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Analysis of a quantum Markov chain

by

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ABSTRACT. — A quantum chain is analogous to a classical stationary Markov chain except that the probability measure is replaced by a complex amplitude measure and the transition probability matrix is replaced by a transition amplitude matrix. After considering the general situation, we study a particular example of a quantum chain whose transition amplitude matrix has the form of a Dirichlet matrix. Such matrices generate a discrete analog of the usual continuum Feynman amplitude. We then compute the probability distribution for these quantum chains.

RÉSUMÉ. — Une chaîne de Markov quantique est analogue à une chaîne de Markov stationnaire classique avec la différence que la mesure de probabilité est remplacée par une mesure d'amplitude complexe, et la matrice des probabilités de transition est remplacée par une matrice d'amplitude de transition. Après avoir considéré le cas général, nous étudions le cas particulier d'une chaîne quantique dont la matrice de transition est une matrice de Dirichlet. De telles matrices conduisent à un analogue discret des amplitudes de Feynman continues. Nous calculons ensuite les distributions de probabilité de ces chaînes quantiques.

INTRODUCTION

The time evolution for a reversible quantum system is usually governed by a one-parameter unitary group $U(t)$. This unitary group can then be used to compute the state of the system at any time from its initial state. To be precise, if the initial state at time $t=0$ is given by a unit vector ψ_0 in Hilbert space, then the state at time t is $U(t)\psi_0$. Moreover, $U(t)$ can be used to find transition amplitudes which are important for calculating scattering cross-sections, decay lifetimes and decay probabilities. If the system is in the state ψ at some time, then $\langle \varphi, U(t)\psi \rangle$ is the transition amplitude that the system goes to state φ after an elapsed time t.

In this paper we consider a simplified discrete version of quantum mechanics. In this case, observables only have a finite number of values and time is discrete. We then have a finite dimensional Hilbert space H and a single unitary operator U on H which generates a discrete unitary group $U(n)=U^n$. We can then interpret $\langle \varphi, U\psi \rangle$ as the transition amplitude from state ψ to state φ in one time step. If ψ_j, $j=0, 1, \ldots, n-1$, is an orthonormal basis for H, we define the $n \times n$ transition amplitude matrix $\hat{A}$ relative to this basis by $\hat{A}_{jk}=\langle \psi_j, U\psi_k)$. It is clear that $\hat{A}$ is a unitary matrix.

From the probabilistic viewpoint, $(\varphi, U\psi)$ is the conditional amplitude that the system is in state φ given that it was in state ψ one time unit previously. The usual transition probability matrix of conventional probability theory is now replaced by the transition amplitude matrix $\hat{A}$. Since we have a probabilistic interpretation, one might suspect that there is an underlying discrete stochastic process that generates these transition amplitudes. This is indeed true and the functions f_j of this process can be interpreted as quantum mechanical observables or as we shall call them, measurements. Since $\hat{A}$ is unitary, we call f_j a unitary process and much of our work in Section 1 can be applied to such general processes. However, we shall be primarily interested in processes with the additional properties of being Markov and stationary. For this to happen, $\hat{A}$ must not only be unitary but it must be stochastic. We then call f_j a quantum chain.

In Section 1 we consider the general theory of quantum chains. Section 2 studies a particular example of a quantum chain whose transition amplitude matrix has the form of a Dirichlet matrix. Such matrices generate a discrete analog of the usual continuum Feynman amplitude ([2], [5], [6]) and might be useful in providing a method of approximation to Feynman integrals. They have also been useful in certain elementary particle studies [7]. We compute the probability distribution for these quantum chains in Section 3. In an appendix we prove some technical results that are used in the proofs of the theorems of Section 3.

There are various other approaches to quantum Markov processes that have been considered in the literature. Although these approaches are quite different in scope from that presented here, the reader may want to consult some of them for a comparison ([1], [3], [4], [8], [9], [11]).

1. QUANTUM N-CHAINS

Let Ω be a nonempty set called a *sample space*. The elements of Ω are called *sample points* and they represent a set of possible configurations for a physical system. Let Λ be a σ-algebra of subsets of Ω and let $A:\Lambda\rightarrow\mathbb{C}$ be a complex measure with $A(\Omega)=1$. We interpret Λ as a set of quantum events. If $|A(\Delta)|\leqq 1$, then $A(\Delta)$ is interpreted as the amplitude that the event $\Delta\in\Lambda$ occurs and $P(\Delta)=|A(\Delta)|^2$ is the probability that Δ occurs. Let $S=\{s_0,\ldots,s_{n-1}\}$ be a finite set and let $f:\Omega\rightarrow S$. We call f a *measurement* if $f^{-1}(s_j)\in\Lambda$, $j=0,\ldots,n-1$, and $\sum_j P[f^{-1}(s_j)]=1$. We interpret $f^{-1}(s_j)$ as the event that f has the outcome s_j. The sample points in $f^{-1}(s_j)$ are interpreted as the set of configurations of the system that result in the outcome s_j upon execution of the measurement f. To avoid measure-theoretic complications, we have assumed that f has only a finite number of outcomes. The more general situation is treated in [5], [6].

Let f and g be measurements with outcome sets $R=\{r_0,\ldots,r_{m-1}\}$ and $S=\{s_0,\ldots,s_{n-1}\}$, respectively. We say that g *does not interfere* with f if for all j, k and for $j=0,\ldots,m-1$

$$P[f^{-1}(r_j)]=\sum_{k=0}^{n-1}P[f^{-1}(r_j)\cap g^{-1}(s_k)].$$

This conditions states that the probability of $f^{-1}(r_j)$ is the sum of the probabilities for the subevents $f^{-1}(r_j)\cap g^{-1}(s_k)$. Thus, if $P[f^{-1}(r_j)]\neq 0$ we have

$$\sum_{k=0}^{n-1}P[g^{-1}(s_k)\,|\,f^{-1}(r_j)]=1$$

so $s_k\mapsto P[g^{-1}(s_k)\,|\,f^{-1}(r_j)]$ is a probability distribution on the outcome space S and we can condition g with respect to f. If g does not interfere with f, as we shall see in Section 2, it is possible that f interferes with g. Notice that a measurement $g:\Omega\rightarrow S$ does not interfere with itself since $g^{-1}(s_j)\cap g^{-1}(s_k)$ equals $g^{-1}(s_j)\in\Lambda$ if $j=k$ and equals $\varphi\in\Lambda$ if $j\neq k$.

