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On super (or pseudo) differential forms as superfunctions

by

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ABSTRACT. — We study the representation of super (or pseudo) differential forms on a C^∞ graded manifold as sections of a vector bundle.

RÉSUMÉ. — Nous étudions les représentations des super (ou pseudo) formes différentielles sur une variété C^∞ graduée comme sections d'un fibré vectoriel.

0. INTRODUCTION

Let M be a graded manifold of dimension $(m|n)$, in the following sense ([1], [2], [3])

– M_0 is a $\mathscr{C}^\infty$ real manifold of dimension m.

– $\mathscr{A}_M$ is a sheaf of Z_2 graded commutative algebras such that:

(i) there exists a surjective sheaf morphism

$$\varepsilon : \quad \mathscr{A}_M \to \mathscr{C}^\infty(M_0)$$

(ii) there exists an open covering $\{V_i\}_{i \in I}$ of M_0 and sheaf isomorphisms $\mathscr{A}(V_i) \to \underline{\Lambda \mathbb{R}^n}(V_i)$.

One knows that there exists a vector bundle E such that $\mathscr{A}_M$ is isomorphic to the sheaf of sections of ΛE^* ([4], [5]).

Pseudodifferential forms (we shall call superdifferential forms analogous objects in this paper) are defined in [6] as superfunctions of a graded manifold, say $\mathscr{M}$, defined by patching supertransformations (*see* [2]) obtained naturally from the patching supertransformations of M.

With an isomorphism $h : \mathscr{A}_M \to \Gamma(\ , \Lambda E^*)$, these last supertransformations can be reduced to those induced by coordinate transformations (vectorial changes of charts) of the bundle E. The same is true for $\mathscr{M}$, with an isomorphism H and a bundle $\mathscr{E}$.

In that paper, we want to use this reduction induced by h whenether it is possible. So we will start with E and construct a sheaf $\Omega_{M,\tau}$ (the dependence in τ, a family of local trivialization of E is only for simplification) isomorphic to $\mathscr{A}_M$. It differs from $\hat{\Omega}_M$ of [6] in the fact that the patching supertransformations which define it are more special in $\Omega_{M,\tau}$ than in $\hat{\Omega}_M$, *i. e.* a reduction has already been done, due to h. An isomorphism $H_1 : \hat{\Omega}_M \to \Omega_{M,\tau}$ is hidden.

As a counterpart to this less intrinsic definition, we are as close as possible to an isomorphism from $\Omega_{M,\tau}$ to $\Gamma(., \Lambda \mathscr{E}^*)$, which allows a somewhat more concrete manipulation of super (or pseudo) differential forms. An example of application could be found in supergravity. We emphasize that this last isomorphism is not canonical and it is the second purpose of this paper to show that what misses to induce such an isomorphism is a connection on E. It is interesting to note that we have here a relatively simple example of a graded manifold the sheaf of which is not given as the sheaf of sections of a vector bundle. If the bundle had a natural connection, for example in supergravity, there would be a prefered isomorphism from Ω_M to $\Gamma(., \Lambda \mathscr{E}^*)$.

In the first part, we define the sheaf Ω_τ with two graduations:

$$\Omega^p_{\tau, i}, \quad p \in \mathbb{N}, \quad i=0,1.$$

For W an open set of E, $\Omega_\tau(W)$ will be an algebra and a bimodule on $\Omega^0_\tau(W)$. We define a differentiation operator , d, of null square, making $\Omega_\tau(W)$ a complex.

In the second part, we shall connect super-differential forms of degree 0 and 1 to those of same degree in [1], by giving the following isomorphisms, for V open set of M_0:

- a ring isomorphism between $\Omega^0_\tau(\pi^{-1}(V))$ and $\Omega^0_K(V) = \Gamma(\mathscr{A}, V)$;
- an isomorphism of bimodule of $\Omega^1_\tau(\pi^{-1}(V))$ and $\Pi\Omega^1_K(V)$ where $\Omega^1_K(V)$ is $\mathrm{Hom}_{\Gamma(\mathscr{A}, V)}(\mathrm{Der}\,\Gamma(\mathscr{A}, V), \Gamma(\mathscr{A}, V))$.

Then, we shall define isomorphisms of $S(\Pi\Omega^1_K(V))$ on a subset, denoted by $\Omega^{\mathbb{N}}_\tau(\pi^{-1}(V))$ of $\Omega_\tau(\pi^{-1}(V))$. Once we have introduced the natural d operator on $S(\Pi\Omega^1_K(V))$, it is an isomorphism of modules, Z_2 graded algebras, and differential complexes.

In the third part, we shall show how Ω_τ is a sheaf on E which can be considered as an extension of $S(\Pi\Omega^1_K)$. This last object is denoted by $\Omega(M)$ in reference [6].

In the fourth part, we shall consider algebras $\Omega^0_E(V)$ and $\Omega^1_E(V)$ appearing in part 2, and, by means of a connection on E, we shall construct morphisms of these algebras into $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$ in order to get morphisms from $\Omega^0_\tau(\pi^{-1}(V))$ and $\Omega^1_\tau(\pi^{-1}(V))$ into $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$. We shall then extend them to isomorphisms from $\Omega_\tau(W)$ onto $\Gamma(W, \Lambda\mathscr{E}^*)$ for W an arbitrary open set in E.

CONTENTS

1. THE COMPLEX OF SUPER DIFFERENTIAL FORMS

1.1. The sheaf $\Omega_{M,\tau}$

Let E be a vector bundle associated to the graded manifold M, $(V_i)_{i\in I}$ an open covering of M_0 by domains of charts c_i, $(W_i=\pi^{-1}(V_i))_{i\in I}$ the covering of E, domains of charts φ_i. We send the reader to the complete list of notations postponed at the end of the paper.

1.1.1. *Definition of* $\Omega_{M,\tau}$

For every open set W in E, an element ω_τ of $\Omega_\tau(W)$ will be defined on the open sets $W\cap W_i$ by means of the trivializations φ_i and we shall give the patching super transformations on $W\cap W_{ij}$. It is a standard construction of a graded manifold [2].

Precisely, let ε^λ denote the canonical linear forms on $\mathbb{R}^{m+n}$, considered as even elements in $\mathscr{C}^\infty(\mathbb{R}^{m+n})\otimes\Lambda\mathbb{R}^{m+n}$ (denoted also by $\Lambda\mathbb{R}^{m+n}(\mathbb{R}^{m+n})$); we shall keep the same symbol for their restrictions to open subsets. Let ϑ^λ be the generators of the Grassmann algebra $\Lambda\mathbb{R}^{m+n}$; they are odd elements. The c_{ij}'s and g_{ij}'s are the coordinate transformations of M_0 and E respectively. We set $I(W)=\{i\in I,\ W\cap W_i\neq\varnothing\}$.

DEFINITION. — $\Omega_\tau(W)$ is the set of $\omega=\{\omega_i\}_{i\in I(W)}$, where

$$\omega_i\in\mathscr{C}^\infty(\varphi_i(W\cap W_i))\otimes\Lambda\mathbb{R}^{m+n}=\Lambda\mathbb{R}^{m+n}(\varphi_i(W\cap W_i)),$$

the following patching conditions being satisfied:

$$\omega_j\,|_{\varphi_j(W\cap W_{ij})}=h_{ji}(\omega_i\,|_{\varphi_i(W\cap W_{ij})})$$

The h_{ji}'s are [restrictions to $\varphi_i(W\cap W_{ij})$ of] morphisms from $\Lambda\mathbb{R}^{m+n}(\varphi_i(W_{ij}))$ to $\Lambda\mathbb{R}^{m+n}(\varphi_j(W_{ij}))$ defined by:

$$(1.1.a)\qquad h_{ji}(\varepsilon_i^\rho)=\varepsilon_i^\rho\circ\underline{c}_{ij},\qquad \rho=1,\ \dots,\ m$$

$$(1.1.b)\qquad h_{ji}(\varepsilon_i^{m+\alpha})=\sum_{\beta=1}^{n}(\underline{\tilde{g}_{ij}})_\beta^\alpha\,\varepsilon_j^{m+\beta}+\sum_{\substack{\rho=1,\ \dots,\ m\\ \gamma=1,\ \dots,\ n}}D_\rho(\underline{\tilde{g}_{ij}})_\gamma^\alpha\,\vartheta_j^\rho\,\vartheta_j^{m+\gamma}$$

$$\alpha=1,\ \dots,\ n$$

(1.1.c) $$h_{ji}(\vartheta_i^\rho)=\sum_{\sigma=1}^{m}(\underline{D}\,\underline{c}_{ij})_\sigma^\rho\,\vartheta_j^\sigma,\qquad \rho=1,\ldots,m$$

(1.1.d) $$h_{ji}(\vartheta_i^{m+\alpha})=\sum_{\beta=1}^{n}(\underline{\tilde g}_{ij})_\beta^\alpha\,\vartheta_j^{m+\beta},\qquad \alpha=1,\ldots,n$$

All these are equalities between elements of $\Lambda\,\mathbb{R}^{m+n}(\varphi_j(W\cap W_j))$. Symbols used above are defined in the list of notations and, besides,

$$\underline{c}_{ij}:\ c_j(V_{ij})\times\mathbb{R}^n\to c_i(V_{ij})\times\mathbb{R}^n$$
$$\underline{c}_{ij}(u,v)=(c_{ij}(u),v)$$

D_ρ is the derivative with respect to the ρ-component in $\mathbb{R}^m$, $(\underline{D}\,\underline{c}_{ij})_\sigma^\rho$ is the function on $c_j(V_{ij})\times\mathbb{R}^n$ pulled back from the function $(D\,c_{ij})_\sigma^\rho=\partial_\sigma c_{ij}^\rho$.

We admit that h_{ji} is actually defined by formulas 1.1, and, in particular that

$$h_{ji}(f)=f(h_{ji}(\varepsilon_i^1),\ldots,h_{ji}(\varepsilon_i^{m+n}))$$

for $f\in\mathscr{C}^\infty\,\mathbb{R}^{m+n})$, the right hand side being defined by means of the Taylor expansion of f, as usual.

Let us explicit restriction morphisms $\rho_{W'W}$ form $\Omega_\tau(W)$ to $\Omega_\tau(W')$ when $W'\subset W$, making Ω_τ, the collection of the $\Omega_\tau(W)$, a sheaf: $\rho_{W'W}(\omega_\tau)$ is the element of $\Omega_\tau(W')$ defined by the family $(\rho_{W'W}\,\omega)_{i,\,i\in I(W')}=\omega_i|_{W_i\cap W'}$.

1.1.2. *The algebra structure on* $\Omega_\tau(W)$

Let ω_τ and ω'_τ be two elements of $\Omega_\tau(W)$. One defines their product $\omega_\tau.\omega'_\tau$ by the collection

$$(\omega_\tau.\omega'_\tau)_i=\omega_i.\omega'_i,$$

where the product in the right-hand side is in $\Lambda\,\mathbb{R}^{m+n}(\varphi_i(W\cap W_i))$. This definition makes sense for $h_{ji}[(\omega.\omega')_i]=(\omega.\omega')_j$.

1.1.3. *The graded manifold defined by* $\Omega_{M,\tau}$

That the h_{ij}'s define a graded manifold follows from [2] because of the relation:

$$h_{ij}|_{\varphi_j(W_{ijk})}=h_{ik}|_{\varphi_k(W_{ijk})}\circ h_{kj}|_{\varphi_j(W_{ijk})}$$

The body of the graded manifold is E since the mapping from $\mathscr{C}^\infty(\varphi_j(W_{ij}))$ on $\mathscr{C}^\infty(\varphi_i(W_{ij}))$ induced by h_{ij} are the coordinate transformations of the bundle E. We denote by $\mathscr{M}$ that graded manifold, and explicit now local charts and the mapping onto the sheaf $\mathscr{C}^\infty(E)$.

1.1.3.1. *Local "trivializations" of $\mathcal{M}$*

Above W_i, we have an isomorphism from $\Omega_\tau(W_i)$ to $\Lambda\mathbb{R}^{m+n}(W_i)$ by the product of the two following maps

$$\kappa_i: \quad \Omega_\tau(W_i) \to \Lambda\mathbb{R}^{m+n}(U_i \times \mathbb{R}^n)$$
$$\kappa_i(\omega_\tau) = \omega_i$$
$$C: \quad \Lambda\mathbb{R}^{m+n}(U_i \times \mathbb{R}^n) \to \Lambda\mathbb{R}^{m+n}(W_i)$$
$$C(f(.)\vartheta_i^{\lambda_1}\dots\vartheta_i^{\lambda_k}) = f(\varphi_i(.))\vartheta_i^{\lambda_1}\dots\vartheta_i^{\lambda_k}.$$

In the following, we will sometimes call κ_i a superchart.

1.1.3.2. *The morphism* ε *from* Ω_τ *to* $\mathscr{C}^\infty(E)$

For $\mathscr{U}$ open set in $\mathbb{R}^{m+n}$, we have a morphism ε_0 from $\Lambda\mathbb{R}^{m+n}(\mathscr{U})$ onto $\mathscr{C}^\infty(\mathscr{U})$. We define, for $\omega_\tau \in \Omega_\tau(W)$, $\varepsilon(\omega_\tau)$ as the element of $\mathscr{C}^\infty(W)$ which is sent to $\varepsilon_0(\omega_i)$ by the chart φ_i, where $\omega_i = \kappa_i(\omega_\tau)$. This definition makes sense for $\varepsilon_0(\omega_i)$ and $\varepsilon_0(\omega_j)$ define the same function on $W \cap W_{ij}$. Indeed, formulas (1.1.*a*) and (1.1.*b*) induce a morphism from $\mathscr{C}^\infty(\varphi_i(W \cap W_{ij}))$ onto $\mathscr{C}^\infty(\varphi_j(W \cap W_{ij}))$ by composition of injection into $\Lambda\mathbb{R}^{m+n}(\varphi_i(W \cap W_{ij}))$, h_{ji} and ε_0. It sends in particular $\varepsilon_i^\rho|_{\varphi_i(W \cap W_{ij})}$, on $\varepsilon_i^\rho \circ \underline{c}_{ij}|_{\varphi_j(W \cap W_{ij})}$, $\varepsilon_i^{m+\alpha}|_{\varphi_i(W \cap W_{ij})}$ on $\sum_{\beta=1}^{n} (\underline{\tilde{g}_{ij}})_\beta^\alpha \varepsilon_j^{m+\beta}|_{\varphi_j(W \cap W_{ij})}$, and $\varepsilon_0(\omega_i)$ on $\varepsilon_0(\omega_j)$.

