below.

Then m^2 is positive for all values of the scalar field σ. However, equations (9 a), (9 b) show that the theory is close to General Relativity

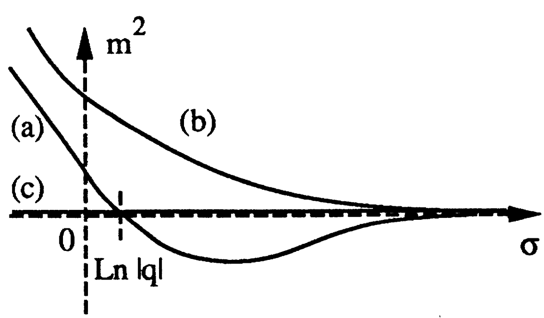
([4]) Actually, $m^2<0$ means that ds^{*2} becomes spacelike, since equation (23 a) was derived assuming $ds^{*2}<0$.

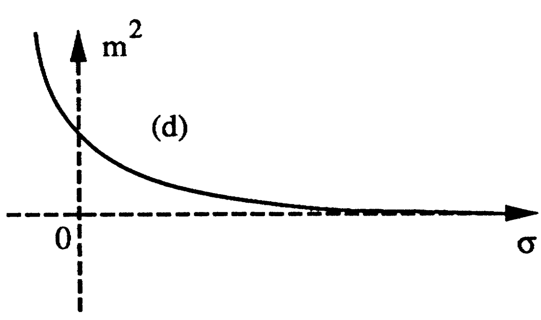
only if $e^{3\sigma} \approx 1$ [equation (31 *b*) will also prove that the electrostatic field created by a charged particle is smaller than the observed one by a factor $e^{3\sigma}$]. Therefore, the scalar field σ must be very small near the Earth. But for $\sigma \approx 0$, we find $m^2 \approx \frac{q^2+1}{q^2} \cdot \frac{e^2}{4G}$ which corresponds to extremely heavy masses, if e does not vanish $\left[\text{a particle with charge 1 in electron units should have a mass } m > \frac{e}{2\sqrt{G}} \approx 5 \times 10^{20}\ \text{MeV}/c^2\right]$. Hence, these 5-dimensional time-like geodesics cannot describe the motion of usual particles.

. $d\sigma^2 = 0$. Then we have two branches:

The first corresponds to $ds^2 = 0$ and $\gamma_{55}\, dx^5 + \gamma_{\mu\nu}\, dx^\mu = 0$; this describes massless neutral particles and is given by curve (*c*) below [the horizontal axis].

The second corresponds to $\frac{1}{q} = 0$, and $ds^2 \neq 0$. This describes massive particles with infinite q, but with finite charge: $\frac{e^2}{4\,G\,m^2} = e^{3\sigma}$. The behaviour of these trajectories looks like the previous case (*b*), and is given by curve (*d*). They also lead to extremely heavy masses when $\sigma \approx 0$, and cannot describe the motion of usual charged particles.





2.2. Action of matter on geometry

The action of matter on geometry is described by the right hand side of Einstein equations – or of their generalization (5) –. Assuming only that this r.h.s. is a symmetrical tensor $\mathbb{T}_{mn}$ (using the 5-dimensional formalism), we will write these equations in full generalty. We will also show how to use these equations to relate the parameters appearing in the Schwarzschild solution (13) to the matter distribution of a star. Finally, we will study the possible descriptions of a perfect fluid.

2.2.1. *Field equations in presence of matter*

• In the 5-dimensional formalism, they can be written:

$$\mathbb{E}_{mn} = 8\pi \mathbb{G} \mathbb{T}_{mn} \tag{30}$$

Using the formulae of sections 1.2.2 and 1.3.3, one can write them in terms of 4-dimensional quantities:

$$E^*_{\mu\nu} = 8\pi G\,(T^{*\text{e.m.}}_{\mu\nu} + \tau^*_{\mu\nu} + T^{*\text{mat.}}_{\mu\nu}) \tag{31a}$$

$$\nabla^*_\nu H^{*\mu\nu} = 4\pi J^{*\mu} \tag{31b}$$

$$\Box^* \sigma = G H^*_{\mu\nu} F^{*\mu\nu} + S \tag{31c}$$

where:

$$T^{*\text{mat.}}_{\mu\nu} = \mathbb{T}_{\mu\nu} - \kappa A_\mu \mathbb{T}_{\nu 5} - \kappa A_\nu \mathbb{T}_{\mu 5} + \kappa^2 A_\mu A_\nu \mathbb{T}_{55} \tag{32a}$$

$$J_\mu = \kappa(\mathbb{T}_{\mu 5} - \kappa A_\mu \mathbb{T}_{55}) \tag{32b}$$

$$S = \frac{8\pi G}{3}(T^{*\mu\text{mat.}}_{\mu} - 2e^{-3\sigma}\mathbb{T}_{55}) \tag{32c}$$

(the constant κ was found in section 1.3.4: $\kappa = 2\sqrt{G}$.)