For Δ_1, $\Delta_2\in\Lambda$ with $A(\Delta_2)\neq 0$ we interpret $A(\Delta_1\,|\,\Delta_2)=A(\Delta_1\cap\Delta_2)/A(\Delta_2)$ as the conditional amplitude of Δ_1 given Δ_2. If $A(\Delta_2)=0$, we define $A(\Delta_1\,|\,\Delta_2)=0$. Care must be taken if $A(\Delta_2)=0$

since there are examples in which $A(\Delta_2)=0$ but $A(\Delta_1 \cap \Delta_2) \neq 0$ [3]. In general, the formula $A(\Delta_1 \cap \Delta_2)=A(\Delta_2)A(\Delta_1 | \Delta_2)$ only holds when $A(\Delta_2) \neq 0$.

Let $f_0, \ldots, f_N$ be a finite sequence of measurements with the same outcome space $S=\{s_0, \ldots, s_{n-1}\}$. We call $\{f_t : 0 \leq t \leq N\}$ an N-*chain* if the following conditions hold:

(C1) $f_0^{-1}(s_0)=\Omega$.

(C2) For all $t \in \{1, \ldots, N\}$ and $s_{jt}, s_{jt-1}, \ldots, s_{j1}$ we have

$$A[f_t^{-1}(s_{jt}) \cap f_{t-1}^{-1}(s_{jt-1}) \cap \ldots \cap f_1^{-1}(s_{j1})] \neq 0.$$

Property (C1) fixes the initial condition of the process f_{t_j}(C2) permits the use of conditional amplitude formulas. An N-chain $\{f_t\}$ is *stationary* if

$$A[f_{t+1}^{-1}(s_j) | f_t^{-1}(s_k)] = A[f_2^{-1}(s_j) | f_1^{-1}(s_k)]$$

for every $j, k=0, \ldots, n-1$ and $t=1, \ldots, N-1$ and

$$A[f_1^{-1}(s_j) | f_0^{-1}(s_0)] = A[f_2^{-1}(s_j) | f_1^{-1}(s_0)].$$

Of course, $A[f_1^{-1}(s_j) | f_0^{-1}(s_k)]=0$ if $k \neq 0$ for any N-chain. An N-chain $\{f_t\}$ is *Markov* if

$$A[f_{t+1}^{-1}(s_j) | f_t^{-1}(s_{jt}) \cap f_{t-1}^{-1}(s_{jt-1}) \cap \ldots \cap f_1^{-1}(s_{j_1})] = A[f_{t+1}^{-1}(s_j) | f_t^{-1}(s_{j_t})]$$

for any $j, j_t, \ldots, j_1=0, \ldots, n-1$ and $t=1, \ldots, N-1$.

We interpret a stationary Markov N-chain $\{f_t\}$ as a repeated measurement using the same measuring apparatus and f_t corresponds to the measurement at the discrete time t. Then $A[f_2^{-1}(s_j) | f_1^{-1}(s_k)]$ corresponds to the amplitude of a transition of the system from outcome s_k to outcome s_j in one time step. The complex conjugate $\bar{A}[f_2^{-1}(s_j) | f_1^{-1}(s_k)]$ is interpreted as the transition amplitude from s_j to s_k in minus one time step; that is, the amplitude that if the outcome s_j results at $t=2$ then at time $t=1$ the outcome was s_k. An N-chain is *unitary* if f_2 does not interfere with f_1 and for $j \neq k$ we have

$$\sum_{r=0}^{n-1} A[f_2^{-1}(s_r) | f_1^{-1}(s_j)] \bar{A}[f_2^{-1}(s_r) | f_1^{-1}(s_k)] = 0. \tag{1}$$

Equation (1) means that the system cannot instantaneously jump from "state" s_j to "state" s_k if $j \neq k$. It is reasonable that f_2 does not interfere with f_1 since the measurement f_2 is performed at a later time than f_1. However, since f_1 is performed at an earlier time, it is possible that f_1 interferes with f_2. We interpret

$$P(s_k, s_j) = |A[f_2^{-1}(s_j) | f_1^{-1}(s_k)]|^2$$

as the transition probability from s_k to s_j. If f_2 does not interfere with f_1 we obtain

$$\sum_{j=0}^{n-1} P(s_k, s_j) = \frac{1}{P[f_1^{-1}(s_k)]} \sum_{j=0}^{n-1} P[f_2^{-1}(s_j) \cap f_1^{-1}(s_k)] = 1. \qquad (2)$$

From (2) we conclude that

$$\sum_{j=0}^{n-1} P[f_2^{-1}(s_j) \mid f_1^{-1}(s_k)] = 1. \qquad (3)$$

A unitary, stationary, Markov N-chain is called a *quantum* N-*chain*. The $n \times n$ matrix $[A_{jk}]$ with entries $A_{jk} = A[f_2^{-1}(s_j) \mid f_1^{-1}(s_k)]$ is the *transition amplitude matrix* for $\{f_t\}$.

We now show that the transition amplitude matrix $[A_{jk}]$ for a quantum N-chain satisfies some quite restrictive properties. First, $[A_{jk}]$ is unitary. Indeed, from (1) we have $\sum_r A_{rj} \bar{A}_{rk} = 0$ if $j \neq k$, and if $j = k$, since f_2 does not interfere with f_1, from (3) we have

$$\sum_r |A_{rj}|^2 = \sum_r |A[f_2^{-1}(s_r) \cap f_1^{-1}(s_k)]|^2 = 1.$$

Second, it follows from (C2) that the entires A_{jk} are nonzero, j, $k = 0, \ldots, n-1$. Third, $[A_{jk}]$ is a *stochastic matrix* in the sense that $\sum_j A_{jk} = 1$, $k = 0, \ldots, n-1$. This follows from

$$\sum_j A_{jk} = \frac{1}{A[f_1^{-1}(s_k)]} \sum_j A[f_2^{-1}(s_j) \cap f_1^{-1}(s_k)] = 1.$$

Notice that the last two properties hold for any N-chain, while the first property holds for any unitary N-chain.

For any N-chain $\{f_t\}$, f_t does not interfere with f_0 for all $t = 0, \ldots, N$ since $f_0^{-1}(s_j) \cap f_t^{-1}(s_k) = f_t^{-1}(s_k)$ if $j = 0$ and equals φ otherwise. If $\{f_t\}$ is stationary and f_2 does not interfere with f_1, then f_{t+1} does not interfere with f_t for any $t = 0, \ldots, N-1$. Indeed, for $t \geq 1$ we have

$$\begin{aligned} \sum_k P[f_t^{-1}(s_j) \cap f_{t+1}^{-1}(s_k)] &= \sum_k |A[f_t^{-1}(s_j) \cap f_{t+1}^{-1}(s_k)]|^2 \\ &= P[f_t^{-1}(s_j)] \sum_k |A[f_{t+1}^{-1}(s_k) \mid f_t^{-1}(s_j)]|^2 \\ &= P[f_t^{-1}(s_j)] \sum_k |A[f_2^{-1}(s_k) \mid f_1^{-1}(s_j)]|^2 \\ &= P[f_t^{-1}(s_j)]. \end{aligned}$$

For a quantum N-chain we have the following stronger result. In the sequel, we use the notation $P_t(j) = P[f_t^{-1}(s_j)]$, $\hat{A} = [A_{jk}]$.

