ANNALES DE L'I. H. P., SECTION A

J. P. FRANÇOISE
O. RAGNISCO

Matrix second-order differential equations and hamiltonian systems of quartic type

Annales de l'I. H. P., section A, tome 50, nº 3 (1989), p. 369-375
<http://www.numdam.org/item?id=AIHPA_1989__50_3_369_0>

Ann. Inst. Henri Poincaré,

Vol. 49, n° 3, 1989, p. 369-375. *Physique théorique*

Matrix second-order differential equations and hamiltonian systems of quartic type

by

J. P. FRANÇOISE

Bâtiment de Mathématiques n° 425, Université de Paris-XI, 91405 Orsay, France

and

O. RAGNISCO

Dipartimento di Fisica, Università degli Studi, « La Sapienza »,
Pl. A. Moro 2, 00185 Roma, Italia

ABSTRACT. — We show the existence of conserved quantities for matrix differential equations and Hamiltonian systems of quartic type.

RÉSUMÉ. — Nous montrons l'existence de quantités conservées pour des équations différentielles matricielles et des hamiltoniens quartiques.

I. INTRODUCTION

We study second-order differential equations of the form:

$$\ddot{X} = Q(X) = -hh'(X) \tag{1}$$

where $h(X) = \lambda X + \mu X^2, (\lambda, \mu) \in \mathbb{R}^2$, and the unknown function X is a $m \times m$ symmetric real matrix.

Annales de l'Institut Henri Poincaré - Physique théorique - 0246-0211
Vol. 49/89/03/369/7/$2,70/© Gauthier-Villars

The method we use is the same which has allowed to prove the existence of a symplectic action of the torus associated to the Olshanetsky-Perelomov systems of the class V ([F1], [F2]).

We associate to (1) a Hamiltonian system for a symplectic form introduced by Kazhdan-Kostant-Sternberg [K-K-S] and used by J. Moser [M]. For a quadratic h, we get a quartic anharmonic flow on the cotangent bundle of the Lie algebra $u(m)$. We show the existence for this system of a $m \times m$ Lax pair whose eigenvalues are in involution for the symplectic form.

The Calogero-Moser Hamiltonian describes an integrable system of m particles on the line interacting pairwise via an inverse-quadratic potential ([C], [M]).

It has been observed by M. Adler [A] that the system stays integrable under the influence of an external quadratic potential.

Following the work of Kazhdan-Kostant-Sternberg, the systems of Calogero-Moser type can be seen as symplectic reductions of harmonic flows on the cotangent bundle of a simple Lie algebra.

The second-order differential equations that we consider here give after a symplectic reduction the Calogero-Moser system with a quartic external potential. We find as a consequence of the main theorem that the reduced system is completely integrable in the Arnol'd-Liouville sense. This last result had been obtained previously by Wojciechowski [W] and independently Inozemtzev [I]. The novelty of our approach is in the use of the Kazhdan-Kostant-Sternberg machinery which makes all the subject more coherent.

Hamiltonian systems of quartic type have been also studied in [F-S-W] in relation with the non-linear Schrödinger equation.

II. SECOND-ORDER MATRIX DIFFERENTIAL EQUATIONS AND HAMILTONIAN SYSTEMS FOR THE KAZHDAN-KOSTANT-STERNBERG SYMPLECTIC FORM

We are interested in second-order differential equations of the form:

$$\ddot{X} = Q(X) = -hh'(X)$$

where h is polynomial with scalar coefficients and the unknown X is a $m \times m$ matrix.

We assume that X varies in a vector space V of matrices so that the trace provides an identification with the dual V^* through the mapping $A \mapsto (B \mapsto \mathrm{Tr}(AB))$. The cotangent bundle $T^*V \simeq V \times V^*$ is equipped with the symplectic form $\omega = \mathrm{Tr}(dX \wedge dY)$ following [K-K-S]. The Hamiltonian

flow of the function $H : T^* V \mapsto \mathbb{R}$, defined by:
$H(X, Y) = (1/2)\,\mathrm{Tr}\,(h(X)^2 + Y^2)$ is given by

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= -hh'(X)) \end{aligned} \tag{2}$$

and it coïncides with (1) for solutions X in V. Hereafter V will be the space of real symmetric matrices.

III. HAMILTONIAN SYSTEMS OF QUARTIC TYPE

We use the matrices

$$Z = \sqrt{-1}\,h(X) + L,\ Z^* = -\sqrt{-1}\,h(X) + L, \tag{3}$$

In the following, (X, Y) are assumed to be real symmetric matrices so that:

$$Z^* = {}^t\bar{Z} \qquad \text{(transposed of the complex conjugated of Z).}$$

We are concerned with the Hamiltonian system

$$H(X, Y) = (1/2)\,\mathrm{Tr}\,(h(X)^2 + Y^2),\ \omega = \mathrm{Tr}\,(dX \wedge dY) \tag{4}$$

for $h(X) = \lambda X + \mu X^2$.