[In an adapted frame, the formulae giving $T^{*\text{mat.}}_{\mu\nu}$ and J^*_μ are obvious:

$$T^{*\text{mat.}}_{\underline{\mu\nu}} = \mathbb{T}_{\underline{\mu\nu}} \text{ and } J^*_{\underline{\mu}} = \kappa\, \mathbb{T}_{\underline{\mu 5}}.]$$

• The stress-energy tensor $T^{*\text{mat.}}_{\mu\nu}$ depends on the properties of matter (one can consider a perfect fluid, for example), and is given by experiment. Similarly, the electric current J^*_μ depends on the charge distribution, and can be measured. But our usual knowledge of matter does not allow us to find the value of the scalar S. The only property we know, is that S must vanish where matter and charges are absent. Then, the only way to know the value of this scalar is to contruct models of matter in the 5-dimensional space, and derive the corresponding S.

• Equations (31) derive from the lagrangian:

$$L = \frac{1}{16\pi G}\mathbb{R}\sqrt{-\gamma} + L_{\text{mat.}}$$

which can be written in terms of 4-dimensional quantities:

$$L=\frac{1}{16\pi G}\left(R^*-\frac{3}{2}\partial_\rho\sigma\,\partial^{*\rho}\sigma-GH^*_{\mu\nu}F^{*\mu\nu}\right)\sqrt{-g^*}+L_{mat}. \tag{33}$$

The only difference here with section 1.3.4 is the lagrangian describing matter $L_{mat.}$. It depends on the description of matter one chooses in the 5-dimensional space. But one can also choose the usual $L_{mat.}$ of General Relativity (depending on $g^*_{\mu\nu}$ but not on the scalar field σ), in which case $S=0$.

2.2.2. *Field equations inside a star*

In this section, we write the field equations inside a static and spherically symmetrical star. The metric (12) will be used, and the constants b and d introduced for the generalized Schwarzschild solution (13) will be related to the matter distribution of the star.

Using the calculations of section 1.4.2, the field equations (31) read now:

$$\begin{gathered}2\lambda''+\lambda'^2+\nu'^2+3\sigma'^2=32\pi G e^{-\nu}p\\(e^\lambda)''=2+32\pi G e^{\lambda-\nu}p\\(\nu' e^\lambda)'=8\pi G e^{\lambda-\nu}(\mu+3p)\\(\sigma' e^\lambda)'=S e^{\lambda-\nu}\end{gathered}$$

The last two allow us to find the functions $\nu' e^\lambda$ and $\sigma' e^\lambda$ inside the star. Their values at the star radius R_* give the constant b and d, since μ, p and S vanish outside the star:

$$b=8\pi G\int_0^{R_*}(\mu+3p)e^{\lambda-\nu}dr+b_0 \tag{34a}$$

$$d=\int_0^{R_*}S e^{\lambda-\nu}dr+d_0 \tag{34b}$$

where b_0 and d_0 are additive constants.

To assure that the metric and the scalar field do not diverge when $r\to 0$, we must take:

$$b_0=d_0=0$$

Notice that the scalar field σ is completely determined by the properties of the star, and is not an arbitrary field of the theory.

2.2.3. *Jordan-Thiry description of a perfect fluid*

This description is the natural generalization of a perfect fluid in the 5-dimensional space:

$$\mathbb{T}_{mn}=\pm(\mu+p)\,\mathbb{U}_m\,\mathbb{U}_n+p\,\gamma_{mn} \tag{35}$$

where μ is the matter density, p is the pressure, and the sign $\pm$ is introduced because $\mathbb{U}_m \mathbb{U}^m$ may be positive.

We can use the expression of the 5-unitary velocity in terms of u^*_μ and q (22) to find:

$$E^*_{\mu\nu} = 8\pi G (T^{*\text{e.m.}}_{\mu\nu} + \tau^*_{\mu\nu} + [\mu + p] e^{-\sigma} [1 \pm e^{-2\sigma} q^2] u^*_\mu u^*_\nu + p e^{-\sigma} g^*_{\mu\nu})$$

Then, we must redefine μ and p, to obtain the usual stress-energy tensor of a perfect fluid:

$$\mu^* + p^* = (\mu + p) e^{-\sigma} (1 \pm e^{-2\sigma} q^2)$$
$$p^* = e^{-\sigma} p$$

The fields equations are then given by (31), where:

$$T^{*\text{mat.}}_{\mu\nu} = (\mu^* + p^*) u^*_\mu u^*_\nu + p^* g^*_{\mu\nu} \tag{36a}$$

$$J^*_\mu = \rho u^*_\mu = 2q \sqrt{\frac{e^\sigma G}{e^{-2\sigma} q^2 \pm 1}} (\mu^* + p^*) u^*_\mu \tag{36b}$$

$$S = -\frac{8\pi G}{3} (\mu^* - p^*) - \frac{4\pi}{3} e^{-3\sigma} \frac{\rho^2}{\mu^* + p^*} \tag{36c}$$

Remark: Note that the redefinition of μ and p gives for $\dfrac{\rho}{\mu^* + p^*}$ the same formula as (23 *b*):

$$\frac{\rho}{\mu^* + p^*} = 2q \sqrt{\frac{e^\sigma G}{e^{-2\sigma} q^2 \pm 1}} \tag{37}$$

Hence, the Jordan-Thiry description of a perfect fluid, and the 5-dimensional geodesics considered as equations of motion, lead to the same formula for $\dfrac{e}{m}$. The description of matter fields in the 5-dimensional manifolds is then consistent with the description of particles moving in this space.