THEOREM 1.1. — *For a quantum* N-*chain* $\{f_t\}$, $f_{t'}$ *does not interfere with* f_t *any* $0 \leqq t \leqq t' \leqq N$.

Proof. — We can assume that $t \geqq 1$. By Markovicity and stationarity we have

$$\begin{aligned} &A[f_t^{-1}(s_j) \cap f_{t'}^{-1}(s_k)] \\ &= \sum_{j_1, \ldots, j_{t'-1}} A[f_{t'}^{-1}(s_k) \cap f_{t'-1}^{-1}(s_{j_{t'-1}}) \cap \ldots \cap f_t^{-1}(s_j) \cap \ldots \cap f_t^{-1}(s_{j_1})] \\ &= \sum_{j_1, \ldots, j_{t'-1}} A[f_{t'-1}^{-1}(s_{j_{t'-1}}) \cap \ldots \cap f_1^{-1}(s_{j_1})] A[f_{t'}^{-1}(s_k) | f_{t'-1}^{-1}(s_{j_{t'-1}})] \\ &= \sum_{j_{t'-1}} A[f_t^{-1}(s_j) \cap f_{t'-1}^{-1}(s_{j_{t'-1}})] A_{kj_{t'-1}}. \end{aligned}$$

Iterating this equation gives

$$\begin{aligned} A[f_t^{-1}(s_j) \cap f_{t'}^{-1}(s_k)] &= \sum_{j_{t'-1}} \sum_{j_{t'-2}} A[f_t^{-1}(s_j) \cap f_{t'-2}^{-1}(s_{j_{t'-2}})] A_{kj_{t'-1}} A_{j_{t'-1} j_{t'-2}} \\ &= \sum_{j_{t'-2}} A[f_t^{-1}(s_j) \cap f_{t'-2}^{-1}(s_{j_{t'-2}})] \hat{A}^2_{kj_{t'-2}} \\ &\vdots \\ &= \sum_{j_t} A[f_t^{-1}(s_j) \cap f_t^{-1}(s_{j_t})] \hat{A}^{t'-t}_{kj_t} \\ &= A[f_t^{-1}(s_j)] \hat{A}^{t'-t}_{kj}. \end{aligned}$$

Since $\hat{A}^{t'-t}$ is unitary, we have

$$\sum_k P[f_t^{-1}(s_j) \cap f_{t'}^{-1}(s_k)] = P_t(j) \sum_k |\hat{A}^{t'-t}_{kj}|^2 = P_t(j). \quad \square$$

COROLLARY 1.2. — *For a quantum* N-*chain* $\{f_t\}$, f_t *does not interfere with* $f_{t'}$ *for* $0 \leqq t \leqq t' \leqq N$ *if and only if*

$$P_{t'}(k) = \sum_j P_t(j) |\hat{A}^{t'-t}_{kj}|^2.$$

Let $\{f_t\}$ be a quantum N-chain with transition amplitude matrix $\hat{A} = [A_{jk}]$. The *amplitude at time* $t = 0, \ldots, N$ is given by the unit vector

$$\hat{f}_t = (A[f_t^{-1}(s_0)], \ldots, A[f_t^{-1}(s_{n-1})]) \in \mathbb{C}^n$$

and the *distribution at time* $t = 0, \ldots, N$ is given by the probability distribution

$$P_t(k) = |A[f_t^{-1}(s_k)]|^2 = |(\hat{f}_t)_k|^2.$$

Notice that $\hat{f}_0 = (1, 0, \ldots, 0)$ and $P_0(k) = \delta_{k,0}$. We now show that $\hat{f}_t$ and P_t can be computed from $\hat{A}$.

THEOREM 1.3. — *For a quantum* N-*chain* $\{f_t\}$, $\hat{f}_t = \hat{A}^t \hat{f}_0$ *and*

$$P_t(k) = |(\hat{A}^t \hat{f}_0)_k|^2. \tag{4}$$

Proof. – By Markovicity and stationarity we have

$$
\begin{aligned}
A[f_t^{-1}(s_k) \cap f_{t-1}^{-1}(s_{j_{t-1}}) \cap \ldots \cap f_0^{-1}(s_{j_0})]
&= A[f_{t-1}^{-1}(s_{j_{t-1}}) \cap \ldots \cap f_0^{-1}(s_{j_0})]\, A[f_t^{-1}(s_k) \,|\, f_{t-1}^{-1}(s_{j_{t-1}})] \\
&= A_{kj_{t-1}}\, A[f_{t-1}^{-1}(s_{j_{t-1}}) \cap \ldots \cap f_0^{-1}(s_{j_0})] \\
&= A_{kj_{t-1}} A_{j_{t-1}\, j_{t-2}} \ldots A_{j_2\, j_1}\, A[f_1^{-1}(s_{j_1}) \,|\, f_0^{-1}(s_{j_0})] \\
&= A_{kj_{t-1}} A_{j_{t-1}\, j_{t-2}} \ldots A_{j_1\, j_0} (\hat{f}_0)_{j_0}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\hat{f}_t)_k &= A[f_t^{-1}(s_k)] \\
&= \sum_{j_{t-1}, \ldots, j_0} A_{kj_{t-1}} A_{j_{t-1}\, j_{t-2}} \ldots A_{j_1\, j_0} (\hat{f}_0)_{j_0} \\
&= (\hat{A}^t \hat{f}_0)_k.
\end{aligned}
$$

It follows that $\hat{f}_t = \hat{A}^t \hat{f}_0$ and $P_t(k) = |(\hat{A}^t \hat{f}_0)_k|^2$. □

Let $\lambda_0, \ldots, \lambda_{n-1}$ be the (possibly repeated) eigenvalues of the unitary matrix $\hat{A}$ and let $\psi_0, \ldots, \psi_{n-1}$ be the corresponding orthonormal basis of eigenvectors. We can now find an explicit expression for $P_t(k)$. In fact,

$$
\begin{aligned}
\hat{A}^t \hat{f}_0 &= \hat{A}^t \sum_j \langle \hat{f}_0, \psi_j \rangle \psi_j = \sum_j \overline{(\psi_j)}_0\, \hat{A}^t \psi_j \\
&= \sum_j \overline{(\psi_j)}_0\, \lambda_j^t \psi_j.
\end{aligned}
$$