1.1.3.3. *Note*

Let us suppose that the fiber of E is reduced to the null vector so that M is identified to M_0. Then formulas (1.1.*a*) to (1.1.*d*) reduce to

$$(1.1.e) \qquad h_{ji}(\varepsilon_i^\rho) = \varepsilon_i^\rho \circ c_{ij}, \qquad h_{ji}(\vartheta_i^\rho) = \sum_{\sigma=1}^{m} (Dc_{ij})_\sigma^\rho \vartheta_j^\sigma$$

These are formulas for changes of coordinates in the tangent bundle TM_0; ϑ_i^ρ is the image in the chart i of the ρ-component function on the base $e_{i,\rho}$. Ω_M is, in that case, the sheaf of differential forms on M_0.

Formulas (1.1.*e*) are obtained by writing the change of coordinates for the tangent bundle (functions x^ρ and $\dot{x}^\rho$ with usual notations), and by changing the parity of the $\dot{x}^\rho$'s. In an analogous way, formulas (1.1.*a*) to (1.1.*d*) are obtained by considering locally "superfunctions" x_i^ρ, $\vartheta_i^{m+\alpha}$, $\dot{x}_i^\rho$, $\dot{\vartheta}_i^{m+\alpha}$ on TE, with the patching transformations, and by changing parity of $\dot{x}$'s and $\dot{\vartheta}$'s ([6], [7]).

1.1.3.4. *Comment*

One can be tempted to consider usual differential forms on M_0 as superfunctions of the graded manifold defined by the pair (M_0, TM_0). But this view would not be perfectly correct, because the interest of considering graded manifolds is to forget the N-grading of the algebras of sections of the bundle from which the graded manifold is constructed.

It is to be noted that, even though our super differential forms will be superfunctions in the above strong sense, we shall define later an $\mathbb{N}$-graduation on a subspace $\Omega_\tau^{\mathbb{N}}$ of Ω_τ, *i.e.* we shall define the degree of some forms. When an isomorphism will be constructed from Ω_τ to $\Gamma(., \Lambda\mathscr{E}^*)$, that $\mathbb{N}$-graduation will not be that induced by the graduation on $\Lambda\mathscr{E}^*$, if rank (E)$\neq 0$.

When M is reduced to M_0. $\Omega_{M,\tau}$ and $\Omega_{M,\tau}^{\mathbb{N}}$ will be reduced to usual Ω_{M_0}, the graduation on $\Omega_{M,\tau}^{\mathbb{N}}$ going to the usual one. That follows the general result, established in part four that a connection on E (trivially given if E is M_0 itself) gives a morphism of $\Omega_{M,\tau}$ onto sections of a bundle $\Lambda\mathscr{E}^*$ on E. (In that case $\mathscr{E}=TM_0$.)

Should we want an analogy between the graded manifold defined by the bundle TM_0 and that defined by $\Omega_{M,\tau}$, we would have to define a sheaf isomorphic to $\Omega_{M,\tau}$ but not naturally endowed with the $\mathbb{N}$ graduation.

1.1.4. *The bigraduation on* $\Omega_\tau(W)$ *and the definition of* $\Omega_\tau^{\mathbb{N}}(W)$

(*a*) *The* $\mathbb{Z}_2$ *graduation*

It was understood in preceding sections; let us explicit it now, setting notations:

$(\Lambda\mathbb{R}^{m+n}(\mathscr{U}))_0$ and $(\Lambda\mathbb{R}^{m+n}(\mathscr{U}))_1$ denote even and odd parts of $\Lambda\mathbb{R}^{m+n}(\mathscr{U})$.

$$\Omega_{\tau,\,0}(W)=\{\omega:\omega_i\in(\Lambda\mathbb{R}^{m+n}(\varphi_i(W\cap W_i)))_0,\ \forall i\in I(W)\}$$
$$\Omega_{\tau,\,1}(W)=\{\omega;\ \omega_i\in(\Lambda\mathbb{R}^{m+n}(\varphi_i(W\cap W_i)))_1,\ \forall i\in I(W)\}$$
$$\Omega_\tau(W)=\Omega_{\tau,\,0}(W)+\Omega_{\tau,\,1}(W)$$

(*b*) *The* $\mathbb{N}$ *graduation*

Let us first define elements of degree p in $\Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n)$:

$$(\Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n))^{(p)}=\{\omega_i:\omega_i=\sum_{J,\,K,\,L} g_{J,\,K,\,L}\,\varepsilon^K\,\vartheta^J\,\vartheta^L,\ |J|+|K|=p\}$$

where

$$
\begin{gathered}
g_{J,\,K,\,L} \in \mathscr{C}^\infty(U_i) \to \mathscr{C}^\infty(U_i \times \mathbb{R}^n) \\
J = \{\rho_1, \ldots, \rho_j\} \subset \{1, \ldots, m\}, \qquad \rho_1 < \rho_2 \ldots < \rho_j \\
K = \{m+\alpha_1, \ldots, m+\alpha_k\} \subset \{m+1, \ldots, m+n\}, \\
m+\alpha_1 \leqq m+\alpha_2 \leqq \ldots \leqq m+\alpha_k \\
L = \{m+\beta_1, \ldots, m+\beta_l\} \subset \{m+1, \ldots, m+n\}, \\
m+\beta_1 \leqq m+\beta_2 \leqq \ldots \leqq m+\beta_l \\
\varepsilon^K = \varepsilon^{m+\alpha_1} \ldots \varepsilon^{m+\alpha_k}
\end{gathered}
$$

(ordinary products of even elements)

$$
\vartheta^J = \vartheta^{\rho_1} \ldots \vartheta^{\rho_j}, \qquad \vartheta^L = \vartheta^{m+\beta_1} \ldots \vartheta^{m+\beta_l}
$$

(products in the Grassmann algebra $\Lambda\,\mathbb{R}^{m+n}$).

Let us denote by (p, a) the bidegree; $p \in \mathbb{N}$, $a \in \mathbb{Z}_2$.

ε_i^ρ, $\varepsilon_i^{m+\alpha}$, ϑ_i^ρ, $\vartheta_i^{m+\alpha}$ have degree respectively (0, 0), (1, 0), (1, 1), (0, 1).

DEFINITION. – For V an open set in M_0, let us set :

$$
\begin{aligned}
\Omega_\tau^p(\pi^{-1}(V)) = \{ \omega \in \Omega_\tau(\pi^{-1}(V)),& \\
& \omega_i \in (\Lambda\,\mathbb{R}^{m+n}(c_i(V \cap V_i) \times \mathbb{R}^n))^{(p)},\ \forall i \in I(\pi^{-1}(V)) \}
\end{aligned}
$$

Note that h_{ji} maps elements of degree p on elements of same degree.

For W an open set in E, we set

$$
\Omega_\tau^p(W) = \rho_{W,\,\pi^{-1}(\pi W)}\, \Omega_\tau^p(\pi^{-1}(\pi W)).
$$

$$
(c)\ \ \Omega_\tau^{\mathbb{N}}(W) = \bigoplus_{p=0}^{\infty} \Omega_\tau^p(W)
$$

Unlike elements in $\Omega_\tau(W)$, elements in $\Omega_\tau^{\mathbb{N}}(W)$ can always be extended to $\Omega_\tau^p(\pi^{-1}(\pi(W)))$. An arbitrary element ω_τ in $\Omega_\tau(W)$ reads in (super) chart i :

$$
\kappa_i(\omega_\tau) = \omega_i = \sum_{J,\,L} f_{J,\,L}\, \vartheta^J \vartheta^L
$$

where $f_{J,\,L} \in \mathscr{C}^\infty(\varphi_i(W \cap W_i))$ is not in general a polynomial in the $\mathbb{R}^n$ variables, and may have singularities in these directions.

On $\Lambda\,\mathbb{R}^{m+n}$, with the graduation presented in (*b*), we have a structure of bigraded algebra. Therefore $\Omega_\tau(W)$ has the same structure: the product is a map from $\Omega_\tau^p \times \Omega_\tau^{p'}$ into $\Omega_\tau^{p+p'}$. In particular $\Omega_\tau^p(W)$ and $\Omega_\tau(W)$ are modules over $\Omega_\tau^0(W)$.

1.1.5. *Dependence with respect to* τ

$\Omega_\tau^0(W)$ is the set of ω_τ which in chart i write

$$
\omega_i = \sum_L g_L\, \vartheta_i^L, \qquad L = \{m+\alpha_1, \ldots, m+\alpha_l\} \subset \{m+1, \ldots, m+n\}
$$

where $g_L \in \mathscr{C}^\infty(c_i(\pi(W) \cap V_i))$. [Actually, g_L has to be considered as a function on $\varphi_i(W \cap W_i)$, constant in the $\mathbb{R}^n$ variables. As we said previously, ω_τ in $\Omega^0_\tau(W)$ can be extended trivially to an element in $\Omega^0_\tau(\pi^{-1}(\pi(W)))$, with no dependence in the fiber $\mathbb{R}^n$.] The trivializations τ allow us to identify $\Omega^0_\tau(\pi^{-1}(V))$ with $\Gamma(V, \Lambda E^*)$ (*see* section 2.2.1 for details), not depending on τ.

Let us consider an other collection τ' of local trivializations. $\Omega^1_\tau(W)$ and $\Omega^1_{\tau'}(W)$ are two different algebras, for the families $\omega_{\tau,\, i}$ and $\omega_{\tau',\, i}$ defining elements ω_τ and $\omega_{\tau'}$ in each one are defined by different patching relations. Nevertheless, they are isomorphic as we shall see in section 2.2, where we construct an isomorphism from $\Omega^1_\tau(\pi^{-1}(V))$ onto

$$\Omega^1_E(V) = \Pi \operatorname{Hom}_{\Gamma(V,\, \Lambda E^*)}(\operatorname{Der}\Gamma(V, \Lambda E^*), \Gamma(V, \Lambda E^*))$$

and in part 4 with an isomorphism from Ω_τ onto the sheaf of sections of a bundle on E not depending on τ.

In the case where τ and τ' differ only by a family $(g_i)_{i \in I}$ of linear transformations, let us give explicitly the isomorphism. It is given by a family of supertransformations

$$\nu_i:\quad \Lambda\mathbb{R}^{m+n}(U_i \times \mathbb{R}^n) \to \Lambda\mathbb{R}^{m+n}(U_i \times \mathbb{R}^n)$$

satisfying $\nu_j \circ h_{ji} = h_{ji} \circ \nu_i$.

$$\nu_i(\varepsilon_i^{m+\alpha}) = D_\rho(\tilde{g}_i)^\alpha_\beta\, \vartheta^\rho_i\, \vartheta_i^{m+\beta} + (\tilde{g}_i)^\alpha_\beta\, \varepsilon_i^{m+\beta}$$
$$\nu_i(\vartheta_i^{m+\alpha}) = (\tilde{g}_i)^\alpha_\beta\, \vartheta_i^{m+\beta}$$
$$\nu_i(\varepsilon^\rho_i) = \varepsilon^\rho_i$$
$$\nu_i(\vartheta^\rho_i) = \vartheta^\rho_i$$

$(g_i)^\alpha_\beta$ is the real function on M_0 which gives $e^{m+\alpha}_{\tau,\, i} = \sum_{\beta=1}^{n} (g_i)^\alpha_\beta\, e^{m+\beta}_{\tau',\, i}$, $(\tilde{g}_i)^\alpha_\beta = (g_i)^\alpha_\beta \circ c_i^{-1} \circ p_1$, where p_1 is the projection $U_i \times \mathbb{R}^n \to U_i$. Note that the ν_i's preserve the $\mathbb{N}$ degree of section 1.1.4(*b*), but do not preserve the number of ϑ's.

It is clear that the definition we gave of superdifferential forms is not the right one. First the dependence on τ could be avoided by considering simultaneously, in the definition of a form, all the τ's and equivalence relations between the ω_τ's. More deeply, Ω^0 as it would be so defined would still be canonically isomorphic to $\Gamma(.,\Lambda E^*)$ which is not adequate. We send to [6] for a more general construction, as we mentioned in the introduction.

1.2. Exterior differentiation in Ω_τ

In $\Omega_\tau(W)$, we are going to define an exterior differentiation of bidegree (1, 1), using such a differentiation in the superchargts, *i.e.* on

$\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$, $\mathcal{U}$ open set in $\mathbb{R}^{m+n}$. We shall see later in part two that the subalgebra $\Omega_\tau^{\mathbb{N}}(\mathrm{W})$ of $\Omega_\tau(\mathrm{W})$ is isomorphic to an algebra $\mathrm{S}(\Pi\Omega_{\mathrm{K}}^1(\mathrm{W}))$ on which there exists a natural d operator generalizing the one in ordinary geometry. The d operator we shall define on $\Omega_\tau(\mathrm{W})$ will make $\Omega_\tau^{\mathbb{N}}(\mathrm{W})$ a complex isomorphic to $\mathrm{S}(\Pi\Omega_{\mathrm{K}}^1(\mathrm{W}))$. If the graded manifold M is $\mathbb{R}^{m\mid n}$, $\mathcal{M}$, the graded manifold defined by Ω_{M} is $\mathbb{R}^{m+n\mid m+n}$ and our d operator on Ω_{M} is of course the one we defined on $\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$.