2.2.4. *Other descriptions of a perfect fluid*

• A fluid which appears to be "perfect" in the 4-dimensional space can be described in several ways. The most straightforward description is the one that we just discussed. However, the scalar S appearing in equations (37) is not given by our usual knowledge of matter. Each description of matter in the 5-dimensional space leads to a different value of this scalar. For instance, the following choice of $\mathbb{T}_{mn}$:

$$\mathbb{T}_{mn} = \pm (\mu + p) \mathbb{U}_m \mathbb{U}_n + p \gamma_{mn} - \frac{\mu}{2} X_{mn} \tag{38a}$$

where

$$X_{mn} = \frac{\gamma_{m5}\,\gamma_{n5}}{\gamma_{55}} \quad \text{in any adapted coordinate system} \tag{38 b}$$

leads to the usual 4-dimensional $T^{*\text{mat.}}_{\mu\nu}$ (36 *a*), and to equation (37) for the ratio $\frac{\rho}{\mu^* + p^*}$, but S vanishes for a neutral and pressureless fluid – whereas the previous description of matter (35) gives a value of S depending on μ^* –.

[Notice that even without using the equation of geodesics, a description of matter in the 5-dimensional space leads to the variable value of $\frac{e}{m}$ given by equation (23 *b*)]

• Yet another possibility is to describe matter as in Einstein-Maxwell theory (in the 4-dimensional space), *i.e.* to write that $L_{\text{mat.}}$ does not depend on the scalar field σ. This last choice leads to a vanishing S and to a constant value of $\frac{e}{m}$. [For example, the lagrangian of an isentropic perfect fluid in General Relativity is given in [22], and can be used in equation (33).]

2.3. Impossibility of a 5-dimensional description of matter

We already saw that geodesics of the 5-dimensional space with $d\sigma^2 \leqq 0$ (time-like or light-like) could not describe usual particles. The only possibility left is to assume $d\sigma^2 > 0$. This corresponds to 5-dimensional tachyons. Such a particle would appear as timelike in dimension 4 if the value of the scalar field σ is small enough. This was the hypothesis made by Jordan and Thiry.

Nowadays, it is clear that quantum field theorists would be reluctant to build a unified field theory out of tachyonic particles. Even in the case where such a construction would be possible, the hypothesis that usual particles correspond to 5-dimensional tachyons should be rejected for at least two (experimental) reasons:

(1) Charged particles acquire a gravitational mass different from their inertial mass, and this difference is too big.

(2) It leads to variations of the effective mass which are not compatible with what we observe.

The first problem has already been noticed in the past as a probable difficulty of the theory, but we have not found in the literature any comparison with experimental data. Moreover, only the Jordan-Thiry perfect fluid was considered in the past to describe the Sun. In section 2.3.1, we show that any description of the Sun leads to experimental inconsistencies. Section 2.3.2 is devoted to the second problem, which is a consequence of the choice of 5-dimensional tachyons to describe usual particles. [A third problem of less importance is discussed in Appendix A: the precession of Mercury perihelion.]

2.3.1. *Gravitational masses of charged particles*

• *Newtonian limit of geodesic equation:*

. Newton's gravitation law can be obtained by writing the equations of motion for a static neutral particle (*cf.* [23], § 3.4). The generalized geodesics equation reads now:

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{tt} = \frac{1}{2} g^{\lambda r} \sigma'$$

If we use the spherically symmetrical metric (12), we find for the radial variable:

$$\frac{d^2 r}{dt^2} = \frac{\sigma' - \nu'}{2} e^{2\nu} \quad \Rightarrow \quad \frac{d^2 r}{dt^2} = \frac{d-b}{2r^2}$$

Newton's theory is then obtained if:

$$2\mathrm{GM} = b - d \tag{39}$$

[b and d are the constants introduced in equations (13)].