Hence, from (4) we have

$$
P_t(k) = \Big| \sum_j \overline{(\psi_j)}_0\, \lambda_j^t (\psi_j)_k \Big|^2. \tag{5}
$$

We have shown that corresponding to a quantum N-chain there is a transition amplitude matrix $\hat{A}$ where $\hat{A}$ is stochastic, unitary and has all nonzero entries. Conversely, any $n \times n$ matrix $\hat{A} = [A_{jk}]$ with these three properties is the transition amplitude matrix of a quantum N-chain $\{f_t\}$ with a given outcome space $S = \{s_0, \ldots, s_{n-1}\}$. We can construct $\{f_t\}$ as follows. Let $\Omega = \{s_0\} \times S^N$ be the set of "sample paths". For $\omega = (s_0, s_j, \ldots, s_{j_N}) \in \Omega$ define

$$
A(\omega) = A_{j_1\, 0} A_{j_2\, j_1} \ldots A_{j_N\, j_{N-1}}.
$$

Let Λ be the power set on Ω and define the complex mesure $A: \Lambda \to \mathbb{C}$ by $A(\Delta) = \sum \{ A(\omega) : \omega \in \Delta \}, \Delta \in \Lambda$. Since $\hat{A}$ is stochastic, we have $A(\Omega) = 1$. For $t = 0, \ldots, N$ define $f_t: \Omega \to S$ by $f_t(s_0, s_{j_1}, \ldots, s_{j_N}) = s_{j_t}$, where $j_0 = 0$.

We first show that $\{f_t\}$ is an N-chain. Clearly, (C1) holds. To verify (C2) we have from stochasticity and the nonzero condition that

$$\begin{aligned}
&A[f_t^{-1}(s_{j_t})\cap\ldots\cap f_1^{-1}(s_{j_1})]\\
&=\sum_{j_{t+1},\ldots,j_N} A[f_N^{-1}(s_{j_N})\cap\ldots\cap f_{t+1}^{-1}(s_{j_{t+1}})\cap f_t^{-1}(s_{j_t})\cap\ldots\cap f_1^{-1}(s_{j_1})]\\
&=\sum_{j_{t+1},\ldots,j_N} A(s_0, s_{j_1},\ldots,s_{j_{t+1}},\ldots,s_{j_N})\\
&=A_{j_1 0}A_{j_1 j_2}\ldots A_{j_t j_{t-1}}\sum_{j_{t+1},\ldots,j_N} A_{j_{t+1} j_t}\ldots A_{j_N j_{N-1}}\\
&=A_{j_1 0}A_{j_1 j_2}\ldots A_{j_t j_{t-1}}\neq 0.
\end{aligned}\tag{6}$$

For stationarity we have for $t\geqq 1$

$$\begin{aligned}
A[f_{t+1}^{-1}(s_j)\cap f_t^{-1}(s_k)] &= \sum_{\substack{j_1,\ldots,j_{t-1}\\ j_{t+2},\ldots,j_N}} A(s_0, s_{j_1},\ldots,s_{j_{t-1}}, s_k, s_j, s_{j_{t+2}},\ldots,s_{j_N})\\
&=A_{jk}\sum_{\substack{j_1,\ldots,j_{t-1}\\ j_{t+2},\ldots,j_N}} A_{j_t 0}A_{j_2 j_1}\ldots A_{k j_{t-1}}A_{j_{t+2} j}\ldots A_{j_N j_{N-1}}\\
&=A_{jk}\sum_{j_1,\ldots,j_{t-1}} A_{j_t 0}A_{j_2 j_1}\ldots A_{k j_{t-1}}\\
&=A_{jk}A[f_t^{-1}(s_k)].
\end{aligned}$$

In particular,

$$A[f_2^{-1}(s_j)\cap f_1^{-1}(s_k)]=A_{jk}A[f_1^{-1}(s_k)].$$

It follows that

$$A[f_{t+1}^{-1}(s_j)\,|\,f_t^{-1}(s_k)]=A[f_2^{-1}(s_j)\,|\,f_1^{-1}(s_k)]=A_{jk}$$

so $\hat{A}$ is the transition amplitude matrix for $\{f_t\}$. Moreover, by (6) we have

$$A[f_1^{-1}(s_j)\,|\,f_0^{-1}(s_o)]=A[f_1^{-1}(s_j)]=A_{j0}=A[f_2^{-1}(s_j)\,|\,f_1^{-1}(s_0)]$$

so $\{f_t\}$ is stationary. For Markovicity, we apply (6) and stationary to obtain

$$\begin{aligned}
A[f_{t+1}^{-1}(s_j)\,|\,f_t^{-1}(s_{j_t})\cap\ldots\cap f_1^{-1}(s_{j_1})]=A_{jj_t}&=A[f_2^{-1}(s_j)\,|\,f_1^{-1}(s_{j_t})]\\
&=A[f_{t+1}^{-1}(s_j)\,|\,f_t^{-1}(s_{j_t})].
\end{aligned}$$

The unitarity of $\{f_t\}$ easily follows from the unitarity of $\hat{A}$.

2. DIRICHLET MATRICES

In the previous section we showed that an $n\times n$ matrix $\hat{A}$ is the transition amplitude matrix for a quantum N-chain if and only if $\hat{A}$ is stochastic, unitary and has all entries nonzero. We now give an example of such a matrix and find an explicit expression for the distribution $P_t(k)$ for any $t=0, 1, \ldots$

Let n and a be positive integers that are relatively prime. The *Dirichlet matrix* M (n, a) is the $n \times n$ matrix with entries

$$A_{jk} = \frac{1}{\sqrt{n}} e^{i\pi a (j-k)^2/n}, \qquad j, k = 0, 1, \ldots, n-1.$$

It is shown in [5], [6] that M (n, a) generates a discrete analog of the usual continuum Feynman amplitude for a free particle [2] and as $n \to \infty$ this analog approaches the Feynman amplitude. Moreover, it can be shown that M (n, a) is unitary ([5], [10]). Clearly, M (n, a) has all nonzero entries.

Let S (n, a) be the *Dirichlet sum*

$$S(n, a) = \sum_{j=0}^{n-1} e^{i\pi a j^2/n}.$$

The following result is proved in [10].

LEMMA 2.1. — *(a) If na is even, then*

$$\sum_{j=0}^{n-1} e^{i\pi a (j-k)^2/n} = S(n, a)$$

for every integer $0 \leqq k \leqq 2n-2$. *(b) if na is odd, then*

$$\sum_{j=0}^{n-1} e^{i\pi a (j-k-(1/2))^2/n} = \frac{1}{2} S(4n, a)$$

for every integer $0 \leqq k \leqq 2n-2$.