1.2.1. *A d-operator in* $\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$

$\mathcal{U}$ is an open set in $\mathbb{R}^{m+n}$. Let F be in $\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$.

$$\mathrm{F}=\sum f_{\mathrm{J,\,L}}\,\vartheta^{\mathrm{J}}\vartheta^{\mathrm{L}},\quad \text{with symbols as in 1.1.4}(b)$$

$$f_{\mathrm{J,\,L}}\in\mathscr{C}^\infty(\mathcal{U}),\qquad \mathrm{L}=\{m+\alpha_1,\,\ldots,\,m+\alpha_l\}.$$

We define $d\mathrm{F}$ in $\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$:

$$d\mathrm{F}=\sum_{\rho=1}^{m}(\partial_\rho f_{\mathrm{J,\,L}})\,\vartheta^\rho\vartheta^{\mathrm{J}}\vartheta^{\mathrm{L}}+(-1)^{|\mathrm{J}|+p+1}f_{\mathrm{J,\,L}}\sum_{p=1}^{l}\varepsilon^{m+\alpha_p}\vartheta^{\mathrm{J}}\vartheta^{\mathrm{L}-\{m+\alpha_p\}}$$

d is in fact entirely determined by its value on functions on $\mathcal{U}$, on $\vartheta^{m+\alpha}$ and by the properties:

$$d^2=0$$
$$d(\mathrm{F}\,.\,\mathrm{F}')=d\mathrm{F}\,.\,\mathrm{F}'+(-1)^{|\mathrm{F}|}\,\mathrm{F}\,d\mathrm{F}'$$

where $|\mathrm{F}|$ denotes the $\mathbb{Z}_2$ degree of F.

We have $d\varepsilon^\rho=\vartheta^\rho$, $d\vartheta^{m+\alpha}=\varepsilon^{m+\alpha}$ and therefore $d\vartheta^\rho=d\varepsilon^{m+\alpha}=0$.

In other terms, $\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$ can be regarded as the sheaf of superdifferential forms of the graded manifold $\mathbb{R}^{m\mid n}$. The grading of $\Lambda\,\mathbb{R}^{m+n}(\mathcal{U})$ in 1.1.4(b) defines the degree of the forms. A superfunction of $\mathbb{R}^{m\mid n}$ is identified with a 0-form and the d operator defined above sends p-forms on forms of degree $p+1$. We shall now patch together these differentiations.

1.2.2. *The d operator in* $\Omega_\tau(\mathrm{W})$

Let ω_τ be in $\Omega_\tau(\mathrm{W})$. $d\kappa_i(\omega_\tau)\in\Lambda\,\mathbb{R}^{m+n}(\varphi_i(\mathrm{W}\cap\mathrm{W}_i))$. The sequence $(d(\kappa_i(\omega_\tau)))_{i\in\mathrm{I(W)}}$ defines an element in $\Omega_\tau(\mathrm{W})$ because one verifies that

$$h_{ij}((d\omega_j)\,|_{\varphi_j(\mathrm{W}\cap\mathrm{W}_{ij})})=d\omega_i\,|_{\varphi_i(\mathrm{W}\cap\mathrm{W}_{ij})}.$$

We denote this element by $d\omega_\tau$.

So, $\Omega_\tau(\mathrm{W})$ is a bigraded algebra with a differentiation operator of bidegree (1.1).

2. RELATION TO DIFFERENTIAL FORMS IN $\mathrm{Hom}_{\mathscr{A}}(\mathrm{Der}\,\mathscr{A}, \mathscr{A})$

The aim of this chapter is to connect Ω_τ to the graded differential forms of Kostant [1] that we denote by Ω_K. Note that $\Omega_K(V)$ is constructed for V open set in M_0 whereas $\Omega_\tau(W)$ is defined for W open set in E. We shall give an isomorphism μ from $S(\Pi\Omega_K^1(V))$ to $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$.

But, unfortunately, the definition of $\Omega_\tau(W)$ does not lie directly on the $\mathscr{A}(W)$. So it is more natural, at our present level, to forget $\mathscr{A}$ and work on the bundle E, using an arbitrary isomorphism h from $\mathscr{A}$ to $\Gamma(.,\Lambda E^*)$. We shall then connect $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$ to

$$S(\Pi\,\mathrm{Hom}_{\Gamma(V,\Lambda E^*)}(\mathrm{Der}\,\Gamma(V,\Lambda E^*), \Gamma(V,\Lambda E^*)))$$

by an isomorphism μ_E^{-1}.

The isomorphism μ will be $\mu_E \circ h^*$.

2.1. Definition of Ω_K^1, $\Pi\,\Omega_K^1$, $S(\Pi\,\Omega_K^1)$

2.1.1. *Definition of* Ω_K^1 ([1], [8])

For V an open set in M_0, we set:

$$\Omega_K^1(V) = \mathrm{Hom}_{\Gamma(\mathscr{A},V)}(\mathrm{Der}\,\Gamma(\mathscr{A},V), \Gamma(\mathscr{A},V))$$

We use here the following conventions for homomorphisms

$$\omega(a\xi) = a\,\omega(\xi) \quad \text{if} \quad \xi \in \mathrm{Der}\,\mathscr{A}(V), \quad \omega \in \Omega_K^1(V).$$

We set also $\Omega_K^0(V) = \Gamma(\mathscr{A}, V)$. $\Omega_K^1(V)$ is a bimodule over $\Omega_K^0(V)$ by

$$\omega a(\xi) = \omega(\xi)\,a$$
$$a\omega = (-1)^{|a|\cdot|\omega|}\,\omega a$$

where $|a|$ is the $\mathbb{Z}_2$ degree of a, and $|\omega|$ that of ω.

Definition of $\Pi\,\Omega_K^1$

We denote by $\Pi\,\Omega_K^1$ the following module over Ω_K^0: It is the same set Ω_K^1, but the parity of the elements is changed.

$$(\Pi\,\Omega_K^1)_0 = (\Omega_K^1)_1, \; (\Pi\,\Omega_K^1)_1 = (\Omega_K^1)_0$$

The bimodule structure is given by

$$(\Pi\,\omega)\,a = \Pi(\omega a)$$
$$a(\Pi\,\omega) = (-1)^{|a|}\,\Pi(a\omega).$$

2.2. Algebra isomorphism between $\Omega_\tau^{\mathbb{N}}$ and $S(\Pi\,\Omega_K^1)$

Let us denote by h an isomorphism from $\mathscr{A}$ to $\Gamma(.,\Lambda E^*)$, and by τ_i the isomorphism from $\Gamma(V_i, \Lambda E^*)$ to $\Lambda\mathbb{R}^n(U_i)$ induced by φ_i. We shall first compare the module structures of $\Pi\,\Omega_K^1(V)$ and $\Omega_\tau^1(V)$.

2.2.1. *A ring isomorphism* (μ_0) *from* $\Omega_K^0(V)$ *onto* $\Omega_\tau^0(V)$

Let a be in $\Omega_K^0(V)$; $h(a)\in\Gamma(V, \Lambda E^*)$

$$h(a)|_{V\cap V_i}(m_0)=\sum_{\alpha_1,\ldots,\alpha_p\in[1,\ldots,n]} f_{i,\alpha_1,\ldots,\alpha_p}\times(c_i^1(m_0),\ldots,c_i^m(m_0))\times e_{i,m_0}^{\alpha_1}\wedge\ldots\wedge e_{i,m_0}^{\alpha_p}$$

$$\tau_i(h(a)|_{V\cap V_i})\in\Lambda\mathbb{R}^n(U_i)$$

$$\tau_i(h(a)|_{V\cap V_i})=\sum f_{i,\alpha_1,\ldots,\alpha_p}\,\vartheta_i^{m+\alpha_1}\ldots\vartheta_i^{m+\alpha_p}$$

One verifies, with formulas $(1.1.a)$ and $(1.1.b)$, that

$$\tau_j(h(a)|_{V\cap V_j})|_{c_j(V_{ij})}=h_{ji}(\tau_i(h(a)|_{V\cap V_i})|_{c_i(V_{ij})}$$

from which if follows that the collection of $\tau_i(h(a)|_{V\cap V_i})$, considered as elements in $\Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n)$, define an element of $\Omega_\tau^0(V)$. We denote it by $\mu_0(a)$. In particular,

$$\mu_0(h^{-1}(c_i^\rho))=\kappa_i^{-1}(\varepsilon_i^\rho)$$
$$\mu_0(h^{-1}(e_i^\alpha))=\kappa_i^{-1}(\varepsilon_i^{m+\alpha})$$

2.2.2. *An isomorphism* (μ_τ) *between the* $\Pi\,\Omega_K^1(V)$*'s,* bimodules over $\Omega_K^0(V)$ *and the* $\Omega_\tau^1(\pi^{-1}(V))$*'s, bimodules over* $\Omega_\tau^0(V)$

We shall consider more precisely a family $\mu_{\tau,V}$.

Let us set $\Omega_E^0(V)=\Gamma(V,\Lambda E^*)$

$$\Omega_E^1(V)=\Pi\ \mathrm{Hom}_{\Gamma(V,\Lambda E^*)}(\mathrm{Der}\,\Gamma(V,\Lambda E^*),\Gamma(V,\Lambda E^*))$$

We have to be careful that the graduation on Ω_E^1 is such that $\deg(\omega_E(X))\neq\deg(\omega_E)+\deg X$. Constructing our isomorphism μ_τ is equivalent to construct a collection of isomorphisms $\mu_{E,\tau,V}$

$$\mu_{E,\tau,V}:\ \Omega_E^1(V)\to\Omega_\tau^1(\pi^{-1}(V))$$

compatible with restriction morphisms, $\rho_{V'V}$ and $\rho_{\pi^{-1}(V'),\pi^{-1}(V)}$ respectively, for $V'\subset V$.

We denote by h_V^* the morphisms from $\Omega_K^1(V)$ to $\Pi\,\Omega_E^1(V)$ induced by $h_V(h^*(\Pi\,\omega_K)=\Pi\,h^*(\omega_K))$.

We shall define $\mu_{\tau,V}$ by

$$(2.2.2.1)\qquad \mu_{\tau,V}=\mu_{E,\tau,V}\circ h_V^*$$

We now proceed to construct $\mu_{E,\tau}$.

τ_i induces a morphism τ_i^*:

$$\tau_i^*:\ \mathrm{Hom}_{\Gamma(V_i,\,\Lambda E^*)}(\mathrm{Der}\,\Gamma(V_i,\,\Lambda E^*),\,\Gamma(V_i,\,\Lambda E^*)) \to \mathrm{Hom}_{\Lambda\mathbb{R}^n(U_i)}(\mathrm{Der}\,\Lambda\mathbb{R}^n(U_i),\,\Lambda\mathbb{R}^n(U_i))$$

We keep the same notation for the morphism between the Π-transformed modules.

We define now an even morphism χ_i which connects the following two modules on $\Lambda\mathbb{R}^n(U_i)$:

$$\chi_i:\ \Pi\,\mathrm{Hom}_{\Lambda\mathbb{R}^n(U_i)}(\mathrm{Der}\,\Lambda\mathbb{R}^n(U_i),\,\Lambda\mathbb{R}^n(U_i)) \to (\Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n))^1$$

[*see* 1.1.4 (*b*) for the notation; $(\Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n))^0$ is identified with $\Lambda\mathbb{R}^{m+n}(U_i)$]. ε_i^ρ, even coordinate functions, and $\vartheta_i^{m+\alpha}$, Grassmann variables being generators of $\Lambda\mathbb{R}^n(U_i)$, we denote by $\partial/\partial\varepsilon_i^\rho$ and $\partial/\partial\vartheta_i^{m+\alpha}$ the associated derivations:

$$\partial/\partial\varepsilon_i^\rho(\varepsilon_i^\sigma)=\delta_\rho^\sigma,\qquad \partial/\partial\varepsilon_i^\rho(\vartheta_i^{m+\alpha})=0,$$
$$\partial/\partial\vartheta_i^{m+\alpha}(\vartheta_i^{m+\beta})=\delta_\alpha^\beta,\qquad \partial/\partial\vartheta_i^{m+\alpha}(\varepsilon_i^\rho)=0$$

We denote by $d\varepsilon_i^\rho$, $d\vartheta_i^{m+\alpha}$ the dual basis. χ_i is then given by:

$$\chi_i(\Pi\,d\varepsilon_i^\rho)=\vartheta_i^\rho,\qquad \chi_i(\Pi\,d\vartheta_i^{m+\alpha})=\varepsilon_i^{m+\alpha}$$

The differentiation in $\Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n)$ has been defined in such a way that the following diagram is commutative:

$$\begin{array}{ccc}
\Lambda\mathbb{R}^n(U_i) & \xrightarrow[\text{injection}]{\text{natural}} & \Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n) \\
\Big\downarrow{\scriptstyle \Pi\circ d} & & \Big\downarrow{\scriptstyle d \text{ (defined in 1 . 2 . 1)}} \\
\Pi\,\mathrm{Hom}(\mathrm{Der}\,\Lambda\mathbb{R}^n(U_i),\,\Lambda\mathbb{R}^n(U_i)) & \xrightarrow{\chi_i} & \Lambda\mathbb{R}^{m+n}(U_i\times\mathbb{R}^n)
\end{array}$$

The d operator acting on $\Lambda\mathbb{R}^n(U_i)$ is the natural one, defined by $df(\xi)=\xi(f)$, for $f\in\Lambda\mathbb{R}^n(U_i)$, $\xi\in\mathrm{Der}\,\Lambda\mathbb{R}^n(U_i)$.