. If the generalized geodesics equation is written for a static charged particle, one finds for the radial coordinate:

$$\frac{d^2 r}{dt^2} = \frac{\sigma' - \nu'}{2} + \frac{e^2 \sigma'}{4\mathrm{G}m^2} + \frac{e}{m}\mathrm{F}_{rt} = \frac{d-b}{2r^2} + \frac{e^2 d}{4\mathrm{G}m^2 r^2} + \frac{e}{m}\mathrm{E}_r$$

[where E_r is the radial component of the electric field]. The first term is the usual gravitational term $\frac{-\mathrm{GM}}{r^2}$, but the second term is also an interaction in $\frac{1}{r^2}$ (this had already been noticed in [1], and more recently in [15]). Since the gravitational mass of charged particles is known to be very close to their inertial mass, this second term must be negligible with respect to the first one. Then, we find an upper bound for the derivative of σ:

$$|\sigma'| \ll \frac{2\mathrm{G}m^2}{e^2} |\sigma' - \nu'| \tag{40}$$

For usual charged particles, $\frac{e^2}{4\mathrm{G}m^2}$ is very large with respect to 1. For an electron, its value is about 10^{42}. Hence, equation (40) can be written:

$$|d| \ll 10^{-42} |b| \tag{41}$$

The ratio $\frac{d}{b}$ depends upon the description that we adopt for the 5-dimensional perfect fluid.

. The Jordan-Thiry description of a perfect fluid (35) gives $\frac{|d|}{|b|} \approx 1$.

The second description we studied (38) gives $\frac{|d|}{|b|}$ with the order of $\frac{p}{\mu}$ [which is about 10^{-6} at the Sun center]. Then, these descriptions are not compatible with equation (41).

. It can be seen that a description leading to $\mathrm{S}=0$ is also incompatible with what we measure. Indeed, the electromagnetic corrections (§ 1.4.4) imply that the condition $|\mathscr{E}_\mathrm{M} - \mathscr{E}_\mathrm{E}| \ll 10^{-42}\,\mathrm{M}$ should be satisfied, if $\mathscr{E}_\mathrm{E}$ and $\mathscr{E}_\mathrm{M}$ denote the electric and the magnetic energies of the Sun respectively. This is certainly not the case experimentally.

• *Conclusion of section* 2.1.3:

If the equations of motion are supposed to be given by the geodesics of the 5-dimensional space, the gravitational masses of charged particles are found to be very different from their inertial masses, because of a "5th force" due to the scalar field.

If the motion of usual particles is described as usual by 4-dimensional geodesics with a Lorentz force, this problem obviously disappear, although the scalar field is still present, unified to gravity and electromagnetism (in the non-physical 5-dimensional space).

2.3.2. *The variation of* $\frac{e}{m}$

• If one assumes that the geodesics of the 5-dimensional space give the equations of motion, one finds equation (23 b): $\frac{e}{m} = 2q\sqrt{\frac{e^\sigma \mathrm{G}}{e^{-2\sigma}q^2 \pm 1}}$. The same formula is found for $\frac{\rho}{\mu^* + p^*}$ [equation (37)] if one uses a 5-dimensional description of matter fields, like the Jordan-Thiry perfect fluid (35) or the description (38), Then, these ratios are variable, and one must check if these variations are small in the Solar system, where $\frac{e}{m}$ is found to be (almost) constant.

. We showed in section 2.1.3 that only space-like (*i.e.* tachyonic) geodesics of the 5-dimensional manifold can describe the motion of usual (light) particles. Therefore, the minus sign must be used in the square root (23 *b*), (37).

. If σ is close enough to 0, so that $m \approx m_0$ [where m_0 is the mass of the particle when σ vanishes (29)], one can compute the constant q for electron and proton, for example:

$$\frac{e}{m} \approx \frac{e}{m_0} = 2\sqrt{G}\frac{q}{\sqrt{q^2-1}} \Rightarrow |q| \approx \frac{1}{\sqrt{1-\frac{(4\,Gm^2}{e^2}}} \approx 1 + \frac{2\,Gm^2}{e^2}$$

For electron, one finds:

$$q \approx -1 - 4 \times 10^{-43} \approx -1$$

For proton:

$$q \approx 1 + 2 \times 10^{-36} \approx 1$$

Then q *is very close to* 1 *or* -1 *for usual charged particles*.

• Since $q^2 - 1$ is very small, $e^{-2\sigma} q^2 - 1 \approx (q^2 - 1) - 2\sigma$ may be very different from $q^2 - 1$, even when σ is close to 0. For a small σ, equation (29) yields:

$$m \approx m_0 \sqrt{\frac{(q^2-1)-2\sigma}{q^2-1}} \tag{42}$$

When we derived the generalized Schwarzschild solution (13), we found that σ varies like $\frac{1}{r}$:

$$\sigma = \frac{d}{a} \ln\left(1 - \frac{a}{r}\right) \approx -\frac{d}{r}$$

It must be negligible with respect to $q^2 - 1$, so that m is almost a constant in the Solar system. Therefore, we find an upper bound for σ:

$$\sigma \ll (q^2 - 1) \tag{43}$$

The smallest value of $q^2 - 1$ is found for electron:

$$q^2 - 1 \approx \frac{4G\,m_e^2}{e^2} \approx 10^{-42}$$

Then, the condition (43) reads:

$$\frac{|d|}{r} \ll 10^{-42} \tag{44}$$

. This condition is less restrictive than (41), because b is almost equal to the Schwarzschild radius of the star [*cf.* equation (14)], and is negligible with respect to usual values of r. However, it also leads to experimental inconsistencies for the Jordan-Thiry perfect fluid, and for the other possible descriptions. Actually, if r is an astronomical unit (*i.e.* the distance Earth-Sun), one finds: $\frac{GM}{r} \approx 10^{-8}$. Then the results of section 2.3.1 can be used, with multiplying the upper bonds by 10^8.