It follows from Lemma 2.1 a that if na is even then the column (and row) sums of M (n, a) all equal $n^{-1/2}$ S (n, a). Moreover, it is shown in [10] that $|S(n, a)| = n^{1/2}$. Therefore, the matrix

$$M'(n, a) = \frac{\overline{S(n, a)}}{\sqrt{n}} M(n, a)$$

is stochastic. The factor $n^{-1/2}\overline{S(n, a)}$ does not affect unitary, so $M'(n, a)$ is still unitary and of course has all nonzero entries. In the sequel we shall not bother to multiply M (n, a) by $n^{-1/2}\overline{S(n, a)}$ and shall just work with M (n, a) since the distribution P_t is unaffected. Unfortunately, if na is odd, a similar trick does not work and M (n, a) cannot be made stochastic. For example, if $n=3$ and $a=1$ we have

$$M(3, 1) = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \alpha & -\alpha \\ \alpha & 1 & \alpha \\ -\alpha & \alpha & 1 \end{bmatrix}, \qquad \alpha = e^{i\pi/3}.$$

The column sums are $1/\sqrt{3}$ and $(1+\alpha)/\sqrt{3}$ so no multiple of M(3, 1) is stochastic. However, using the construction of Section 1, M (n, a) can still be interpreted as the transition amplitude matrix of a unitary process $\{f_t\}$. Although $\{f_t\}$ is not a quantum N-chain, it is still of interest ([7], [10]). Since M (n, a) is unitary, P_t is a probability distribution. For these reasons, we shall consider M (n, a) for arbitrary (relatively prime) n and a in our computation of P_t.

In order to apply (5) we need the eigenvalues and eigenvectors of M (n, a). This has been done in [10].

THEOREM 2.2. — *(a) If na is even, then for $j=0, \ldots, n-1$, the eigenvalues of* M (n, a) *are*

$$\lambda_j = n^{-1/2} S(n, a)\, e^{-i\pi a j^2/n}$$

and a corresponding orthonormal basis of eigenvectors is

$$\psi_j = n^{-1/2}\,(e^{-i\,2\pi a j k/n}), \qquad k=0, \ldots, n-1.$$

(b) If na is odd, then for $j=0, \ldots, n-1$, the eigenvalues of M (n, a) *are*

$$\lambda_j = \frac{1}{2n^{1/2}} S(4n, a)\, e^{-i\pi a (j+(1/2))^2/n}$$

and a corresponding orthonormal basis of eigenvectors is

$$\psi_j = n^{-1/2}\,(e^{-i\,2\pi a (j+(1/2))\,k/n}), \qquad k=0, \ldots, n-1.$$

Applying Theorem 2.2 for na even, we obtain

$$\sum_j (\bar{\psi}_j)_0\, \lambda_j^t\, (\psi_j)_k = \frac{S(n, a)^t}{n^{(t/2)+1}} \sum_j e^{-i\pi a t j^2/n}\, e^{-i\,2\pi a j k/n}$$

and for na odd we have

$$\sum_j (\bar{\psi}_j)_0\, \lambda_j^t\, (\psi_j)_k = \frac{S(4n, a)^t}{2^t\, n^{(1/2)+1}} \sum_j e^{-i\pi a t (j+(1/2))^2/n}\, e^{-i\,2\pi a (j+(1/2))\,k/n}.$$

Since $|S(n, a)| = \sqrt{n}$ we have from (5) that, for $t>0$

$$P_t(k) = \frac{1}{n^2} \left| \sum_{j=0}^{n-1} e^{-i\pi a (tj+k)^2/nt} \right|^2 \tag{7}$$

for na even and

$$P_t(k) = \frac{1}{n^2} \left| \sum_{j=0}^{n-1} e^{-i\pi a [t(j+(1/2))+k]^2/nt} \right|^2 \tag{8}$$

for na odd. Although (7) and (8) give explicit expressions for P_t, they are not in closed form and they do not give us much information about the dynamics of the system. We perform the technical work of evaluating certain summations in the appendix and we apply these results to compute P_t in the next section.

3. PROBABILITY DISTRIBUTIONS

We now compute the probabilities $P_t(k)$ given by (7) and (8) of Section 2. For an integer t, let t_2 denote the number of times a factor 2 appears in the prime decomposition of t and by convention $0_2=0$. We denote the greatest common divisor of two integers n and t by (n, t). If an integer d divides an integer k we write $d \mid k$.

THEOREM 3.1. – *Let $(n, t)=d$ and $t>0$. (a) If n is even, then $P_t(k)=d/n$ if $t_2 \neq n_2$ and $d \mid k$ or if $t_2=n_2$ and $2k/d$ is odd. Otherwise, $P_t(k)=0$. (b) if n is odd, then $P_t(k)=d/n$ if $d \mid k$. Otherwise, $P_t(k)=0$.*

Proof. – (*a*) Applying (7) and Lemma A1 (*a*) of the appendix we have

$$P_t(k)=\frac{1}{n^2}\left|\sum_{m=0}^{d-1}(-1)^{amnt/d^2} e^{-i2\pi akm/d}\right|^2\left|\sum_{j=0}^{n/d-1} e^{-ia\pi(tj+k)^2/nt}\right|^2.$$

Suppose $t_2 \neq n_2$. Then nt/d^2 is even so

$$\sum_{m=0}^{d-1}(-1)^{amnt/d^2} e^{-i2\pi akm/d}=\sum_{m=0}^{d-1}(e^{-i2\pi ak/d})^m.$$

The geometric series has sum d if $d \mid k$ and sum 0 otherwise. Suppose $t_2=n_2$. Then nt/d^2 is odd so

$$\sum_{m=0}^{d-1}(-1)^{amnt/d^2} e^{-i2\pi akm/d}=\sum_{m=0}^{d-1}(-e^{i2\pi ak/d})^m.$$

If $2k/d$ is an odd integer, the geometric series has sum d. Otherwise, we have

$$\sum_{m=0}^{d-1}(-e^{-i2\pi ak/d})^m=\frac{1-(-1)^d}{1+e^{-i2\pi ak/d}}.$$

Since $t_2=n_2$ and n is even, we have d is even so the last expression vanishes. We conclude that when $P_t(k)$ does not vanish we have

$$P_t(k)=\frac{d^2}{n^2}\left|\sum_{j=0}^{n/d-1} e^{-i\pi a(tj+k)^2/nt}\right|^2$$

$$=\frac{d^2}{n^2}\left|\sum_{j=0}^{n'-1} e^{-i\pi a(t'j+(1/2))^2/nt}\right|^2$$

where $n'=n/d$ and $t'=t/d$. It follows that $(n',t')=1$. If $t_2 \neq n_2$ and $d \mid k$, then nt/d^2 is even and $n'at'=ant/d^2$ is even. Applying Lemma A 2 (*a*) of the appendix gives

$$\left| \sum_{j=0}^{n'-1} e^{-i\pi a(t'j+(k/d))/n't'} \right|^2 = |S(n',at')|^2 = n'.$$