Noticing that $\chi_j\circ\tau_j^*=h_{ji}\circ\chi_i\circ\tau_i^*$ allows us to define $\mu_{E,\,\tau,\,V_i}$ by the equality

$$\kappa_i\circ\mu_{E,\,\tau,\,V_i}=\chi_i\circ\tau_i^*$$

and $\mu_{\tau,\,V_i}$ by relation (2.2.2.1).

With basis, if c_i^ρ and e_i^α are the generators of $\Gamma(V_i,\,\Lambda E^*)$, $\partial/\partial c_i^\rho$ and $\partial/\partial e_i^\alpha$ the associated derivations, generators of the left module $\mathrm{Der}(\Gamma(V_i,\,\Lambda E^*))$, dc_i^ρ and de_i^α, the associated (by duality) generators of the right module $\Pi\,\Omega_E^1(V_i)$, d being the mapping from $\Gamma(V_i,\,\Lambda E^*)$ to

$$\mathrm{Hom}_{\Gamma(V_i,\,\Lambda E^*)}(\mathrm{Der}\,\Gamma(V_i,\,\Lambda E^*),\,\Gamma(V_i,\,\Lambda E^*))$$

defined by $df(X)=X(f)$, we have:

$$\tau_i^*(\Pi\,dc_i^\rho)=\Pi\,d\varepsilon_i^\rho,\qquad \chi_i(\Pi\,d\varepsilon_i^\rho)=\vartheta_i^\rho;$$

so

$$\mu_{E,\tau,V_i}(\Pi\, dc_i^\rho)=\kappa_i^{-1}(\vartheta_i^\rho)$$
$$\tau_i^*(\Pi\, de_i^\alpha)=\Pi\, d\vartheta_i^{m+\alpha},\qquad \chi_i(\Pi\, d\vartheta_i^{m+\alpha})=\varepsilon_i^{m+\alpha};$$

so

$$\mu_{E,\tau,V_i}(\Pi\, de_i^\alpha)=\kappa_i^{-1}(\varepsilon_i^{m+\alpha})$$

τ_i^* and χ_i being even right modules morphisms, $\mu_{E,\tau}$ and μ_τ are bimodules morphisms; in fact, for $\omega_K\in\Omega_K^1(V)$, $a\in\Gamma(V,\mathscr{A})$

$$\mu_\tau((\Pi\,\omega_K)\,a)=\mu_\tau(\Pi\,\omega_K)\,\mu_0(a)$$

with μ_0 of section 2.2.1.

2.2.3. $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$ *is isomorphic to* $S(\Pi\,\Omega_K^1(V))$

$\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$ is isomorphic to $S(\Omega_\tau^1(\pi^{-1}(V)))$ as algebras and modules on $\Omega_\tau^0(\pi^{-1}(V))$. As $\mu_{\tau,V}$ of the preceding section defines an isomorphism from $\Pi\,\Omega_K^1(V)$, as a module over $\Omega_K^0(V)=\mathscr{A}(V)$, to $\Omega_\tau^1(V)$ as a module on $\mu_0(\mathscr{A}(V))=\Omega_\tau^0(V)$, we can extend it to an isomorphism between the graded symmetric tensor products of these modules, preserving the algebra structures.

Note that $S(\Pi\,\Omega_K^1)$ and $\Lambda\,\Omega_K^1$ are not isomorphic as algebras.

2.3. μ_τ is an isomorphism of differential complexes

2.3.1. *A differentiation on* $S(\Omega_E^1)$

There exists on $S(\Omega_E^1)$ a unique d_E-operator of degree (1, 1) such that :

(1) $$\text{if } f\in\Omega_E^0,\qquad d_E f=\Pi\, df,$$

df being the usual differentiation:

$$\langle X, df\rangle = X(f)\qquad \text{for}\quad X\in \mathrm{Der}\,\Gamma(.,\Lambda E^*)$$

(2) $$d_E(\omega_E\,.\,\omega_E')=d_E\,\omega_E\,.\,\omega_E'+(-1)^{\deg\omega_E}\,\omega_E\, d_E\,\omega_E'$$

for $\omega_E,\omega_E'\in S(\Omega_E^1)$, $\deg\omega_E$ is the $\mathbb{Z}_2$ degree.

(3) $$d_E^2=0$$

A general element in $S(\Omega_E^1)$ writes in chart i:

$$\omega_E|_{V_i}=\sum_{J,K} d_E(c_i^{\rho_1})\ldots d_E(c_i^{\rho_j})\,d_E(e_i^{\alpha_1})\ldots d_E(e_i^{\alpha_k})f_{JK}$$
$$J=\{\rho_1,\ldots,\rho_j\},\qquad K=\{\alpha_1,\ldots,\alpha_K\}.$$

where $f_{J,K} \in \Gamma(V_i, \Lambda E^*)$. If

$$f_{JK} = \sum_L f_{JKL}\, e_i^{\alpha_1} \dots e_i^{\alpha_l}, \qquad f_L \in \mathscr{C}^\infty(V_i),$$

$$d_E f_{J,K} = \sum_L \sum_{\sigma=1}^{m} (\partial_\sigma f_{JKL}\, d_E\, c_i^\sigma)\, e_i^{\alpha_1} \dots e_i^{\alpha_l} + \sum_L f_{JKL}\, d_E (e_i^{\alpha_1} \dots e_i^{\alpha_l})$$

where $d_E(e_i^{\alpha_1} \dots e_i^{\alpha_l})$ is deduced also from rule (2).

We then have from the same rule:

$$d_E\, \omega_E = \sum_{J,K} d_E\, c_i^{\rho_1} \dots d_E\, c_i^{\rho_j}\, d_E\, e_i^{\alpha_k}\, d_E f_{JK}.$$

Note. – $d_E\, e_i^\alpha$ is not the 1-form on E, differential in the ordinary sense of the function $\tilde{e}_i^\alpha$ on E $(\tilde{e}_i^\alpha(x) = e_{i,\pi(x)}^\alpha(x))$.

2.3.2. *Isomorphism of differential complex from* $S(\Pi\, \Omega_K^1)$ *to* Ω_τ^N

We have to exhibit a complex isomorphism from $\Omega_\tau^N(\pi^{-1}(V))$ to $S(\Omega_E^1(V))$, or equivalently from $\Lambda\, \mathbb{R}^{m+n}(U_i \times \mathbb{R}^n)$ with the d of 1.2.1. to

$$S(\Pi\, \mathrm{Hom}_{\Lambda\, \mathbb{R}^n(U_i)}(\mathrm{Der}\, \Lambda\, \mathbb{R}^n(U_i), \Lambda\, \mathbb{R}^n(U_i)).$$

It is χ_i^{-1} defined in section 2.2.2. We sum up notations by the following diagram:

$$\begin{array}{ccccc}
\Omega_K^1(V_i) & \xrightarrow{h^*} & \Pi\, \Omega_E^1(V_i) & \xrightarrow{\tau^*} & \mathrm{Hom}_{\Lambda\, \mathbb{R}^n(U_i)}(\mathrm{Der}\, \Lambda\, \mathbb{R}^n(U_i), \Lambda\, \mathbb{R}^n(U_i)) \\
\downarrow{\scriptstyle \mu_{\tau, V_i}} & & \downarrow{\scriptstyle \mu_{E,\tau,V_i}} & & \downarrow{\scriptstyle \chi_i} \\
\Pi\, \Omega_\tau^1(\pi^{-1}(V_i)) & & \xrightarrow{\quad \varkappa_i \quad} & & \Pi\, \Lambda\, \mathbb{R}^{m+n}(U_i \times \mathbb{R}^n)
\end{array}$$

(we denote by the same symbols an even morphism from A to B and the morphism from Π A to Π B that it induces)

$$S(\Pi\, \Omega_K^1(V_i)) \underset{\mu_{\tau, V_i}}{\rightarrow} \Omega_\tau^N(\pi^{-1}(V_i)).$$

3. THE SHEAF Ω_τ

3.1. Localization properties on E

In the sheaf Ω_τ, $\Omega_\tau(W)$ is a graded algebra for W an arbitrary open set in E [*i.e.* even if $\pi^{-1}(\pi(W)) \neq W$]. We have already seen that $\Omega_\tau^N(\pi^{-1}(V))$ is strictly included in $\Omega_\tau(\pi^{-1}(V))$. Note that elements in

$\Omega_\tau(W)$, for W non cylindrical, cannot be obtained by repeated applications of our differential operator d acting on superfunctions of M, or on forms in $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$.

Of course, an element in $\Omega_\tau(W)$ may fail to have an extension to $\Omega_\tau(\pi^{-1}(\pi(W))$. We can also define the support of ω_τ in E in the usual way: we shall say that $\omega_\tau \in \Omega_\tau(W)$ vanishes on $W' \subset W$ if $\forall i \in I(W)$, $\kappa_i(\omega)$ vanishes on $\varphi_i(W' \cap W_i)$; the support of ω_τ will be the complementary in E of the greatest open set on which ω_τ vanishes. Note that an element in $\Omega_\tau(\pi^{-1}(V))$ may have it's support smaller than $\pi^{-1}(V)$: if $\omega_\tau \in \Omega_\tau(W_i)$, it reads in superchart i:

$$\kappa_i(\omega_\tau) = \sum_{J, L} f_{i, J, L}\, \vartheta_i^J \vartheta_i^L \quad \text{[notations of 1.1.4 } (b)]$$

where $f_{i, J, L} \in \mathscr{C}^\infty(U_i \times \mathbb{R}^n)$ may for instance have compact support. Elements in $\Omega_\tau^{\mathbb{N}}(W_i)$ are of the above form, but with the restriction that the $f_{i, J, L}$ are polynomials in the $\mathbb{R}^n$ variables. When $\omega_\tau \in \Omega_\tau(W)$, with $W \neq \pi^{-1}(\pi(W))$, the $f_{i, J, L}$ are only defined in an open set $\varphi_i(W \cap W_i)$ of $\mathbb{R}^{m+n}$.

Elements ω_τ in $\Omega_\tau(\pi^{-1}(V))$ may be decreasing along the fibers of E. We mean that, $\forall i \in I\ (\pi^{-1}(V))$, $\kappa_i(\omega_\tau)$ [in $\Lambda\, \mathbb{R}^{m+n}(c_i(V) \cap U_i \times \mathbb{R}^n)$] have coefficient (of $\vartheta_i^{\lambda_1} \ldots \vartheta_i^{\lambda_r}$) functions which, together with their derivatives, have a decrease of a certain order in the $\mathbb{R}^n$ directions. Precisely:

$$\kappa_i(\omega_\tau) = \sum_\lambda f_\lambda\, \vartheta_i^{\lambda_1} \ldots \vartheta_i^{\lambda_r}$$

$$\exists k; \quad \forall q = (q_1, \ldots, q_n),$$

$$\forall y = (y^1, \ldots, y^m) \in c_i(V) \cap U_i, \quad \forall y' = (y'^1 \ldots, y'^n) \in \mathbb{R}^n$$

$$|(\partial_{y'}^q f_\lambda)(y, y')| < C_q(y)(1 + \|y'\|^2)^{-k/2}$$

This definition is reasonnable, for if decreasing is true for $\omega_i \in \Lambda\, \mathbb{R}^{m+n}(U_{ij} \times \mathbb{R}^n)$, it is true for $h_{ji}(\omega_i)$.

Let us note that the definition would not be correct if the patching supertransformations had coefficient functions with arbitrary behaviour along the fibres of E. So, we could not speak about the behaviour in those direction by looking at an arbitrary family of (super) trivializations of $\mathscr{M}$. That is perhaps not very satisfactory if one has in vew a more direct construction of a $\Omega(W)$. For the time being, our definition makes sense.

3.2. Computations in Ω_τ

For ω_τ in $\Omega_\tau(W)$ and f in $\mathscr{C}^\infty(\mathbb{R})$, one can define $f(\omega_\tau)$ as an element in $\Omega_\tau(W)$. Inded, in a graded manifold, a superfunction may always, by a partition of unity, be written as the sum of superfunctions whose images by supercharts are in $\Lambda\, \mathbb{R}^k(\mathscr{U})$. These images have finite and "infinitesimal" parts and one can consider $f(a)$ for such an element a, defined by the Taylor expansion of f.

In $S(\Pi\Omega^1_K)$, what is to be meant by $f(\omega_K)$ is indicated in [6] since $S(\Pi\Omega^1_K(V))$ is embedded in the sheaf of a graded manifold.

At our level. $\Omega^{\mathbb{N}}_\tau(\pi^{-1}(V))$, which is isomorphic to it, does belong to such a sheaf and therefore, one can use the isomorphism μ_τ and define $f(\mu_\tau(\omega_K))$ for any ω_K in $S(\Pi\Omega^1_K(V))$. The point is that if f is not a polynomial, $f(\mu_\tau(\omega_K))$ is only in $\Omega_\tau(\pi^{-1}(V))$, not in $\Omega^{\mathbb{N}}_\tau(\pi^{-1}(V))$, so that $\mu_\tau^{-1}(f(\mu(\omega_K)))$, expected to define $f(\omega_K)$ has no meaning. For example, $\mu_{\varepsilon,\tau}(\Pi\, de^\alpha_i)$ is the super differential form ω_τ in $\Omega^{\mathbb{N}}_\tau(W_i)$ which reads $\varepsilon^{m+\alpha}_i$ in superchart i. By h^*, it has an image ω_K in $\Pi\Omega^1_K(V_i)$. $\operatorname{Exp}(-\omega^2_K)$ has no meaning, but $\operatorname{Exp}(-\omega^2_\tau)$ is the element in $\Omega_\tau(W_i)$ which reads $\operatorname{Exp}(-\varepsilon^2_i)$ in chart i.

We shall see in next part another way of computing $f(\omega_\tau)$ when we have an isomorphism from Ω_τ to the sheaf of sections of a bundle $\mathscr{E}$, where the operation is allowed.