. If one uses a description of matter leading to $S=0$, the electromagnetic corrections must be taken into account. The condition (44) reads:

$$|\mathscr{E}_M - \mathscr{E}_E| \ll 10^{-34} M \tag{45}$$

This condition is certainly not verified either.

• *Conclusion of section* 2.3.2

Both the assumption that particles move on the geodesics of the 5-dimensional manifold, and the descriptions of matter fields in this space (35), (38), lead to the same formula for $\frac{e}{m}$, which depends on the scalar field σ, and whose variation is inconsistent with what is measured.

Of course, this problem is cured when one chooses the usual description of matter in the 4-dimensional space, like in Einstein-Maxwell Theory. In this case, $\frac{e}{m}$ is defined as a constant, and matter fields are assumed not to depend on the scalar field σ.

APPENDIX A
PERIHELION SHIFT OF PLANETARY ORBITS

In order to rule out the original theory of Jordan-Thiry which incorporates artificially matter fields in the 5-dimensional space, we need only one experimental contradiction, and we have already underlined two of them in section 2.3. However, we want to discuss here the problem of the perihelion shift of planetary orbits, because it is sometimes claimed in the literature that *this* is the experimental test which rules out the theory. The aim of this section is to clarify three points:

– The previous calculation by K. Just [13] is in fact incorrect.

– If the Sun is not described as a Jordan-Thiry Perfect fluid, it is possible to predict a perihelion shift which is consistent with experiment. Hence this test is not sufficient to rule out the original theory.

– If planets are assumed to follow usual 4-dimensional geodesics, one finds the *same* prediction as in General Relativity, although the scalar field modifies the Schwarzschild solution.

In this section, we use the results of section 1.4, *i.e.* the generalized Schwarzschild solution. The 4-dimensional line element is given by (12), and no stars (*) are written, though the conformal metric is used. We first study the predictions of the theory when 5-dimensional geodesics are assumed to give the equations of motion. The last paragraph of this section will be devoted to the predictions when the 4-dimensional geodesics are considered.

A.1. Expression of the perihelion shift

We study the motion of a (neutral) planet around a star, assuming that the equations of motion are given by the 5-dimensional geodesics equation:

$$u^\nu \nabla_\nu u^\mu = \frac{u^\mu u^\nu \partial_\nu \sigma}{2} + \frac{\partial^\mu \sigma}{2}$$

If $\theta = \frac{\pi}{2}$, they lead to:

$$\frac{dt}{|ds|} = \mathrm{E}\, e^{-\nu + (\sigma/2)} \quad \text{where E is a constant}$$

$$\frac{d\varphi}{|ds|} = \mathrm{J}\, e^{\nu - \lambda + (\sigma/2)} \quad \text{where J is a constant}$$

$$\left(\frac{dr}{|ds|}\right)^2 = \mathrm{E}^2 e^{-\nu} - \mathrm{J}^2 e^{\nu - \lambda} - e^{-\sigma}$$

If one looks for a precessing ellipse: $r = \frac{\mathrm{L}}{1 + e . \cos \omega\varphi}$ (where e is the eccentricity and L the *semilatus rectum*), one finds at first approximation in $\frac{b}{\mathrm{L}}$ or $\frac{d}{\mathrm{L}}$:

$$\omega = 1 - \frac{3b + d}{2\mathrm{L}}$$

Therefore, the perihelion shift is given by:

$$\Delta\varphi = \frac{3b + d}{\mathrm{L}} \pi = \frac{6\pi \mathrm{MG}}{\mathrm{L}} \left(\frac{3b + d}{3b - 3d}\right)$$

since $2\mathrm{MG} = b - d$ [equation (39)].

This is the General Relativity result with a corrective factor

$$f = \frac{3b + d}{3b - 3d}. \tag{46}$$

A.2. The Sun as a Jordan-Thiry perfect fluid

• If one describes the Sun as a Jordan-Thiry perfect fluid, one finds:

$$S = -\frac{8\pi G}{3}(\mu - p) - \frac{4\pi}{3}\frac{\rho^2}{(\mu+p)}e^{-3\sigma} \tag{47}$$

But the Sun in on average neutral, then the second term can be neglected. Equation (34 *b*) allows us to compute the constant d introduced in the generalized Schwarzschild solution (13):

$$d = -\frac{8\pi G}{3}\int_0^{R*}(\mu - p)\,e^{\lambda-\nu}\,dr \tag{48}$$

Formula (34 *a*) also gives b in terms of μ and p. Then, the generalized Schwarzschild solution is completely determined.