Hence,

$$P_t(k) = \frac{d^2}{n^2} n' = \frac{d}{n}.$$

If $t_2 = n_2$ and $2k/d$ is odd, then nt/d^2 is odd and $n'at'=ant/d^2$ is odd. Letting $2k/d=2\mu+1$ and applying Lemma A 2 (*b*) we have

$$\left| \sum_{j=0}^{n'-1} e^{-i\pi a(t'j+\mu+(1/2))^2/n't'} \right|^2 = \frac{1}{2} |S(4n',at')|^2 = n'.$$

Again,

$$P_t(k) = \frac{d^2}{n^2} n' = \frac{d}{n}.$$

(*b*) Let n be odd and a even. Applying (7) and Lemma A 1 (*a*) we have $P_t(k)$ as in (*a*). If $t_2 \neq n_2$, then nt/d^2 is even so as in (*a*) the geometric series has sum d if $d \mid k$ and sum 0 otherwise. If $t_2 = n_2$, then nt/d^2 is odd. Since a is even we have

$$\sum_{m=0}^{d-1} (-1)^{amnt/d^2} e^{-i2\pi akm/d} = \sum_{m=0}^{d-1} (e^{-i2\pi ak/d})^m.$$

As before the geometric series has sum d if $d \mid k$ and sum 0 otherwise. Again, as in (*a*), $P_t(k)=d/n$ when it does not vanish. Finally, let na be odd. Applying (8) and Lemma A 1 (*b*) we have

$$P_t(k) = \frac{1}{n^2} \left| \sum_{m=0}^{d-1} e^{-i2\pi akm/d} \right|^2 \left| \sum_{j=0}^{n/d-1} e^{-i\pi a[t(j+(1/2))+k]^2/nt} \right|^2.$$

The geometric series has sum d if $d \mid k$ and sum 0 otherwise. When $P_t(k)$ does not vanish we have

$$P_t(k) = \frac{d^2}{n^2} \left| \sum_{j=0}^{n'-1} e^{-i\pi a[t'(j+(1/2))+(k/d)]^2/n't'} \right|^2$$

where $n'=n/d$ and $t'=t/d$. Again, we have $(n',t')=1$. Applying Lemma A 2 (*c*) and (*d*) we conclude that

$$P_t(k) = \frac{d^2}{n^2} n' = \frac{d}{n}. \quad \square$$

Theorem 3.1 gives the surprising fact that P_t is independent of a. We now consider discrete times at which the probability distributions coincide.

COROLLARY 3.2. – *(a) If n is odd, then $P_t = P_s$ if and only if $(n, t) = (n, s)$. (b) If n is even, then $P_t = P_s$ if and only if $(n, t) = (n, s)$ and t_2, $s_2 \neq n_2$ or $t_2 = s_2 = n_2$.*

Proof. – Let $d_t = (n, t)$ and $d_s = (n, s)$. (a) Sufficiency is clear. For necessity, there exists a k such that $P_t(k) \neq 0$. Then

$$\frac{d_t}{n} = P_t(k) = P_s(k) = \frac{d_s}{n}.$$

Hence, $d_t = d_s$. (b) Sufficiency is clear. For necessity, assume $P_t = P_s$. Suppose $t_2 \neq n_2$ and $d_t \mid k$. Then

$$\frac{d_t}{n} = P_t(k) = P_s(k) = \frac{d_s}{n}.$$

Hence, $d_t = d_s$. If $s_2 = n_2$, then since $P_s(k) \neq 0, 2k/d_t$ is odd. But then $2(k/d_t)$ is odd which is a contradiction. Hence, $s_2 \neq n_2$ so t_2, $s_2 \neq n_2$. Suppose $t_2 = n_2$ and $2k/d_t$ is odd. Then as before, $d_t = d_s$. If $s_2 \neq n_2$, then since $P_s(k) \neq 0$, $d_s \mid k$. But then $2(k/d_s)$ is odd which is a contradiction. Hence, $t_2 = s_2 = n_2$. □

We call p the *probability period* of the process if p is the smallest positive integer such that $P_{t+p} = P_t$ for all t. In the next proof we shall need the following well known fact. For any nonnegative integers m, n, t, $(n, t) = (n, mn + t)$.

COROLLARY 3.3. – *(a) If n is odd, the probability period is n. (b) If n is even, the probability period is $2n$.*

Proof. – (a) Since $(n, t) = (n, n + t)$, by Corollary 3.2, $P_t = P_{t+n}$ for every t. The smallest positive integer such that $n = (n, 0) = (n, p)$ is $p = n$. Hence, n is the smallest positive integer such that $P_n = P_0$. Therefore, n is the probability period. (b) First, $(n, t) = (n, 2n + t)$. Suppose $t_2 = n_2 = m$. Then $t = 2^m p$, $n = 2^m q$ where p and q are odd. Hence,

$$2n + t = 2^{m+1} q + 2^m p = 2^m (2q + p).$$

Since $2q + p$ is odd, $(2n + t)_2 = m$. Next suppose $(2n + t)_2 = n_2 = m$. Then $2n + t = 2^m p$, $n = 2^m q$ where p and q are odd. Hence,

$$t = 2^m p = 2^{m+1} q = 2^m (p - 2q).$$

Since $p - 2q$ is odd, $t_2 = m$. By Corollary 3.2, $P_t = P_{t+2n}$ for every t. Now suppose $p > 0$ and $P_p = P_0$. By Corollary 3.2, $n = (n, 0) = (n, p)$. Hence, $n \mid p$ so $p = rn$ for some positive integer r. If $r = 1$, then $p_2 = n_2$ but $0_2 \neq n_2$ which contradicts Corollary 3.2. Hence, $r \neq 1$. If follows that $2n$ is the smallest positive integer satisfying $P_{2n} = P_0$. Therefore, $2n$ is the probability period. □

If follows from Corollary 3.3 that we need not compute P_t for $t \geqq n$ if n is odd and for $t \geqq 2n$ if n is even. The next corollary shows that for n even we also need not compute P_t for $t \geqq n$.

COROLLARY 3.4. — *(a)* $P_0(k) = \delta_{0,k}$. *(b) If n is even,* $P_n(k) = \delta_{n/2,k}$ *and* $P_t = P_{2n-t}$, $0 \leqq t \leqq 2n$.