We sent the reader to reference [9] for a more general setting of the problem.

4. AN ISOMORPHISM $\Omega_\tau \to \Gamma(.,\Lambda\mathscr{E}^*)$ GIVEN BY A CONNECTION ON E.

The aim of this section is to present, using a connection on E, the superdifferential forms as sections of a vector bundle $\Lambda\mathscr{E}^*$ over E.

We shall construct, depending on an arbitrary connection χ on E, a family $\{\gamma_{\tau,W}\}$ of morphisms from $\Omega_\tau(W)$ into $\Gamma(W,\Lambda\mathscr{E}^*)$, for each open set W of E. We shall begin by their definition on the $\Omega^{\mathbb{N}}_\tau(\pi^{-1}(V))$, using the isomorphism μ_E of (2.2.2), that is to say we shall construct morphisms $\gamma_{E,V}$ from $S(\Omega^1_E(V))$ into $\Gamma(\pi^{-1}(V),\Lambda\mathscr{E}^*)$. Starting from their definition as even mappings on $\Omega^1_E(V)$, we shall extend them to $S(\Omega^1_E(V))$ and set

$$\gamma_{\tau,\pi^{-1}(V)}=\gamma_{E.V}\circ\mu^{-1}_{E,\tau} \tag{4.1}$$

Those will be morphisms of graded algebras defined on $\Omega^{\mathbb{N}}_\tau(\pi^{-1}(V))$. Lastly, we shall define extensions $\gamma_{\tau,W}$ on $\Omega_\tau(W)$.

4.1. The bundle $\mathscr{E}$

$\mathscr{E}$ is a vector bundle on the manifold E. It is the pull-back, via the projection $E\xrightarrow{\pi}M_0$, of the vector bundle $TM_0\oplus E$ on M_0.

4.2. Image of $\Omega_E^0(V)$ in $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$

$$\Omega_E^0(V) = \Gamma(V, \Lambda\mathscr{E}^*)$$

A section of ΛE^* is easily transformed into a section of $\Lambda\mathscr{E}^*$; let us denote the map by γ_E [with a slight abuse of notation: we should write $\{\gamma_{E,V}\}$ and the same symbol γ_E will denote morphisms on Ω_E^1 and $S(\Omega_E^1)$]:

Let $\varphi \in \Gamma(V, \Lambda^p E^*)$, $\gamma_E(\varphi)$ is the element of $\Gamma(\pi^{-1}(V), \Lambda^p \mathscr{E}^*)$:

$$\gamma_E(\varphi)(x; t_1, y_1, \ldots, t_p, y_p) = \varphi(\pi(x); y_1, \ldots, y_p)$$

for $x \in E$, $t_i \in T_{\pi(x)}M_0$, $y_i \in E_{\pi(x)}$ *i. e.* $(t_i, y_i) \in \mathscr{E}_x$.

γ_E is a morphism of algebras. If we denote by $\mathscr{E}_2$ the bundle on E with fiber $E_{\pi(x)}$ at x, and also its image in $\mathscr{E}$, the image $\gamma_E(\Omega_E^0(V))$ is in $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}_2^*)$, but not every section is obtained.

Let us denote by $\Gamma_c(\pi^{-1}(V), \Lambda\mathscr{E}^*)$ the set of sections of $\Lambda\mathscr{E}^*$ which are constant along the fibers of E. It is an algebra, a module on $\mathscr{C}_c^\infty(\pi^{-1}(V))$, the set of functions constant along the same fibers.

$$\gamma_E(\Omega_E^0(V)) = \Gamma_c(\pi^{-1}(V), \Lambda\mathscr{E}_2^*)$$

with obvious notation.

4.3. Image of $\Omega_\tau^N(\pi^{-1}(V))$ in $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$ for V open subset of M_0

4.3.1. *A graduation in* Ω_E^1

Every element ω_E in $\Omega_E^1(V_i)$ can be written in term of maps c_i^ρ and sections e_i^α in $\Gamma(V_i, E^*)$:

$$\omega_E = \sum_{\alpha=1}^{n} (\Pi\, de_i^\alpha)\, a_\alpha + \sum_{\rho=1}^{m} (\Pi\, dc_i^\rho)\, b_\rho$$

where a_α and b_ρ are in $\Gamma(V_i, \Lambda E^*)$.

DEFINITION. – We shall denote by $\Omega_{E,p}^1(V)$ the set of elements of $\Omega_E^1(V)$ which have restrictions to the $V \cap V_i$ of the form above, with

$$a_\alpha = 0 \quad \text{if } p = 0$$
$$a_\alpha \in \Gamma(V \cap V_i, \Lambda^{p-1} E^*) \quad \text{if } p \geqq 1$$
$$b_\rho \in \Gamma(V \cap V_i, \Lambda^p E^*) \quad \text{if } p \geqq 0$$

We have of course: $\Omega_E^1 = \bigoplus_{p=0}^{\infty} \Omega_{E,p}^1$.

4.3.2. *What properties should the morphisms* $\gamma_{E,V}$: $\Omega^1_E(V)) \to \Gamma(\pi^{-1}(V), \Lambda \mathscr{E}^*)$ *satisfy?*

4.3.2.1. P_1. *Parity*

γ_E is to be even on Ω^0_E and Ω^1_E for we want γ_τ, induced by γ_E, to be even from $\Omega^{\mathbb{N}}_\tau(\pi^{-1}(V))$ to $\Gamma(\pi^{-1}(V), \Lambda \mathscr{E}^*)$.

4.3.2.2. P_2. *It is a morphism of graded bimodule*

We want: $\forall \omega_E, \omega'_E \in \Omega^1_E, \forall s \in \Omega^0_E$,

$$\gamma_E(\omega_E + \omega'_E) = \gamma_E(\omega_E) + \gamma_E(\omega'_E)$$
$$\gamma_E(\omega_E s) = \gamma_E(\omega_E)\gamma_E(s).$$

That γ_E is a morphism of graded bimodule is then a consequence of property P_1.

4.3.2.3.

$\gamma_{E,V}$ should satisfy the following property: First note that every element s of $\Gamma(V, E^*)$ induces a fonction C^∞ on $\pi^{-1}(V) \subset E$, denoted by $\tilde{s}$, and defined by

$$\forall x \in \pi^{-1}(V), \quad \tilde{s}(x) = s(\pi(x))(x).$$

Property P_3. — For such an s,

$$\gamma_{E,V}(\Pi\, ds) = \tilde{s} + \gamma_{E,V,1}(\Pi\, ds)$$

where $\gamma_{E,V,1}(\Pi\, ds) \in \Gamma(\pi^{-1}(V), \bigoplus_{p \geq 2} \Lambda^p \mathscr{E}^*)$.

That property is the consequence of one we want for $\gamma_\tau : \Omega_\tau \to \Gamma(., \Lambda \mathscr{E}^*) : \gamma_\tau$ is supposed to induce the identity map on E. For W an arbitrary (*i.e.* non necessarily cylindrical in the direction of the fibers of E) open set in E.

$$\gamma_{\tau,W} : \quad \Omega_\tau(W) \to \Gamma(W, \Lambda \mathscr{E}^*)$$

will be defined in section 4.4. It induces a morphism from $\mathscr{C}^\infty(W)$ into itself, denoted by $\text{proj}(\gamma_{\tau,W})$, via the maps

$$\varepsilon : \quad \Omega_\tau(W) \to \mathbb{C}^\infty(W) \qquad (\textit{see } 1.1.3.2)$$
$$\varepsilon_0 : \quad \Gamma(W, \Lambda \mathscr{E}^*) \to \mathscr{C}^\infty(W) \qquad \text{(natural projection)}$$

It satisfies

$$(\varepsilon_0 \circ \gamma_{\tau,W})(\omega_\tau) = \text{proj}(\gamma_{\tau,W}) \circ \varepsilon(\omega_\tau)$$

We want $\text{proj}(\gamma_{\tau,W})$ to be the identity.

So, for $\omega_\tau = \mu_E(\Pi\, de^\alpha_i)$, the above equality gives

$$\varepsilon_0 \circ \gamma_{E,V_i}(\Pi\, de^\alpha_i) = \varepsilon_0 \circ \gamma_{\tau,W_i}(\omega_\tau) = \tilde{e}^\alpha_i.$$

(from section 1.1.3.2 and relation 4.1).

4.3.2.4. *Consequences*

Property P_1 implies that

$$\forall s \in \Gamma(V, E^*) \subset \Omega^0_E,$$
$$\gamma_E(\Pi\, ds) \in \Gamma(\pi^{-1}(V), \bigoplus_{p \geqq 0} \Lambda^{2p} \mathscr{E}^*).$$

Properties P_1 and P_2 imply that $\gamma_E(\Pi\, ds)$ cannot be a function on E, for we must have, if $f \in \mathscr{C}^\infty(V)$, from P_2,

(4.3.2.4.1) $\quad \gamma_E(\Pi\, d(fs)) = \gamma_E(\Pi\, df)\, \gamma_E(s) + \gamma_E(f)\, \gamma_E(\Pi\, ds).$

$\gamma_E(s)$ is odd; so the right-hand side could not be a function if $\gamma_E(\Pi\, ds)$ were one. Therefore, we shall look for morphisms satisfying, for the sake of simplicity:

PROPERTY P_4:

$$\gamma_{E,V}(\Omega^1_{E,1}(V)) \subset \Gamma(\pi^{-1}(V), \Lambda^\circ \mathscr{E}^* \oplus \Lambda^2 \mathscr{E}^*)$$

and even more:

PROPERTY P'_4:

$$\gamma_{E,V}(\Omega^1_{E,1}(V)) \subset \Gamma(\pi^{-1}(V), \Lambda^\circ \mathscr{E}^* \oplus \mathscr{E}^*_1 \wedge \mathscr{E}^*_2)$$

where $\mathscr{E}_1$ is the bundle over E with fiber $T_{\pi(x)} M_0$ at x, $\mathscr{E}_2$ is the bundle over E with fiber $E_{\pi(x)}$ at x and $\mathscr{E}^*_1 \wedge \mathscr{E}^*_2$ is in $\Lambda^2(\mathscr{E}^*_1 \oplus \mathscr{E}^*_2)$.

We shall denote by $\gamma_{E,1}$ the component of γ_E on $\bigoplus_{p>0} \Lambda^p \mathscr{E}^*$, and by $\gamma_{E,0}$ that on $\mathscr{C}^\infty(E)$.

4.3.2.5. *Compatibility with ordinary differentiation*

If $n=0$, $\mathscr{E}^*$ is reduced to $T^* M_0$; so we would like that, if f is in $C^\infty(V)$, $\gamma_E(\Pi\, df)$ reduces in that case to the usual element df of $\Gamma(V, T^* M_0)$. So we ask for:

PROPERTY P_5. — if $f \in \mathscr{C}^\infty(V)$, $\gamma_E(\Pi\, df) \in \Gamma(\pi^{-1}(V), \mathscr{E}^*)$

$$\forall x \in E, \quad t \in T_{\pi(x)} M_0, \quad y \in E_{\pi(x)},$$
$$\gamma_E(\Pi\, df)(x, t, y) = (df)_{\pi(x)}(t).$$

4.3.2.6.

We shall now ask for a property which concern the peculiar nature of the bundle $\mathscr{E}$: It is given with an identification of all fibers $\mathscr{E}_x$ along the same fiber $E_{\pi(x)}$. So, still for the sake of simplicity, we ask for:

PROPERTY P_6. – $\forall \omega_E \in \Omega^1_{E,1}$, $\gamma_{E,1}(\omega_E)$ is constant along the fibers of E. Note that, in account of P_3, this will not be true for $\gamma_E(\omega_E)$; thus, for ω_E arbitrary in Ω^1_E, $\gamma_{E,1}(\omega_E)$ will not be constant along the fibers of E.

It is understood that the collection of morphisms $\gamma_{E,V}$ must be compatible with restriction morphisms.

4.3.2.7.

It is natural to ask oneself whether, given a family of morphisms $\gamma_{E,V}$ from $\Omega_E(V)$ to $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$, compatible with restrictions, even morphisms of Z_2 graded algebras, and satisfying properties P_3, P_5 and perhaps P_6, there exists a super transformation of $\Gamma(., \Lambda\mathscr{E}^*)$ onto itself which converts the $\gamma_{E,V}$'s into a new family satisfying property P_4. In other words, is it possible to factorize a reasonnable morphism $\gamma_{\pi^{-1}(V)}$: $\Omega_\tau(\pi^{-1}(V)) \to \Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$ by a morphism $\gamma_{E,V}$ satisfying properties P_1 to P_6? We leave it as an open question.

4.3.3. *Connection on* E *defined by* γ_E

PROPOSITION. – *Let* $\gamma_{E,V}$ *be a family of morphisms satisfying properties* P_1 *to* P_6 *above: It defines a linear connection* χ *on the bundle* E.

Proof. – Let us show that we can construct, with the $\gamma_{E,V}$'s, a covariant derivation ∇ on E*, *i.e.* a linear map from $\Gamma(V, E^*)$ to $\Omega(V)\otimes_{C^\infty(M_0)}\Gamma(V, E^*)$ satisfying $\nabla(fs) = df\otimes s + f\nabla s$.