Notice that since $2\,MG = b - d$, the solar mass is not given by an integral of μ as usual, but of:

$$\left(\mu + 3p - \frac{S}{8\pi G}\right) = \frac{4}{3}(\mu + 2p)$$

This comes only from the fact that the radial coordinate r is not the standard one.

• Formula (46) gives the factor f correcting the General Relativity result for Mercury perihelion shift. Using the values of b and d corresponding to the Jordan-Thiry description of a perfect fluid, one finds:

$$\frac{2}{3} \leqq f \leqq \frac{13}{15} \tag{49}$$

The higher limit is obtained for an extremely hot star $\left(p \approx \frac{\mu}{3}\right)$, while the lower one corresponds to a "cold" star ($p \approx 0$).

• The value of f corresponding to the Sun is close to the lower bound of this interval because it is "cold". Actually, even at the center of the Sun, where the temperature is about 15×10^6 K, the pressure is negligible with respect to the matter density:

$$p \ll \mu \qquad \left[\frac{p}{\mu} \text{ is about } 10^{-6}, \text{ there}\right].$$

Thus, the Jordan-Thiry description of matter leads to:

$$f \approx \frac{2}{3} \qquad \left[f - \frac{2}{3} \text{ is about } 10^{-6}\right].$$

• Experimental results for Mercury ([24], [25]) give:

$$f = 1.003 \pm 0.005 \tag{50}$$

Hence, if the Sun is considered as a neutral perfect fluid, the Jordan-Thiry description of matter predicts a value of f which is not compatible with experiment. K. Just [13] pointed out this problem, but his value for $f > \frac{5}{4}$ was wrong. (He worked on an earlier version of Jordan-Thiry theory, with a variable gravitational "constant" G; but our result is also valid in his case.)

• One may try to improve this numerical result for f by considering the electromagnetic field of the Sun. To get a rough estimate of the electromagnetic corrections, one can use the results of section 1.4.4 for a sherically symmetrical field, which becomes negligible for r greater than a given value R. One can take for R the typical length of the problem, *i.e.* the *semilatus rectum* L of Mercury orbit.

$$b = 8\pi G \int_0^{R*} (\mu + 3p) e^{\lambda - \nu} dr + 2G \int_0^{L} e^{\lambda - \nu + 3\sigma} (\mathbf{E}^2 + \mathbf{B}^2) dr \quad (51\,a)$$

$$d = -\frac{8\pi G}{3} \int_0^{R*} (\mu - p) e^{\lambda - \nu} dr + 2G \int_0^{L} e^{\lambda - \nu + 3\sigma} (\mathbf{B}^2 - \mathbf{E}^2) dr \quad (51\,b)$$

For the Sun, the pressure p can be neglected in the first integral. If one denotes the electric and the magnetic energies of the Sun respectively as $\mathscr{E}_E$ and $\mathscr{E}_M$, one finds that f is shifted towards 1 if: $2\mathscr{E}_M \geqq \mathscr{E}_E$.

This condition is not absurd inside the Sun, and is actually verified outside it. Then, electromagnetic corrections draw the corrective factor f towards 1. But to find an experimentally consistent perihelion shift, $(\mathscr{E}_M - \mathscr{E}_E)$ should be of the same order of magnitude as $\frac{M}{6}$, which is really too large! Therefore, electromagnetic corrections are not large enough to lead to an experimentally compatible perihelion shift.

• Another way to improve the numerical value of f is to consider a cosmological constant in the equations. One can use the results of section 1.4.5 to compute f, at first order in Λ:

$$f = \frac{3b + d}{3b - 3d} + \frac{4\Lambda L^4}{9(b-d)^2} \quad (52)$$

For our "cold" sun, it reads:

$$f = \frac{2}{3}\left(1 + \frac{\Lambda L^4}{6G^2 M^2}\right) \quad (52')$$

(If one calculates the cosmological constant influence on the perihelion shift *in General Relativity*, one obtains a corrective factor: $1 + \frac{\Lambda L^4}{6G^2M^2}$.

Hence, the cosmological correction in Jordan-Thiry theory is exactly the same as the General Relativity one, the extra factor $\frac{2}{3}$ being a consequence of the Jordan-Thiry description of a perfect fluid.$\Big)$

The corrective factor f to General Relativity can be drawn closer to 1 if $\Lambda>0$, *i.e.* if the vacuum quantum energy is positive, as it is thought to be. With our sign conventions, a positive cosmological constant corresponds to gravitational forces vanishing faster than $\frac{1}{r^2}$ for large r $\Big($as if the graviton had a mass $\sqrt{2\Lambda}\Big)$. But the cosmological correction cannot lead to an experimentally compatible perihelion shift, because the matter density of the Universe would have to be negligible with respect to $\frac{\Lambda}{8\pi G}$, which is clearly not the case.