Proof. — The proof of *(a)* is clear. *(b)* By Theorem 3.1, $P_n(k) = 1$ if and only if $2k/n$ is odd. But $2k/n$ odd is equivalent to $2k = nr$ for r odd which is equivalent to $k = (n/2)r$. Since $0 \leqq k \leqq n-1$, this holds if and only if $k = n/2$. For the second part, $(n, t) = (n, 2n-t)$. Suppose $t_2 = n_2 = m$. Then $t = 2^m p$, $n = 2^m q$ where p and q are odd. Hence,

$$2n - t = 2^m (2q - p).$$

Since $2q-p$ is odd, $(2n-t)_2 = m$. Next suppose $(2n-t)_2 = n_2 = m$. By a similar argument $t_2 = m$. By Corollary 3.2, $P_t = P_{2n-t}$. □

We now apply our previous results to compute P_t for the case $n = 12$. This is given in Table.

TABLE. — ($P_t(k)$ for $n = 12$).

$t \diagdown k$	0	1	2	3	4	5	6	7	8	9	10	11
0	1	0	0	0	0	0	0	0	0	0	0	0
1	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
2	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0
3	1/4	0	0	1/4	0	0	1/4	0	0	1/4	0	0
4	0	0	1/3	0	0	0	1/3	0	0	0	1/3	0
5	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
6	1/2	0	0	0	0	0	1/2	0	0	0	0	0
7	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
8	1/3	0	0	0	1/3	0	0	0	1/3	0	0	0
9	1/4	0	0	1/4	0	0	1/4	0	0	1/4	0	0
10	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0
11	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
12	0	0	0	0	0	0	1	0	0	0	0	0
13	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
14	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0
15	1/4	0	0	1/4	0	0	1/4	0	0	1/4	0	0
16	1/3	0	0	0	1/3	0	0	0	1/3	0	0	0
17	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
18	1/2	0	0	0	0	0	1/2	0	0	0	0	0
19	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
20	0	0	1/3	0	0	0	1/3	0	0	0	1/3	0
21	1/4	0	0	1/4	0	0	1/4	0	0	1/4	0	0
22	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0	1/6	0
23	1/12	1/12	1/12	1/2	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
24	1	0	0	0	0	0	0	0	0	0	0	0

4. APPENDIX

In this appendix we prove some technical results that are needed to prove Theorem 3.1. As usual, we assume that $(n, a)=1$ and $k=0, \ldots, n-1$.

LEMMA A.1. – *Let* $(n, t)=d$. (*a*) *Then*

$$S_1=\sum_{j=0}^{n-1} e^{-i \pi a (tj+k)^2/nt} = \sum_{m=0}^{d-1} (-1)^{amnt/d^2} e^{-i 2 \pi akm/d} \sum_{j=0}^{n/d-1} e^{-i \pi a (tj+k)^2/nt}.$$

(*b*) *If na is odd, then*

$$S_2=\sum_{j=0}^{n-1} e^{-i \pi a [t (j+(1/2))+k]^2/nt} = \sum_{m=0}^{d-1} e^{-i 2 \pi akm/d} \sum_{j=0}^{n/d-1} e^{-i \pi a [t (j+(1/2))+k]^2/nt}.$$

Proof. – (*a*) Split the sum S_1 into d parts to obtain

$$S_1=\sum_{m=0}^{d-1} \sum_{j=mn/d}^{(m+1)n/d-1} e^{-i \pi a (tj+k)^2/nt}.$$

Letting $j=s+mn/d$ gives

$$S_1=\sum_{m=0}^{d-1} \sum_{s=0}^{n/d-1} e^{-i \pi a [t (s+(mn/d))+k]^2/nt}.$$

Since $d \mid t$ and $d^2 \mid nt$ we have

$$e^{-i \pi a [t (s+(mn/d))+k]^2/nt}=e^{-i \pi a (ts+k)^2/nt} (-1)^{amnt/d^2} e^{-i 2 \pi akm/d}$$

and the result follows.

(*b*) Again, split the sum into d parts to obtain

$$S_2=\sum_{m=0}^{d-1} \sum_{s=0}^{n/d-1} e^{-i a \pi [t (s+(1/2))+k+mnt/d]^2/nt} = \sum_{m=0}^{d-1} e^{-i \pi am^2 nt/d} \sum_{s=0}^{n/d-1} e^{-i \pi a [t (s+(1/2))+k]^2/nt} e^{-i 2 \pi am [t (s+(1/2))+k]/d}.$$

Since $d \mid t$ the last exponential term equals

$$e^{-i \pi amt/d} e^{-i 2 \pi akm/d}=(-1)^{amt/d} e^{-i 2 \pi akm/d}.$$

Since na is odd, $(-1)^{amt/d}=(-1)^{mt/d}$ and

$$e^{-i \pi am^2 nt/d}=(-1)^{mt/d}$$

so the product of these two terms is unity. The result now follows. □

LEMMA A.2. – *Let* $(n, t)=1$, $k=0, \ldots, n-1$. (*a*) *If nat is even,*

$$\sum_{j=0}^{n-1} e^{-i\pi a (tj+k)^2/nt} = e^{-i\pi a k^2 (\alpha t - 1)^2/nt}\,\overline{S(n, at)}$$

where α *is the integer defined by* $\alpha t = 1 \pmod n$, $0 \leqq \alpha \leqq n-1$.

(*b*) *If nat is odd.*

$$\sum_{j=0}^{n-1} e^{-i\pi a (tj+k+(1/2))^2/nt} = \frac{1}{2} e^{-i\pi a (2k+1)^2 (\alpha t - 1)^2/4nt}\,\overline{S(4n, at)}$$

where α *satisfies* $\alpha t = 1 \pmod{4n}$, $0 \leqq \alpha \leqq 4n-1$.

(*c*) *If na is odd and t is even,*

$$\sum_{j=0}^{n-1} e^{-i\pi a [t(j+(1/2))+k]^2/nt} = e^{-i\pi a [((t/2)+k)^2 - \rho^2]/nt}\, e^{-i\pi a (\alpha t - 1)^2 \rho^2/nt}\,\overline{S(n, at)}$$

where ρ *is the remainder of* $\frac{1}{2} + k \pmod n$.

(*d*) *If nat is odd,*

$$\sum_{j=0}^{n-1} e^{-i\pi a [t(j+(1/2))+k]^2/nt} = \frac{1}{2} e^{-i\pi a (\alpha t - 1)^2 k^2/nt}\, e^{-i\pi a (\alpha t - 1) k/n}\,\overline{S(4n, at)}$$

where α *satisfies* $\alpha t = 1 \pmod n$, $0 \leqq \alpha \leqq n-1$.