For $s\in\Gamma(V, E^*)$, T vector field on V, $Y\in\Gamma(V, E)$, let us define ∇s by

$$\nabla s(T, Y)(m) = \gamma_{E,1}(\Pi\, ds)(m, T_m, 0, 0, Y_m) + \gamma_{E,1}(\Pi\, ds)(m, 0, -Y_m, T_m, 0)$$

The notation $\gamma_{E,1}(\Pi\, ds)(m, \ldots)$ is justified by P_6. From (4.3.2.4.1) and P_3, we have

$$\gamma_E(\Pi\, d(fs)) = \gamma_E(\Pi(df))\,\gamma_E(s) + \gamma_E(f)\,[\tilde{s} + \gamma_{E,1}(\Pi\, ds)].$$

Then, from the equality $fs = (f\circ\pi)\,\tilde{s}$, section 4.2, we have

$$\gamma_{E,1}(\Pi\, d(fs)) = \gamma_E(\Pi\, df)\,\gamma_E(s) + \gamma_E(f)\,\gamma_{E,1}(\Pi\, ds).$$

By computing $\nabla(fs)$ using the above definition of ∇, we check the required equality.

4.3.4. *A connection* χ *on* E *induces morphisms* $\gamma_{E,V}$ *from* $\Omega^1_E(V)$ *to* $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$

A connection on E gives a peculiar isomorphism from $\mathscr{E}$ to TE. As we shall construct, by means of χ, an isomorphism from Ω_τ to $\Gamma(., \Lambda\mathscr{E}^*)$ we shall get, by the way, an isomorphism onto $\Gamma(., T^*E)$. It is to be noted that this last one is not a canonical one, *i.e.* it will still depend on the connection. In fact the non existence of such a canonical isomorphism is due to the impossibility of getting a section of $\bigoplus_{k=1}^{\infty} \Lambda^{2k} T^* E$ using only a section s of E^*, to be the image of $\Pi\, ds$.

Let us construct the morphisms $\gamma_{E, V}$ as the sum of two morphisms $\gamma_{E, V, a}$ and $\gamma_{E, V, b}$.

$$\gamma_{E, V, a} : \ \Omega^1_E(V) \to \Gamma(\pi^{-1}(V), \Lambda \mathscr{E}^*)$$
$$\gamma_{E, V, b} : \ \Omega^1_E(V) \to \Gamma(\bigoplus_{p>0} \Lambda^p \mathscr{E}^*)$$

which we are going to define now.

4.3.4.1. *Definition of* $\gamma_{E, V, a}$

. With the basis e_i^α of $\Gamma(V_i, E^*)$, an element ω_E of $\Omega^1_E(V_i)$ writes:

$$\omega_E = \sum_{\alpha=1}^{n} (\Pi\, de_i^\alpha)\, a_{i,\alpha} + \sum_{\rho=1}^{m} (\Pi\, dc_i^\rho)\, b_{i,\rho},$$
$$a_{i,\alpha},\ b_{i,\rho} \in \Gamma(V_i, \Lambda E^*)$$

We define:

$$\gamma_{E, V_i, a}(\omega_E) = \sum_{\alpha=1}^{n} \gamma_{E, V_i}(a_{i,\alpha})\, \tilde{e}_i^\alpha,$$

where $\gamma_{E, V_i}(a_{i,\alpha})$ is defined in 4.2. $\gamma_{E, V_i, a}$ does not depend on the choice of basis. For an arbitrary open set V, we decompose $\omega_E \in \Omega^1_E(V)$ by a partition of unity:

$$\omega_E = \sum \omega_{E, i}, \qquad \operatorname{supp} \omega_{E, i} \subset W_i$$

and define

$$\gamma_{E, V, a}(\omega_E) = \sum_i \gamma_{E, V_i, a}(\omega_{E, i}|_{V_i})$$

which does not depend on the choice of partition. Note that $\gamma_{E, V, a}(\Omega^1_{E, 0}) = 0$ and $\gamma_{E, V, a}(\Omega^1_{E, 1}) \subset \mathscr{C}^\infty(\pi^{-1}(V))$. The given connection χ does not enter in the definition of $\gamma_{E, V, a}$ but will in that of $\gamma_{E, V, b}$.

4.3.4.2. *Definition of* $\gamma_{E, V, b}$.

For T any vector field on M_0, ∇^χ_T, the covariant derivation along T can be extended to a derivation of $\Gamma(M_0, \Lambda E^*)$. Let ω_E be in $\Omega^1_{E, p}(V)$.

$\gamma_{E, V, b}(\omega_E)$ will be the element of $\Gamma(\pi^{-1}(V), \Lambda^{p+1}\mathscr{E}^*)$ defined by:

$$\gamma_{E, V, b}(\omega_E)(x, t_1, y_1, \ldots, t_{p+1}, y_{p+1}) = (p+1)^{-1} \sum_{k=1}^{p+1} (-1)^{k+1} \langle \nabla^{\chi}_{T_k}, \Pi\omega_E \rangle (\pi(x), y_1, \ldots, \hat{y}_k, \ldots, y_{p+1})$$

where $x \in \pi^{-1}(V)$, $t_i \in T_{\pi(x)} M_0$, $y_i \in E_{\pi(x)}$, and T_k is any vector field on M_0 which has value t_k at $\pi(x)$.

Special case. – If $p=0$, $\gamma_{E, V, b}(\omega_E) \in \Gamma(\pi^{-1}(V), \mathscr{E}^*)$.

$$\gamma_{E, V, b}(x, t, y) = \langle \nabla^{\chi}_T, \Pi\omega_E \rangle (\pi(x)).$$

If $\omega_E = \Pi\, df$, with $f \in \mathscr{C}^\infty(V)$,

$$\forall x \in \pi^{-1}(V), \quad t \in T_{\pi(x)} M_0, \quad y \in E_{\pi(x)},$$
$$\gamma_{E, V, b}(\Pi\, df)(x, t, y) = df_{\pi(x)}(t)$$

4.3.4.3. *Definition and properties of* γ_E *on* Ω^1_E

PROPOSITION. – *The two mappings (bearing the same name for simplicity)*

$$\gamma_{E, V} = \gamma_{E, V, a} + \gamma_{E, V, b} : \ \Omega^1_E(V) \to \Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$$
$$\gamma_{E, V} : \ \Omega^0_E(V) \to \Gamma_c(\pi^{-1}(V), \Lambda\mathscr{E}^{*}_{2}) \qquad (\textit{see } 4.2)$$

define an (even) morphism of graded bimodules which satisfies properties P_3, P'_4, P_5 *and* P_6.

Proof. – Property P_1 is satisfied, for $\gamma_{E, b}$ maps elements of $\Omega^1_{E, p}$, which have the same parity as $p+1$, on sections of $\Lambda^{p+1}\mathscr{E}^*$ with the same parity, and $\gamma_{E, a}$ is also even. Property P_4 follows from the definition: for $\omega_E \in \Omega^1_{E, 1}$,

$$\gamma_{E, V, b}(\omega_E)(x, t_1, y_1, t_2, y_2) = 2^{-1} [\langle \nabla^{\chi}_{T_1}, \Pi\omega_E \rangle (\pi(x), y_2) - \langle \nabla^{\chi}_{T_2}, \Pi\omega_E \rangle (\pi(x), y_1)]$$

To verify property P'_4, one first shows that property P_2 is satisfied in the following special case:

$$\gamma_{E, V_i, b}((\Pi\, de^{\alpha}_i) a_{i, \alpha}) = \gamma_{E, V_i, b}(\Pi\, de^{\alpha}_i) \gamma_{E, V_i}(a_{i, \alpha})$$

if

$$a_{i, \alpha} \in \mathscr{C}^\infty(V_i)$$
$$\gamma_{E, V_i, b}((\Pi\, dc^{\rho}_i) b_{i, \rho}) = \gamma_{E, V_i, b}(\Pi\, dc^{\rho}_i) \gamma_{E, V_i}(b_{i, \rho})$$

if

$$b_{i, \rho} \in \Gamma(V_i, E^*)$$

Then we see that right-hand sides are sections of $\Lambda^2\mathscr{E}^*$ which vanish at x on elements of the form $((t, 0), (t', 0))$ or $((0, y), (0, y'))$ in $\mathscr{E}_x \times \mathscr{E}_x$. Property P'_4 follows from the fact that any element in $\Omega^1_{E, 1}$ is a sum of elements of the forms $(\Pi\, de_i^\alpha)\, a_{i,\,\alpha}$ and $(\Pi\, dc_i^\rho)\, b_{i,\,\rho}$ above. Property P_5 is obvious.

Let us now check property P_2; it is sufficient to check it on $\Omega^1_{E,\,p}$, and locally. Let ω_E be in $\Omega^1_{E,\,p}(V_i)$ and s in $\Gamma(V_i, \Lambda^q E^*)$. We have:

$$\omega_E = \sum_\alpha (\Pi\, de_i^\alpha)\, a_{i,\,\alpha} + \sum_\rho (\Pi\, dc_i^\rho)\, b_{i,\,\rho}$$

where $a_{i,\,\alpha} \in \Gamma(V_i, \Lambda^{p-1} E^*)$ and $b_{i,\,\rho} \in \Gamma(V_i, \Lambda^p E^*)$.

From the definition

$$\gamma_{E,\,V_i,\,a}(\omega_E s) = \sum_\alpha \gamma_{E,\,V_i}(a_{i,\,\alpha} s)\, \tilde{e}_i^\alpha$$

So

$$\gamma_{E,\,V_i,\,a}(\omega_E s) = \sum_\alpha \gamma_{E,\,V_i}(a_{i,\,\alpha})\, \tilde{e}_i^\alpha\, \gamma_{E,\,V_i}(s) = \gamma_{E,\,V_i,\,a}(\omega_E)\, \gamma_{E,\,V_i}(s)$$

Let us show now that $\gamma_{E,\,V_i,\,b}(\omega_E s) = \gamma_{E,\,V_i,\,b}(\omega_E)\, \gamma_{E,\,V_i}(s)$; $\omega_E s \in \Omega^1_{E,\,p+q}$; therefore, $\gamma_{E,\,V_i,\,b}(\omega_E s) \in \Gamma(W_i, \Lambda^{p+q+1} \mathscr{E}^*)$:

$$(1)\quad \gamma_{E,\,V_i,\,b}(\omega_E s)(x, t_1, y_1, \ldots, t_{p+q+1}, y_{p+q+1}) = (p+q+1)^{-1} \sum_{k=1}^{p+q+1} (-1)^{k+1} \langle \nabla^\chi_{T_k} (\Pi\, \omega_E) s \rangle (\pi(x), y_1, \ldots, \hat{y}_k, \ldots, y_{p+q+1})$$

Now, $\langle \nabla^\chi_{T_k}, (\Pi\, \omega_E s \rangle = \langle \nabla^\chi_{T_k}, \Pi\, \omega_E \rangle s$. Let P be a partition of the set $\{1, \ldots, k, \ldots, p+q+1\}$ in two subsets P_1 and P_2 of p and q elements respectively: $P_1 = \{k_1 < k_2 \ldots < k_p\}$, $P_2 = \{k_{p+1} < \ldots < k_{p+q}\}$. Let us denote by $\underline{P}$ the number of transpositions necessary to recover the natural order in $P_1 \cup P_2$. The right-hand side of (1) is:

$$(2)\quad (p+q+1)^{-1} \sum_{k=1}^{p+q+1} (-1)^{k+1} p!\, q!\, [p+q)!]^{-1} \sum_P (-1)^{\underline{P}} \times \langle \nabla^\chi_{T_k}, \Pi\, \omega_E \rangle (\pi(x), y_{k_1}, \ldots, y_{k_p}) . s(\pi(x), y_{k_{p+1}}, \ldots, y_{k_{p+q}}).$$

Let us consider now the set of partitions P′ of $\{1, \ldots, p+q+1\}$ in two subsets of $p+1$ and q elements respectively: $P'_1 = \{i_1 < \ldots < i_{p+1}\}$, $P_2 = \{i_{p+2}, \ldots, i_{p+q+1}\}$. For each such partition, we point out successively each element of P'_1 and denote it by i_ρ. We then have, using definition of $\gamma_{E,\,b}$ and formula for exterior product:

$$(3)\quad (\gamma_{E,\,V_i,\,b}(\omega_E)\, \gamma_{E,\,V_i}(s))(x, t_1, y_1, \ldots, t_{p+1}, y_{p+1}) = p!\, q!\, [(p+q+1)!]^{-1} . \sum_{P'} (-1)^{\underline{P}'} \sum_{i_\rho \in P_1}{}' (-1)^{\rho+1} \langle \nabla^\chi_{T_{i_\rho}} . \Pi\, \omega_E \rangle (\pi(x), y_{i_1}, \ldots, \hat{y}_{i_\rho}, \ldots, y_{i_{p+1}}) \times s(\pi(x), y_{i_{p+2}}, \ldots, y_{i_{p+q+1}})$$

Triplets (k, P_1, P_2) and (P'_1, P'_2, ρ) are in bijective relation and

$$\underline{P}+k+1=\underline{P}'+\rho+1.$$

Quantities associated to these two triplets being the same in expressions (2) and (3), we get the expected equality and, linearity of γ_E being obvious, property P_2.

4.3.5. $\gamma_{E,\,V}$ *extends to a morphism from* $S(\Omega^1_E(V))$ *to* $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$

It satisfies

$$\gamma_E(\omega_E \otimes_s \omega'_E)=\gamma_E(\omega_E)\,\gamma_E(\omega'_E).$$

Let us describe $\gamma_{E,\,V_i}$ using the following local basis of sections of $\Gamma(W_i, \Lambda\mathscr{E}^*)$:

$\tilde{c}^\rho_{i,\,*}$ will denote the section of $\Lambda\mathscr{E}^*$ which, at point x, associates to the element (t, y) of $T_{\pi(x)}M_0\oplus E_x=\mathscr{E}_x$ the ρ-component of t on the base induced by chart c_i.

$\tilde{\varphi}^\alpha_i$ will denote the section which associates to the same (t, y) the α-component of y on the base of $E_{\pi(x)}$ induced by φ_i, *i.e.* on $e_{i,\,\alpha,\,\pi(x)}$.