A.3. The Sun as another perfect fluid

If we use other descriptions of the perfect fluid, for example the ones given in section 2.2.4, the conclusions are modified. In the particular case where X_{mn} is given by (38 *b*), one finds $S=\frac{8\pi G}{3}p$, and the value of f is now in the interval: $1\leqq f\leqq\frac{55}{51}$. (Actually $f\approx 1$ for a "cold" sun such as ours.) If a description of the perfect fluid leading to $S=0$ is used, one finds $d=0$ and $f=1$ like in General Relativity. This problem of the value of the perihelion shift could therefore be cured by a (somewhat arbitrary) modification of the expression of $\mathbb{T}_{mn}$. However, the electromagnetic corrections involve the ratio $\frac{\mathscr{E}_M-\mathscr{E}_E}{M}$ which may be of the same order of magnitude as the experimental bounds (50). Hence, this study of Mercury perihelion shift may rule out the original theory, if one gets a good evaluation of $(\mathscr{E}_M-\mathscr{E}_E)$ or if the experimental uncertainties are lowered. But in any case, this possible experimental inconsistency would be far less "spectacular" than the two ones that we pointed out in section 2.3.

A.4. Perihelion shift when planets are assumed to follow 4-geodesics

• *One* experimental inconsistency suffices to rule out a theory. On the contrary, one must check *all* the exprimental results to show that it may be correct. Nowadays, the number of experimental confirmations of General Relativity is large (*cf.* [25]), and we do not claim that Mercury perihelion advance is a sufficient test. However, we will devote this paragraph to this problem, in order to show that Kaluza-Klein-Jordan-Thiry theory predicts the *same* value as General Relativity if planets are assumed to follow usual 4-geodesics, although the scalar field σ modifies the Schwarzschild solution.

• Actually, if the 4-dimensional line-element ds^2 is written in the isotropic form:

$$ds^2 = -\left(1 - 2\frac{GM}{r} + 2\beta\left(\frac{GM}{r}\right)^2 + \ldots\right)dt^2 + \left(1 + 2\gamma\frac{GM}{r} + \ldots\right)[dr^2 + r^2\,d\Omega^2]$$

where β and γ are the *P*arametrized *P*ost *N*ewtonian parameters, it is well known that the perihelion advance of a planetary orbit is the one predicted by General Relativity multiplied by the corrective factor $\frac{2-\beta+2\gamma}{3}$.

• Let us rewrite the line-element (14) in its isotropic form by redefining the radial coordinate. We find:

$$\begin{aligned} ds^2 &= -\left(\frac{1-(a/4r)}{1+(a/4r)}\right)^{2b/a} dt^2 \\ &\quad + \left(1 - \frac{a}{4r}\right)^{2(a-b)/a}\left(1 + \frac{a}{4r}\right)^{2(a+b)/a}[dr^2 + r^2\,d\Omega^2] \\ &= -\left(1 - \frac{b}{r} + \frac{b^2}{2r^2} - \frac{b(3b^2+d^2)}{16r^3} + \ldots\right)dt^2 \\ &\quad + \left(1 + \frac{b}{r} + \frac{3(b^2-d^2)}{8r^2} + \ldots\right)[dr^2 + r^2\,d\Omega^2] \end{aligned} \tag{53}$$

Hence, when planets are assumed to follow 4-geodesics, one finds:

$$b = 2\,GM \qquad [\text{instead of (39): } b - d = 2\,GM] \tag{54}$$

and:

$$\beta = \gamma = 1$$

Therefore, one finds *exactly* the same prediction as General Relativity.

• The presence of $\sigma \approx -\frac{d}{r}$ (13 *c*) modifies the Schwarzschild solution only at order $\frac{1}{r^3}$ for g_{tt} and $\frac{1}{r^2}$ for g_{rr}. Moreover, the corrections at these orders involve the ratio $\frac{d^2}{b^2} \propto \left(\frac{\mathcal{E}_M - \mathcal{E}_E}{M}\right)^2$ [since we consider now the usual

4-dimensional description of matter, S vanishes in equation (31 c), and the source of σ is only the electromagnetic field.] Hence the modifications of Einstein-Maxwell theory created by the scalar field σ not only occur at higher orders than the post-Newtonian ones, but are also very small!

• Let us underline that the other experimental predictions involving only the parameters β and γ will be the same as in Einstein-Maxwell theory. For instance, the deflection of light by the Sun, which depends on γ. Actually, the vacuum behaves for electromagnetic waves like a material body with electric permittivity $\varepsilon = e^{3\sigma}$ and magnetic permeability $\mu = e^{-3\sigma}$, so that $\varepsilon\mu = 1$ (*cf.* section 1.3.4). Therefore, the speed of light remains equal to c, even if σ is not constant, and there is no deflection caused by a variation of index – like in an optical lens –.