Proof. – (*a*) Since $(n, t)=1$, by the Euclidean algorithm there exist unique integers q and α such that $\alpha t = 1 + qn$, $0 \leqq \alpha \leqq n-1$. Then $k = (\alpha t - qn) k$ and we have

$$S = \sum_{j=0}^{n-1} e^{-i\pi a (tj+k)^2/nt} = \sum_{j=0}^{n-1} e^{-i\pi a [(j+\alpha k) t - qnk]^2/nt}$$

$$= \sum_{j=0}^{n-1} e^{-i\pi at (j+\alpha k)^2/n}\, e^{-i\pi a (qn)^2 k^2/nt}.$$

But $(qn)^2 = (\alpha t - 1)^2$ so

$$S = e^{-i\pi a (\alpha t - 1)^2 k^2/nt} \sum_{j=0}^{n-1} e^{-i\pi at (j+\alpha k)^2/n}.$$

Let m be the integer satisfying

$$ak = -m \pmod n, \qquad 0 \leqq m \leqq n-1.$$

Since nat is even

$$e^{-i\pi at (j+\alpha k)^2/n} = e^{-i\pi at (j-m)^2/n}.$$

But by Lemma 2.1 we have

$$\sum_{j=0}^{n-1} e^{i\pi at(j-m)^2/n} = \sum_{j=0}^{n-1} e^{i\pi atj^2/n} = S(n, at)$$

for $0 \leqq m \leqq 2n-1$ and (*a*) is proved.

(*b*) The sum

$$S = \sum_{j=0}^{n-1} e^{-i\pi a(tj+k+(1/2))^2/nt} = \sum_{j=0}^{n-1} e^{-i\pi a(2tj+2k+1)^2/4nt}$$

is a partial sum of

$$T = \sum_{j=0}^{4n-1} e^{-i\pi a(tj+2k+1)^2/4nt}.$$

Now $4nat$ is even and $(4n, t) = 1$ since $(n, t) = 1$ and t is odd. Moreover, $0 \leqq 2k+1 \leqq 4n-1$ and α satisfies $\alpha t = 1 \pmod{4n}$, $0 \leqq \alpha \leqq 4n-1$. If follows from (*a*) that

$$T = e^{-i\pi a(2k+1)^2(\alpha t-1)^2}\, \overline{S(4n, at)}.$$

We now decompose T into its even and odd parts $T = E + U$ where

$$\begin{aligned} E &= \sum_{j=0}^{2n-1} e^{-i\pi a(tj+k+(1/2))^2/nt} \\ &= \sum_{j=0}^{n-1} e^{-i\pi a(tj+k+(1/2))^2/nt} + \sum_{j=n}^{2n-1} e^{-i\pi a(tj+k+(1/2))^2/nt} \\ &= \sum_{j=0}^{n-1} e^{-i\pi a(tj+k+(1/2))^2/nt} + \sum_{j=0}^{n-1} e^{-i\pi a(tj+k+(1/2))+nt)^2/nt} \end{aligned}$$

Since nat is odd, the last summand becomes

$$e^{-i\pi a(tj+k+(1/2))^2/nt}.$$

Hence,

$$E = 2\sum_{j=0}^{n-1} e^{-i\pi a(tj+k+(1/2))^2/nt}.$$

The odd parts is given by

$$U = \sum_{j=0}^{2n-1} e^{-i\pi a[(2j+1)t+2k+1]^2/4nt}.$$

Since nat is odd, we have $(2j+1)\,t+(2k+1)$ even. Hence,

$$\sum_{j=n}^{2n-1} e^{-i\pi a[(2j+1)t+2k+1]^2/4nt} = \sum_{j=0}^{n-1} e^{-i\pi a[(2j+1)t+2k+1+2nt]^2/4nt}$$
$$= \sum_{j=0}^{n-1} e^{-i\pi a[(2j+1)t+2k+1]^2/4nt} e^{-i\pi ant}$$
$$= -\sum_{j=0}^{n-1} e^{-i\pi a[(2j+1)t+2k+1]^2/4nt}.$$

Hence, $U=0$ and

$$\sum_{j=0}^{n-1} e^{-i\pi a(tj+k+(1/2))^2/nt} = \frac{1}{2}T.$$

The result now follows.

(c) Define the sum

$$S = \sum_{j=0}^{n-1} e^{-i\pi a[t(j+(1/2))+k]^2/nt}$$
$$= \sum_{j=0}^{n-1} e^{-i\pi a(tj+(1/2))+k)^2/nt}.$$

Since t is even, $\frac{t}{2}+k$ is an integer and nat is even so we can use the result of (a). Let ρ be the remainder (mod n) of $\frac{t}{2}+k$ so $\frac{t}{2}+k=\rho+\mu$. It now follows that

$$S = e^{-i\pi a\mu^2/nt} e^{-i2\pi a\mu\rho/nt} \sum_{j=0}^{n-1} e^{-i\pi a(tj+\rho)^2/nt}$$
$$= e^{-i\pi a(\mu^2+2\mu\rho)/nt} e^{-i\pi a\rho^2(at-1)^2/nt}\, \overline{S(n,at)}$$
$$= e^{-i\pi a[(1/2)+k)^2-\rho^2]/nt} e^{-i\pi a\rho^2(\alpha t-1)^2/nt}\, \overline{S(4n,at)}.$$

(d) Define S as in (c) and replace k by $(\alpha t-qn)\,k$ as in (a) to obtain

$$S = \sum_{j=0}^{n-1} e^{-i\pi a[t(j+(1/2)+\alpha k)-qnk]^2/nt}$$
$$= e^{-i\pi a(\alpha t-1)^2k^2/nt} e^{-i\pi a(\alpha t-1)k/n} \sum_{j=0}^{n-1} e^{-i\pi at(j+(1/2)+\alpha k)^2/n}.$$

It remains to compute the sum

$$T=\sum_{j=0}^{n-1} e^{-i\pi at(j+\alpha k+(1/2))^2/n}.$$

Let m be the integer defined by

$$\alpha k+1=-m+sn, \qquad 0\leq m\leq n-1.$$

Then

$$j+\alpha k+\frac{1}{2}=j-m+sn-\frac{1}{2}$$

and

$$e^{-i\pi at(j+\alpha k+(1/2))^2/n}=e^{-i\pi at[j-(m+(1/2))+sn]^2/n}$$

$$=e^{-i\pi at(j-m-(1/2))^2/n}e^{-i\pi ats^2 n}e^{i\pi ats}.$$

Since nat is odd we have $e^{i\pi ats}=(-1)^s$ and

$$e^{-i\pi ats^2 n}=(-1)^{s^2}=(-1)^s.$$

Hence,

$$T=\sum_{j=0}^{n-1} e^{-i\pi at(j-m-(1/2))^2/n}.$$

Applying Lemma 2.1 we have

$$T=\frac{1}{2}\overline{S(4n,at)}$$

and the result follows. □

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