A function on E will be expressed in terms of the charts on E: $\underline{c}^\rho_i=c^\rho_i\circ\pi$ and $\tilde{e}^\alpha_i$. (A bar under the symbol of a function on M_0 denotes its lift to a function on E.) We have, from preceding definitions, on Ω^1_E:

$$\gamma_{E,\,V_i}(\Pi\, dc^\rho_i)=\tilde{c}^\rho_{i,\,*}$$
$$\gamma_{E,\,V_i}(\Pi\, de^\alpha_i)=\tilde{e}^\alpha_i+\underline{a}^\alpha_{i,\,\rho,\,\beta}\,\tilde{c}^\rho_{i,\,*}\wedge\tilde{\varphi}^\beta_i$$

if

$$\nabla^\chi(e^\alpha_i)=a^\alpha_{i,\,\rho\beta}\,dc^\rho_i\otimes e^\beta_i.$$
$$\Big(\nabla^\chi_T=\sum_{\rho,\,\alpha,\,\beta} a^\alpha_{i,\,\rho\beta}\,t^\rho e^\beta_i\,\partial/\partial e^\alpha_i+\sum_\rho t^\rho\,\partial/\partial c^\rho_i \text{ if } T=\sum t^\rho\,\partial/\partial c^\rho_i\Big)$$

On Ω^0_E

$$\gamma_{E,\,V_i}(c^\rho_i)=\underline{c}^\rho_i$$
$$\gamma_{E,\,V_i}(e^\alpha_i)=\tilde{\varphi}^\alpha_i.$$

From these formulas, we get:

$$\gamma_{E,\,V_i}(\Pi\, de^\alpha_i-a^\alpha_{i,\,\rho\beta}\,\Pi\, dc^\rho_i\otimes_s e^\beta_i)=\tilde{e}^\alpha_i$$

So, by operations in $S(\Omega^1_E)$, only polynomial functions of $\tilde{e}^\alpha_i$ can be obtained by the map γ_E.

4.3.4.5. *Definition of* $\gamma_{\tau,\,\pi^{-1}(V)}$.

We recall that our aim, in constructing $\gamma_{E,\,V}$, morphism from $S(\Omega^1_E(V))$ to $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$ was to define $\gamma_{\tau,\,\pi^{-1}(V)}$ from $\Omega^N_\tau(V)$ to $\Gamma(\pi^{-1}(V), \Lambda\mathscr{E}^*)$. We set, as we announced in the introduction of this part four:

$$\gamma_{\tau,\,\pi^{-1}(V)} = \gamma_{E,\,V} \circ \mu_{E,\,\tau}^{-1}.$$

Formulas of section 4.3.4.4 give:

$$\begin{aligned} &\gamma_{\tau,\,W_i}(\kappa_i^{-1}(\varepsilon_i^\rho)) = \underline{c}_i^\rho \\ &\gamma_{\tau,\,W_i}(\kappa_i^{-1}(\varepsilon_i^{m+\alpha})) = \tilde{e}_i^\alpha + \underline{a}^\alpha_{i,\,\rho\beta}\, \tilde{c}^\rho_{i,\,*} \wedge \tilde{\varphi}_i^\beta \\ &\gamma_{\tau,\,W_i}(\kappa_i^{-1}(\vartheta_i^\rho)) = \tilde{c}^\rho_{i,\,*} \\ &\gamma_{\tau,\,W_i}(\kappa_i^{-1}(\vartheta_i^{m+\alpha})) = \tilde{\varphi}_i^\alpha. \end{aligned} \tag{4.3.4.5}$$

We are now in a position to define γ_τ on all Ω_τ, which is the purpose of this part. We could have avoided the detour by $S(\Omega_E^1)$ and define directly a morphism from Ω_τ to $\Gamma(.,\Lambda\mathscr{E}^*)$ but it is perhaps illuminating to do this way.

4.4. An isomorphism between Ω_τ and $\Gamma(.,\Lambda\mathscr{E}^*)$

For W an arbitrary open set in E, we shall define a collection $\{\gamma_{\tau,\,W}\}$ of isomorphisms from $\Omega_\tau(W)$ to $\Gamma(W,\Lambda\mathscr{E}^*)$. We emphasize that there are two extensions to be done: First, $\gamma_{\tau,\,\pi^{-1}(V)}$ has been defined on $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$ and will be extended to $\Omega_\tau(\pi^{-1}(V))$. Secondly γ_τ will be defined on $\Omega_\tau(W)$.

$\gamma_{\tau,\,\pi^{-1}(V)}$ was defined in preceding section from $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$ into $\Gamma(\pi^{-1}(V),\Lambda\mathscr{E}^*)$. For $\omega_\tau^{(i)} \in \Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$, $i=1,\ldots,k$ and $f \in \mathscr{C}^\infty(\mathbb{R}^k)$, one can compute, by using supercharts, $f(\omega_\tau^{(1)},\ldots,\omega_\tau^{(k)})$ which will be an element in $\Omega_\tau(\pi^{-1}(V))$, the support of which may not be all $\pi^{-1}(V)$. Also, for $S^{(i)} \in \Gamma(\pi^{-1}(V),\Lambda\mathscr{E}^*)$ one can compute $f(S^{(1)},\ldots,S^{(k)})$ in $\Gamma(\pi^{-1}(V),\Lambda\mathscr{E}^*)$. We expect the extension of $\gamma_{\tau,\,\pi^{-1}(V)}$ to $\Omega_\tau(\pi^{-1}(V))$ to satisfy:

$$\gamma_{\tau,\,\pi^{-1}(V)}(f(\omega_\tau^{(1)},\ldots,\omega_\tau^{(k)})) = f(\gamma_{\tau,\,\pi^{-1}(V)}(\omega_\tau^{(1)}),\ldots,\gamma_{\tau,\,\pi^{-1}(V)}(\omega_\tau^{k}))$$

As for the second extension, we already noted that $\omega_\tau \in \Omega_\tau(W)$ may not have an extension to $\Omega_\tau(\pi^{-1}(V))$ and thus we cannot use only morphisms $\gamma_{\tau,\,\pi^{-1}(V)}$. So we proceed in the three following steps to define $\gamma_{\tau,\,W}$:

(*a*) From $\gamma_{\tau,\,\pi^{-1}(V)}$ on $\Omega_\tau^{\mathbb{N}}(\pi^{-1}(V))$, if $V \subset V_i$, we get morphisms

$$\gamma_{\tau,\,\pi^{-1}(V),\,i}: \bigoplus_{p=1}^{\infty} (\Lambda\mathbb{R}^{m+n}(c_i(V)\times\mathbb{R}^n))^{(p)} \to \Gamma(\pi^{-1}(V),\Lambda\mathscr{E}^*).$$

They extend to $\Lambda\mathbb{R}^{m+n}(c_i(V)\times\mathbb{R}^n)$ with images still in $\Gamma(\pi^{-1}(V),\Lambda\mathscr{E}^*)$.

(*b*) For $W \subset W_i$, we define

$$\gamma_{\tau,\,W,\,i}: \Lambda\mathbb{R}^{m+n}(\varphi_i(W)) \to \Gamma(W,\Lambda\mathscr{E}^*).$$

such that for $W=\pi^{-1}(V)$, we recover the morphisms of (*a*).

(*c*) We show that, for W arbitrary, the $\gamma_{\tau,\, W \cap W_i,\, i}$ patch together and define

$$\gamma_{\tau,\, W}: \quad \Omega_\tau(W) \to \Gamma(W, \Lambda \mathscr{E}^*),$$

i.e. $\gamma_{\tau,\, W \cap W_i,\, i}(\kappa_i(\omega_\tau))$ and $\gamma_{\tau,\, W \cap W_j,\, j}(\kappa_j(\omega_\tau))$ coïncide on $W \cap W_{ij}$.

Step a. Definition of $\gamma_{\tau,\, \pi^{-1}(V),\, i}$. — Let V be an open set in V_i.

$$\gamma_{\tau,\, \pi^{-1}(V),\, i} = \gamma_{\tau,\, \pi^{-1}(V)} \circ \kappa_i^{-1}$$

defines a morphism on $\Lambda \mathbb{R}^{m+n}(c_i(V) \times \mathbb{R}^n)$. Indeed, for $\omega_i \in \Lambda \mathbb{R}^{m+n}(c_i(V) \times \mathbb{R}^n)$, κ_i^{-1} is the element of $\Omega_\tau(\pi^{-1}(V))$ which writes ω_i in superchart *i*. If $V \subset V_i \cap V_j$,

$$\gamma_{\tau,\, \pi^{-1}(V),\, j} = \gamma_{\tau,\, \pi^{-1}(V),\, i} \circ h_{ij}$$
$$[\text{on } \Lambda \mathbb{R}^{m+n}(c_j(V) \times \mathbb{R}^n)].$$

Step b. Definition of $\gamma_{\tau,\, W,\, i}$, *for* $W \subset W_i$. — We define $\gamma_{\tau,\, W,\, i}$ by giving its action on $\mathscr{C}^\infty(\varphi_i(W))$ and on the ϑ_i^λ's. For $f \in \mathscr{C}^\infty(\varphi_i(W))$, we set: (4.4.*b*.1)

$$\gamma_{\tau,\, W,\, i}(f) = f \circ (\varphi_i|_W) + \sum_p \prod_{\lambda=1}^{m+n} (p_\lambda !)^{-1} \times [\gamma_{\tau,\, i,\, b}(\varepsilon_i^\lambda|_W)]^{p_\lambda} (\partial_1^{p_1} \ldots \partial_{m+n}^{p_{m+n}} f)(\varphi_i|_W)$$

where

$$\gamma_{\tau,\, i,\, b} = \gamma_{\tau,\, W_i,\, i,\, b} = \gamma_{\tau,\, W_i,\, b} \circ \kappa_i^{-1}$$
$$\gamma_{\tau,\, W_i,\, b} = \gamma_{E,\, V_i,\, b} \circ \mu_{E,\, \tau}^{-1}.$$

The sum has only a finite number of nonvanishing terms, for $(\gamma_{\tau,\, i,\, b}(\varepsilon_i^\lambda))^N = 0$ if $N > \dfrac{m+n}{2}$. One verifies that $\gamma_{\tau,\, W,\, i}$ is a morphism of algebra from $\mathscr{C}^\infty(\varphi_i(W))$ into $\Gamma(W, \Lambda \mathscr{E}^*)$.

On the generators of $\Lambda \mathbb{R}^{m+n}$, we set:

$$\gamma_{\tau,\, W,\, i}(\vartheta_i^\lambda) = \gamma_{\tau,\, \pi^{-1}(\pi(W)),\, i}(\vartheta_i^\lambda)|_W.$$

Thus we have defined a morphism of algebra $\gamma_{\tau, W, i}$, from $\Lambda \mathbb{R}^{m+n}(\varphi_i(W))$ into $\Gamma(W, \Lambda \mathscr{E}^*)$, where $W \subset W_i$. It is an isomorphism. Let us look at $\gamma_{\tau, W_i, i}$: we have

$$(4.4.b.2) \qquad \begin{aligned} \gamma_{\tau, W_i, i}(\varepsilon_i^\rho) &= \underline{c}_i^\rho \\ \gamma_{\tau, W_i, i}(\varepsilon_i^{m+\alpha}) &= \tilde{e}_i^\alpha + \underline{a}_{i,\rho\beta}^\alpha \tilde{c}_{i,*}^\rho \wedge \tilde{\varphi}_i^\beta \\ \gamma_{\tau, W_i, i}(\vartheta_i^\rho) &= \tilde{c}_{i,*}^\rho \\ \gamma_{\tau, W_i, i}(\vartheta_i^{m+\alpha}) &= \tilde{\varphi}_i^\alpha \end{aligned}$$

Any element in $\Gamma(W_i, \Lambda \mathscr{E}^*)$ is a linear combination, with coefficients in $\mathscr{C}^\infty(W_i)$ of products of the $\tilde{c}_{i,*}^\rho$'s and $\tilde{\varphi}_i^\beta$'s. Let F be in $C^\infty(W_i)$,

$f = F \circ \varphi_i^{-1} \in \mathscr{C}^\infty(\varphi_i(W_i))$.

$$f(c_i^1(x), \ldots, c_i^m(x), \tilde{e}_i^1(x), \ldots, \tilde{e}_i^n(x)) = F(x)$$

That $f(\gamma_{\tau, W_i, i}^{-1}(c_i^1), \ldots, \gamma_{\tau, W_i, i}^{-1}(\tilde{e}_i^n))$ defines an element in $\Lambda \mathbb{R}^{m+n}(U_i \times \mathbb{R}^n)$ will be a consequence of lemma of step *c*; it is $\gamma_{\tau, W_i, i}^{-1}(f)$. So,

$$\gamma_{\tau, W_i, i}^{-1}(f) = f(\varepsilon_i^1, \ldots, \varepsilon_i^m, \varepsilon_i^{m+1} - \underline{a}_{i, \rho\beta}^1 \vartheta_i^\rho \vartheta_i^{m+\beta}, \ldots, \varepsilon_i^{m+n} - \underline{a}_{i, \rho\beta}^n \vartheta_i^\rho \vartheta_i^{m+\beta})$$
$$= f + \text{terms of order} \geqq 1 \text{ in } \vartheta\text{'s}.$$

From our definition of $\gamma_{\tau, W, i}$ on $\Lambda \mathbb{R}^{m+n}(\varphi_i(W))$. it is clear that if $f \in C^\infty(W)$ has an extension $\tilde{f}$ to $\tilde{W} \supset W$, then $\gamma_{\tau, \tilde{W}, i}(\tilde{f})|_W = \gamma_{\tau, W, i}(f)$.

If f is a polynomial in the $\mathbb{R}^n$ variables, with coefficients in $C^\infty(U_i')$, $U_i' \subset U_i$, thus defined in $U_i' \times \mathbb{R}^n$, then $\gamma_{\tau, \varphi_i^{-1}(U_i' \times \mathbb{R}^n), i}(f)$ defined in step *a* or *b* coïncide, for it is true if $f = \varepsilon_i^{m+\alpha}$ or if $f \in C^\infty(U_i')$.