APPENDIX B
CONVENTIONS AND NOTATIONS

B.1. The usual 4-dimensional manifold of General Relativity

B.1.1. *Conventions*

The conventions used in this paper are the MTW ones [26]. The signature of the metric is $(- + + +)$. The speed of light is taken to be unity ($c = 1$), but not the gravitational constant G (5). We also take $\hbar = 1$.

The notation used in this paper for the Dalembertian is: $\Box = \nabla_\mu \nabla^\mu$.

B.1.2. *Fundamental fields of the theory*

(i) The gravitational field: $g_{\mu\nu}$ (which allows us to define the line element: $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$).

(ii) The electromagnetic potential: A_μ.

(iii) The scalar field: σ.

$\left(\text{Sometimes, it is more convenient to use the scalar field } \sigma^* = \frac{3}{2}\sigma, \text{ in which}\right.$ case most of the equations are trivially modified. *Cf.* also the remark at the end of section 1.3.3.)

(5) The gravitational "constant" G is indeed constant in the formulation that we adopt here. We show in section 1.3.3 that a conformal transformation on the 4-dimensional metric is necessary to achieve this.

B.1.3. *Most important tensors*

(*a*) The Einstein tensor

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

(We do not denote it as $G_{\mu\nu}$, in order to distinguish its contraction: $E = E_\mu{}^\mu$ from the gravitational constant.)

(*b*) The tensor of electric field **E** and magnetic induction **B**:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(*c*) The tensor of electric displacement **D** and magnetic field **H**, in the vacuum:

$$H_{\mu\nu} = e^{3\sigma} F_{\mu\nu}$$

(*d*) The modified Maxwell tensor

$$T^{\text{e.m.}}_{\mu\nu} = \frac{1}{4\pi}\left(\frac{1}{2}[H_\mu{}^\rho F_{\nu\rho} + F_\mu{}^\rho H_{\nu\rho}] - \frac{1}{4} g_{\mu\nu} H_{\rho\sigma} F^{\rho\sigma}\right)$$

(*e*) The stress-energy tensor for the scalar field σ

$$\tau_{\mu\nu} = \frac{3}{16\pi G}\left(\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} \partial_\rho \sigma \partial^\rho \sigma\right)$$

B.1.4. *Charged or neutral matter fields*

These are described by:

(*f*) The stress-energy tensor: $T^{\text{mat.}}_{\mu\nu}$.

(*g*) The electric 4-current: $J^\mu = \rho u^\mu$ where ρ is the charge density, and $u^\mu = \dfrac{dx^\mu}{|ds|}$ is the unitary 4-velocity satisfying: $u_\mu u^\mu = -1$.

B.2. The 5-dimensional manifold

B.2.1. *Conventions*

The 5-dimensional coordinates are denoted as $(x^1, x^2, x^3, x^4, x^5)$ where the first four are those which are used in General Relativity. The signature of the metric is $(-, +, +, +, +)$.

B.2.2. *The 5-dimensional line element*

(i) $m, n, \ldots$ vary from 1 to 5, whereas Greek indices vary from 1 to 4 as usual.

(ii) metric: γ_{mn}.

(iii) line element: $d\sigma^2 = \gamma_{mn}\, dx^m\, dx^n$. [This σ is not the scalar field!]

B.2.3. *5-dimensional Einstein tensor*

(*a*) Christoffel symbols: $\Gamma^l_{mn} = \frac{1}{2}\gamma^{lr}(\partial_n \gamma_{mr} + \partial_m \gamma_{rn} - \partial_r \gamma_{mn})$

(*b*) Ricci tensor: $\mathbb{R}_{mn} = \partial_l \Gamma^l_{mn} - \partial_n \Gamma^l_{ml} + \Gamma^r_{mn}\Gamma^l_{lr} - \Gamma^r_{ml}\Gamma^l_{rn}$

(*c*) Einstein tensor: $\mathbb{E}_{mn} = \mathbb{R}_{mn} - \frac{1}{2}\mathbb{R}\,\gamma_{mn}$ where: $\mathbb{R} = \gamma^{mn}\mathbb{R}_{mn} = \mathbb{R}_m{}^m$.

B.2.4. *Matter fields*

(d) Stress-energy tensor describing matter and charges: $\mathbb{T}_{mn}$.

This tensor may depend on the unitary velocity of the 5-dimensional space:

(*c*) Unitary 5-velocity: $\mathbb{U}^m = \dfrac{dx^m}{|d\sigma|}$ [verifying: $\mathbb{U}_m \mathbb{U}^m = \operatorname{sgn}(d\sigma^2)$].

There must be no confusion between the covariant differentiations in the 4-dimensional and the 5-dimensional spaces:

(*f*) Covariant differentiation of $\mathbb{U}^m$:

$$\nabla_n \mathbb{U}^m = \partial_n \mathbb{U}^m + \Gamma^m_{nl}\mathbb{U}^l.$$

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