We use the Hermitian matrix $P = ZZ^*$ that we consider as an element of the Lie algebra $u(m)$.

THEOREM 1. — *The matrix* P *defines a Lax pair for the Hamiltonian system* (4) *and its eigenvalues are in involution for the symplectic form* ω.

Proof. — We get from (4):

$$H = (1/2)\,\mathrm{Tr}\,(P).$$

Hamilton's equations (2) give:

$$\begin{aligned} \dot{Z} &= \sqrt{-1}\,(\lambda Y + \mu(XY + YX)) - hh'(X) \\ \dot{Z}^* &= -\sqrt{-1}(\lambda Y + \mu(XY + YX)) - hh'(X) \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{Z} &= \{\{(\sqrt{-1}/2)\,h'(X),\ Z\}\} \\ \dot{Z}^* &= -\{\{(\sqrt{-1}/2)\,h'(X),\ Z\}\} \end{aligned} \tag{6}$$

where the symbol $\{\{A, B\}\}$ means the anti-commutator of the two matrices A and B.

We get:

$$\begin{aligned} \dot{P} &= \dot{Z}Z^* + Z\dot{Z}^*, \\ \dot{P} &= \{\{(\sqrt{-1}/2)\,h'(X),\ Z\}\}\,Z^* - Z\,\{\{(\sqrt{-1}/2)\,h'(X),\ Z^*\}\}, \\ \dot{P} &= [\sqrt{-1}/2)\,h'(X),\ P]. \end{aligned} \tag{7}$$

So that P is a Lax matrix for the flow of H. As corollary, we obtain that the eigenvalues of P are constants of the motion.

Let Λ be one the eigenvalues of P and Ψ be the corresponding eigenvector. Let T be the projector on the subspace generated by Ψ. The matrix P being Hermitian, we can assume that Ψ is normalized for the standard Hermitian product $\langle\ ,\ \rangle$ on $\mathbb{C}^m$ preserved by P. We get:

$$d\Lambda = \langle d\mathrm{P}\Psi, \Psi \rangle = \mathrm{Tr}(d\mathrm{PT}). \tag{8}$$

Let us now consider the Hamiltonian flow generated by Λ and the symplectic form ω. The dot will designate the time-derivative along this flow.

Hamilton's equations give:

$$\mathrm{Tr}(\dot{\mathrm{X}}\, d\mathrm{Y} - \dot{\mathrm{Y}}\, d\mathrm{X}) = d\Lambda = \mathrm{Tr}(d\mathrm{Z}\mathrm{Z}^*\mathrm{T} + \mathrm{Z}\, d\mathrm{Z}^*\mathrm{T}) \tag{9}$$

and so we get first,

$$\begin{aligned} \dot{\mathrm{X}} &= \mathrm{Z}^*\mathrm{T} + \mathrm{TZ} \\ \dot{\mathrm{Y}} &= -\{\{(\sqrt{-1}/2)\, h'(\mathrm{X}), \mathrm{Z}^*\mathrm{T} - \mathrm{TZ}\}\} \end{aligned} \tag{10}$$

then,

$$\begin{aligned} \dot{\mathrm{Z}} &= \{\{\sqrt{-1}\, h'(\mathrm{X}), \mathrm{TZ}\}\} \\ \dot{\mathrm{Z}}^* &= -\{\{\sqrt{-1}\, h'(\mathrm{X}), \mathrm{Z}^*\mathrm{T}\}\}. \end{aligned} \tag{11}$$

This allows to compute $\dot{\mathrm{P}}$,

$$\begin{aligned} \dot{\mathrm{P}} &= \dot{\mathrm{Z}}\mathrm{Z}^* + \mathrm{Z}\dot{\mathrm{Z}}^*, \\ \dot{\mathrm{P}} &= [\mathrm{T}, \mathrm{Z}(\sqrt{-1}\, h'(\mathrm{X})/2)\, \mathrm{Z}^*] + [(\sqrt{-1}\, h'(\mathrm{X})/2), \mathrm{PT}]. \end{aligned} \tag{12}$$