(*c*) *Definition of* $\gamma_{\tau, W}$. — Let us verify that $\forall f_i \in \Lambda \mathbb{R}^{m+n}(\varphi_i(W \cap W_i)$

$$\gamma_{\tau, W \cap W_{ij}, i}(f_i|_{\varphi_i(W \cap W_{ij})}) = \gamma_{\tau, W \cap W_{ij}, j}(h_{ji}(f_i|_{\varphi_i(W \cap W_{ij})}).$$

We shall then define, for $\omega_\tau \in \Omega_\tau(W)$, $\gamma_{\tau, W}(\omega_\tau)$ by

$$\gamma_{\tau, W}(\omega_\tau)|_{W \cap W_i} = \gamma_{\tau, W \cap W_i, i}(\kappa_i(\omega_\tau|_{W \cap W_i})).$$

Preliminary to our verification, let us set four points.

(1) Let $g^{(\lambda)}$, $\lambda = 1, \ldots, N$ be N sections of ΛF^* where F is a vector bundle over a manifold W. ε_0 being the natural morphism $\Lambda F^* \to \mathscr{C}^\infty(W)$, let us denote by g_0 the map from W to $\mathbb{R}^N$ the λ-component of which is $\varepsilon_0(g^{(\lambda)})$. Lastly, let f be in $\mathscr{C}^\infty(g_0(W))$. Then, we can define an element of $\Gamma(\Lambda F^*)$, which we shall note $f(g)$, by

$$f(g) = f \circ g_0 + \sum_{p_1, \ldots, p_N} \prod_{\lambda=1}^{N} (p_\lambda!)^{-1} (g_1^{(\lambda)})^{p_\lambda} (\partial_1^{p_1} \ldots \partial_N^{p_N} f) \circ g_0$$

where $g_1^{(\lambda)} = g^{(\lambda)} - \varepsilon_0(g^{(\lambda)})$.

(2) Let $\mathscr{O}_i$ be an open set in $U_i \times \mathbb{R}^n$ and $f \in \mathscr{C}^\infty(\mathscr{O}_i)$.

From (1),

$$f(\gamma_{\tau, W_i, i}(\varepsilon_i^1)|_{\varphi_i^{-1}(\mathscr{O}_i)}, \ldots, \gamma_{\tau, W_i, i}(\varepsilon_i^N)|_{\varphi_i^{-1}(\mathscr{O}_i)})$$

is a well defined element in $\Gamma(\varphi_i^{-1}(\mathscr{O}_i), \Lambda \mathscr{E}^*)$.

Take $W = \varphi_i^{-1}(\mathscr{O}_i)$, $F = \mathscr{E}|_W$, $g^{(\lambda)} = \gamma_{\tau, W_i, i}(\varepsilon_i^\lambda)|_{\varphi_i^{-1}(\mathscr{O}_i)}$.

So $g_0 = \varphi_i|_{\varphi_i^{-1}(\mathscr{O}_i)}$.

(3) For $f \in \mathscr{C}^\infty(\mathscr{O}_i)$, $\gamma_{\tau, \varphi_i^{-1}(\mathscr{O}_i), i}(f)$ was defined in (*b*) and we see that:

$$\gamma_{\tau, \varphi_i^{-1}(\mathscr{O}_i), i}(f) = f(\gamma_{\tau, \varphi_i^{-1}(\mathscr{O}_i), i}(\varepsilon_i^1), \ldots, \gamma_{\tau, \varphi_i^{-1}(\mathscr{O}_i), i}(\varepsilon_i^{(m+n)}))$$

This relation can be put more generally on the following form:

(4) LEMMA. — *Let* $\mathscr{U}_i$ *and* $\mathscr{O}_i$ *be two open sets in* $U_i \times \mathbb{R}^n$, $f \in \mathscr{C}^\infty(\mathscr{O}_i)$, $g^{(\lambda)} \in \Lambda \mathbb{R}^{m+n}(\mathscr{U}_i)$, $\lambda = 1, \ldots, m+n$ *such that, with notation of*

(1), $g_0(\mathcal{U}_i)\subset\mathcal{O}_i$; *then, in* $\Gamma(\varphi_i^{-1}(\mathcal{U}_i), \Lambda\mathcal{E}^*)$ *one has:*

$$\gamma_{\tau, \mathcal{U}_i, i}(f(g^1, \ldots, g^{m+n})) = f(\gamma_{\tau, \mathcal{U}_i, i}(g^1), \ldots, \gamma_{\tau, \mathcal{U}_i, i}(g^{m+n})).$$

We are now in a position to do easily our verification.

$$h_{ji}(f_i|_{\varphi_i(W\cap W_{ij})}) \in \Lambda\mathbb{R}^{m+n}(\varphi_j(W\cap W_{ij}))$$

$$\gamma_{\tau, W\cap W_{ij}, j}(h_{ji}(f_i|_{\varphi_i(W\cap W_{ij})}) = \gamma_{\tau, W\cap W_{ij}, j}(f_i(h_{ji}(\varepsilon_i^1|_{\varphi_i(W\cap W_{ij})}), \ldots))$$

To comute the symbols h_{ji} and f_i, we use the fact that the values of

$$(\varepsilon_0(h_{ji}(\varepsilon_i^1|_{\varphi_i(W\cap W_{ij})})), \ldots, \varepsilon_0(h_{ji}(\varepsilon_i^{m+n}|_{\varphi_i(W\cap W_{ij})})))$$

fall in the domain of definition of $f_i|_{\varphi_i(W\cap W_{ij})}$ to give a meaning to the r. h. s., and a "regularity" hypothesis in order to have equality.

By the lemma,

$$\begin{aligned}\gamma_{\tau, W\cap W_{ij}, j}(f_i|_{\varphi_i(W\cap W_{ij})}(\ldots, h_{ji}(\varepsilon_i^\lambda|_{\varphi_i(W\cap W_{ij})}), \ldots)\\ = f_i|(\ldots, \gamma_{\tau, W\cap W_{ij}, j}h_{ji}(\varepsilon_i^\lambda|_{\varphi_i(W\cap W_{ij})}), \ldots)\\ = f_i|(\ldots, \gamma_{\tau, W\cap W_{ij}, i}(\varepsilon_i^\lambda|_{\varphi_i(W\cap W_{ij})}), \ldots)\\ = \gamma_{\tau, W\cap W_{ij}, i}(f_i|_{\varphi_i(W\cap W_{ij})}).\end{aligned}$$

Thus the verification is achieved.

The family $\{\gamma_{\tau, W}\}$ thus defined is obviously a collection of morphisms of Z_2 graded algebras, compatible with restriction morphisms. They define a sheaf isomorphism from Ω_τ to $\Gamma(., \Lambda\mathcal{E}^*)$. The inverse mappings are constructed from formulas (4.3.4.5). If F is in $\mathcal{C}^\infty(W)$ with W arbitrary, $f_i = F\circ\varphi_i^{-1}\in\mathcal{C}^\infty(\varphi_i(W\cap W_i))$.

$$\omega_i = f_i(\gamma_{\tau, W_i, i}^{-1}(c_i^1), \ldots, \gamma_{\tau, W_i, i}^{-1}(\tilde{e}_i^n))$$

is well defined in $\Lambda\mathbb{R}^{m+n}(\varphi_i(W\cap W_i))$ and

$$\gamma_{\tau, W\cap W_i, i}(\omega_i) = F|_{W\cap W_i}$$

We have

$$\omega_j|_{\varphi_j(W\cap W_{ij})} = h_{ji}\,\omega_i|_{\varphi_i(W\cap W_{ij})}.$$

Thus, the family (ω_i) defines an element $\omega_\tau\in\Omega_\tau(W)$ and $\gamma_{\tau, W}(\omega_\tau) = F$.

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NOTATIONS

General objects

$(\mathbb{R}^k)_{i,j}, \ldots$ copies of $\mathbb{R}^k$.

$\varepsilon_1, \ldots, \varepsilon_k$ canonical basis of $\mathbb{R}^k$.
$\varepsilon_{i,1}, \ldots, \varepsilon_{i,k}$ canonical basis of $\mathbb{R}^k_i$.
$\varepsilon_i^1, \ldots, \varepsilon_i^k$ dual basis.

Indices

$i, j, k \ldots \in \mathrm{I}$, label open sets of an open covering of a manifold M_0.
ν, ρ, σ, τ, run from 1 to m.
$\alpha, \beta, \gamma, \delta, \varepsilon$, run from 1 to n.
λ, μ, run from 1 to $m+n$.

Objects relative to M_0

m_0, m, points in M_0.
$(\mathrm{V}_i)_{i \in \mathrm{I}}$ open covering of M_0, $\mathrm{V}_{ij} = \mathrm{V}_i \cap \mathrm{V}_j$, $V_{ijk} = \mathrm{V}_i \cap \mathrm{V}_j \cap V_k$.
$c_i : \mathrm{V}_i \to \mathrm{U}_i \subset \mathbb{R}^m$ chart on M_0.
$c_i^\rho : \mathrm{V}_i \to \mathbb{R}$ it's components.
$c_{ij} : c_j(\mathrm{V}_{ij}) \to c_i(\mathrm{V}_{ij})$, $c_{ij} = c_i \circ c_j^{-1}$.
$c_{ij}^\rho = c_i^\rho \circ c_j^{-1}$.
$(\partial_{i,\rho})_{m_0} = (c_i^{-1})_* \varepsilon_\rho$.

Objects relative to E, *fiber bundle over* M_0

rank of E is n.
π: projection $\mathrm{E} \to \mathrm{M}_0$.
$\mathrm{W}_i = \pi^{-1}(\mathrm{V}_i)$, $\mathrm{W}_{ij} = \mathrm{W}_i \cap \mathrm{W}_j$.
$\psi_i : \mathrm{W}_i \to \mathrm{V}_i \times (\mathbb{R}^n)_i$ local trivialization of E.
$\varphi_i : \mathrm{W}_i \to \mathrm{U}_i \times (\mathbb{R}^n)_i$, $\varphi_i = (c_i \times \mathrm{Id}) \circ \psi_i$.
x point in E.
$\varphi_{i, m_0} : \pi^{-1}(m_0) \to (\mathbb{R}^n)_i$.
$e_{i, m_0, \alpha} = \varphi_{i, m_0}^{-1}(\varepsilon_{i,\alpha})$; $e_i^\alpha \in \Gamma(\mathrm{V}_i, \mathrm{E}^*)$.
$x_i^\rho = c_i^\rho \circ \pi$, real valued functions on W_i.
$x_i^{m+\alpha}(.) = e^\alpha_{i, \pi(.)}(.)$, real valued functions on W_i, also denotes by $\tilde{e}_i^\alpha$.
$g_{ij} : \mathrm{V}_{ij} \to \mathscr{L}((\mathbb{R}^n)_j, (\mathbb{R}^n)_i)$ transition functions

$$g_{ij}(m_0) = \varphi_{i, m_0} \circ \varphi_{j, m_0}^{-1}$$

$(g_{ij}(m_0))^\alpha_\beta = \varepsilon_i^\beta [g_{ij}(m_0)(\varepsilon_{j,\alpha})]$.
$\underline{g}_{ij} = g_{ij} \circ \pi$.
$\tilde{g}_{ij} : \mathrm{U}_j \to \mathscr{L}((\mathbb{R}^n)_j, (\mathbb{R}^n)_i)$, $\tilde{g}_{ij} = g_{ij} \circ c_j^{-1}$.
$\underline{\tilde{g}}_{ij} : \mathrm{U}_j \times \mathbb{R}^n \to \mathscr{L}((\mathbb{R}^n)_j, (\mathbb{R}^n)_i)$.
$\underline{\tilde{g}}_{ij} = \tilde{g}_{ij} \circ p_1$, p_1 projection $\mathrm{U}_j \times \mathbb{R}^n \to \mathrm{U}_j$.

Objects relative to $\mathscr{E} = \pi^{-1}(\mathrm{E} \oplus \mathrm{TM})$

π projection of $\mathscr{E}$ onto E.
$\mathscr{W}_i = \Gamma^{-1}(\mathrm{W}_i)$, $\mathscr{W}_{ij} = \mathscr{W}_i \cap \mathscr{W}_j$.
Φ_i trivialization above W_i

$$\Phi_i : \quad \mathscr{W}_i \to \mathrm{U}_i \times \mathbb{R}^n \times \mathbb{R}^m \times \mathscr{R}^n$$

X points of $\mathscr{E}$, $\mathrm{X}_i^\rho = x_i^\rho \circ \pi$, $\mathrm{X}_i^{m+\alpha} = x_i^{m+\alpha} \circ \pi$.

Graded manifolds, superfunctions, super differential forms

M graded manifold [associated to the pair (M_0, E)].

$\mathscr{A}_M$ sheaf of superfunctions of M.

$\Gamma(\mathscr{A}, U)$ superfunctions of domain U, $\mathscr{A}(U)$ subsheaf of $\mathscr{A}$.

ΛE^* sheaf of sections of the bundle ΛE^*.

$\overline{\Lambda R^k}(U)=\mathscr{C}^\infty(U)\otimes\Lambda\mathbb{R}^k$, U open set in $\mathbb{R}^k$, as a sheaf.

$\mathscr{M}$ graded manifold of superdifferential forms.

$\mathscr{A}_{\mathscr{M}}=\Omega_M$ it's sheaf. A superdifferential form is an element of $\mathscr{A}_{\mathscr{M}}$.

$h:\mathscr{A}_M\to\Gamma(.,\Lambda E^*)$ sheaf isomorphism.

h_{ij} : patching supertransformations defining $\mathscr{M}$.

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