Now we observe that P and T can be codiagonalized so that

$$[\mathrm{P}, \mathrm{T}] = 0.$$

We deduce then that,

$$\dot{\mathrm{P}}\mathrm{P}^k = [\mathrm{T}, \mathrm{Z}(\sqrt{-1}\, h'(\mathrm{X})/2)\, \mathrm{Z}^*\mathrm{P}^k] + [(\sqrt{-1}\, h'(\mathrm{X})/2)\, \mathrm{P}^k, \mathrm{PT}] \tag{13}$$

for all integer k, and then,

$$\mathrm{Tr}(\dot{\mathrm{P}}\mathrm{P}^k) = 0 \tag{14}$$

which implies that the quantities $\mathrm{tr}(\mathrm{P}^k)$ are constants of the motion. Hence, the eigenvalues of P are constants of the motion. But this is true for the flow of any of the eigenvalues of P, so we deduce that the eigenvalues of P are in involution. This ends the proof. □

Example. — Let us consider the case where X is a 2×2 symmetric real matrix.

$$\mathrm{X} = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

The equation (1) gives:

$$\dot{x}_1 = -\lambda^2 x_1 - 3\lambda\mu(x_1^2 + x_3^2) - 2\mu^2(x_1(x_1^2 + x_3^2) + x_3^2(x_1 + x_2))$$
$$\dot{x}_2 = -\lambda^2 x_2 - 3\lambda\mu(x_2^2 + x_3^2) - 2\mu^2(x_2(x_2^2 + x_3^2) + x_3^2(x_1 + x_2))$$
$$\dot{x}_3 = -\lambda^2 x_3 - 3\lambda\mu_3(x_1 + x_2) - 2\mu^2(x_1 x_3(x_1 + x_2) + x_3(x_2^2 + x_3^2)).$$

From the theorem 1, we know that there are two conserved quantities. It would be interesting to decide in that case if the system is integrable.

Remarks. — 1. Let us consider the special case $\mu=0$ (quadratic case) in order to compare with the general case.

For $\mu=0$, $h'(X)=\lambda\,\mathrm{Id}$, and the equation (7) becomes $\dot{P}=0$ identically. This means that all the entries of P are constants of the motion.

In general case when $\mu\neq 0$, the equation (7) tells that only the eigenvalues of P are preserved by the flow. Also (12) becomes:

$$\dot{P}=[T, Z(\sqrt{-1}\,\lambda\,\mathrm{Id}/2)Z^*]+[(\sqrt{-1}\,\lambda\,\mathrm{Id}/2), PT]$$
$$\dot{P}=[T, (\sqrt{-1}\,\lambda\,\mathrm{Id}/2)P]$$

but $[P,T]=0$, so $\dot{P}=0$ identically also. Thus, all the entries of P are preserved by the flow of any eigenvalue.

Going back to (11), we get:

$$\begin{aligned}\dot{Z}&=\sqrt{-1}\,\lambda TZ\\ \dot{Z}^*&=-\sqrt{-1}\,\lambda Z^* T.\end{aligned} \tag{15}$$

One can show [F] that (15) defines a symplectic action of the torus T^m, this is no longer true for $\mu\neq 0$.

2. The proof does not extend as it stands to Hamiltonian systems of the sextic type.

IV. COMPLETE INTEGRABILITY OF THE CALOGERO-MOSER SYSTEM WITH AN EXTERNAL QUARTIC POTENTIAL

Let $\mathcal{U}$ be the vector space of $m\times m$ Hermitian matrices. The cotangent bundle $T^*\mathcal{U}$ may be identified with $\mathcal{U}\times\mathcal{U}^*\simeq\mathcal{U}\times\mathcal{U}$. As a cotangent bundle, it has a symplectic structure which can be written [K-K-S]:

$$\omega=\mathrm{Tr}\,dX\wedge dL, \qquad (X, L)\in\mathcal{U}\times\mathcal{U}.$$

We define on $T^*\mathcal{U}$ the Quartic Anharmonic Flow by the Hamiltonian:

$$H=(1/2)\,\mathrm{Tr}(h(X)^2+L^2).$$

The group $G=U(m)$ acts on $\mathcal{U}$ by the adjoint action; this action lifts into an Hamiltonian action on $T^*\mathcal{U}$. The corresponding moment map is given by:

$$T^*\mathcal{U}\cong\mathcal{U}\times\mathcal{U}\ni(X, L)\mapsto\sqrt{-1}\,[X, L]\in\mathcal{U}\cong\mathcal{U}^*. \tag{16}$$

The Hamiltonian H is invariant under this action.

Following [K-K-S] we proceed to a reduction of $T^* \mathscr{U}$ by the symplectic action of G. By using the identification (16), a fiber of the moment map can be seen as $\{(X, L) : [X, L] = -1\, g\, C\}$.

We choose for C the element $(C_{ij} = 1 - \delta_{ij})$. Let G_c be the isotropy subgroup of C. The reduced manifold

$X_c = \{(X, L) : [X, L] = \sqrt{-1}\, g\, C\}/G_c$ can be parametrized (*see* [K-K-S]) by:

$$X_{ij} = x_i \delta_{ij}$$
$$L_{ij} = y_i \delta_{ij} + \sqrt{-1}\, g/(x_i - x_j)(1 - \delta_{ij}) \qquad (g > 0),$$

and is identified with $T^* W$, $W = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m / x_i - x_j \neq 0,\ i \neq j\}$ equipped with the standard symplectic form $\omega = \sum_{i=1}^{m} dx_i \wedge dy_i$ After the reduction, the Hamiltonian $H = (1/2) \operatorname{Tr}(h(X)^2 + L^2)$ becomes the function

$$H = (1/2) \sum_{i=1}^{m} y_i^2 + g^2 \sum_{i, j;\ i<j} (x_i - x_j)^{-2} + (1/2) \sum_{i=1}^{m} (\lambda x_i + \mu x_i^2)^2. \qquad (17)$$

This System can be viewed as a perturbation of the Calogero-Moser System by an external quartic potential.

Hamilton's equations define a vector field whose flow is a solution of

$$\dot{x}_i = \partial H / \partial y_i = y_i \qquad (18)$$
$$\dot{y}_i = -\partial H / \partial x_i = 2 g^2 \sum_{j=i} (x_i - x_j)^{-3} - \sum_{i=1}^{m} 2((\lambda + 2\mu x_i)(\lambda x_i + \mu x_i^2).$$

The following result is a direct consequence of the theorem proved in the previous paragraph:

COROLLARY 1. — *The Calogero-Moser System with an external quartic potential defined by the Hamiltonian* (17) *is integrable in the Arnol'd-Liouville sense.*

Proof. — The theorem shows that the m quantities $F_k = tr(P^k)$ are in involution. They stay in involution after the reduction. Furthermore $dF_1 \wedge \ldots \wedge dF_k$ is not identically zero since for $\mu = 0$, we have the usual Calogero-Moser System with an external quadratic potential and we know that the functions $F_1, \ldots, F_k$ are generically independent in that case [A]. The symplectic manifold $T^* W$ is $2m$-dimensional, so we have shown the Arnol'd-Liouville integrability of the System (1). □

Note that the hypersurfaces of constant energy are compact and so by the Arnol'd-Liouville theorem, there are invariant tori and the solutions of (18) are quasi-periodic functions of the time.

ACKNOWLEDGEMENTS

The authors thank the Istituto Nazionale di Fisica Nucleare and the Italian Ministero della Pubblica Istruzione for their financial support which made this research possible. They also thank M. Bruschi for enlightening discussions.

REFERENCES

[A] M. ADLER, Some Finite Dimensional Integrable Systems and Their Scattering Behavior; *Commun. Math. Phys.*, Vol. **55**, 1977, pp. 195-230.

[C] F. CALOGERO, Solution of the One-Dimensional n-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials, *J. of Math. Phys.*, Vol. **12**, 1973, pp. 419-436.

[F-W-M] A. P. FORDY, S. WOJCIECHOWSKI and I. MARSHALL, A Family of Integrable Quartic Potentials Related to Symmetric Spaces, *Phys. Letters*, Vol. **1134,** 1986, pp. 395- .

[F1] J. P. FRANÇOISE, Canonical Partition Functions of Hamiltonian Systems and the Stationary Phase Formula, *Comm. Math. Physics*, Vol. **117**, 1, 1988, pp. 37-47.

[F2] J. P. FRANÇOISE, Symplectic Geometry and Integrable m-Body Problems on the Line, *J. Math. Phys.*, Vol. **29**, (5), 1988, pp. 1150-1153.

[K-K-S] J. KAZHDAN, B. KOSTANT and S. STERNBERG, Hamiltonian Group Actions and Dynamical Systems of Calogero type, *Comm. Pure Appl. Math.*, Vol. **31**, 1978, pp. 481-508.

[M] J. MOSER, Various Aspects of Integrable Hamiltonian Systems, *Proc. C.I.M.E. conf. held in Bressanone*, 1978.

[I] V. I. INOZEMTZEV, On the Motion of Classical Integrable Systems of Interacting Particles in an External Field, *Phys. Letters*, Vol. **103A,** 1984, pp. 316- .

[W] S. WOJCIECHOWSKI, On Integrability of the Calogero-Moser System in an External Quartic Potential and Other Many-Body Systems, *Phys. Letters*, Vol. **102A,** 1984, pp. 85- .

(*Manuscrit reçu le* 15 *novembre* 1988.)