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Floquet operators with singular spectrum. II

by

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ABSTRACT. — Floquet operators for time-periodic perturbation of discrete Hamiltonians with increasing gaps between eigenvalues have no absolutely continuous spectrum. An example is

$$-\frac{d^2}{dx^2}+q(x)+v(x, t)$$

with Dirichlet boundary conditions on $[0, \pi]$, where $q(x)$ is bounded and $v(x, t)$ is C^2 and periodic in t.

RÉSUMÉ. — Les opérateurs de Floquet pour des perturbations périodiques d'hamiltoniens discrets ayant des distances entre valeurs propres successives croissantes n'ont pas de spectre absolument continu. Un exemple est donné par l'opérateur

$$-\frac{d^2}{dx^2}+q(x)+v(x, t)$$

avec conditions de Dirichlet sur $[0, \pi]$, où $q(x)$ est borné et $v(x, t)$ est C^2 et périodique en t.

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1. INTRODUCTION

Let H be a *discrete* Hamiltonian; for example, the harmonic oscillator, or the one-dimensional rotor. Let $V(t)$ be a time-periodic perturbation of H, and consider the Floquet operator:

$$K(\beta)=i\frac{d}{dt}+H+\beta V(t)$$

with time-periodic boundary conditions. For $\beta=0$, $K(0)$ is pure point.

What is the nature of the spectrum of $K(\beta)$ *for* $\beta\neq 0$?

There has been considerable recent interest in this question, and the reader is referred to [1], [4] for references. For the rotor, Bellissard [1] proved that the spectrum is pure point for *small* β and generic values of certain parameters. This was accomplished by means of an operator version of the KAM theorem. In [4], the author proved that for "almost every H" in an appropriate probabilistic sense, $K(\beta)$ is pure point for *all* β, provided that

$$\Delta\lambda_n=\lambda_n-\lambda_{n-1}\sim n^{2+\varepsilon},$$

a condition which unfortunately eliminates systems like the rotor, where $\Delta\lambda_n\sim n$. The proof involved methods from localization theory and an *adiabatic analysis* of the time-dependent Hamiltonian

$$H(t,\beta)=H+\beta V(t).$$

The present paper extends the results of [4] to include the Schrödinger case. It is shown that if

$$\Delta\lambda_n\sim n^\alpha$$

for $\alpha=1$ and $V(t)\in C^2$, then $K(\beta)$ has *no absolutely continuous part.* If $\alpha>0$ is smaller, increasing differentiability is required.

This result is not generic, but holds for *all* operators satisfying the hypotheses.

Our method consists of making an "operator-valued gauge transformation", suggested by the operator-theoretic KAM method of [1]. However, because we make only a finite number of such transformations, and make no connection, analytic or otherwise, of the perturbed eigenvalues with the unperturbed, the proof is understandably much simpler.

2. MAIN THEOREMS

Let H be a self-adjoint operator on $\mathscr{H}$ which is positive and discrete, with eigenvalues $0<\lambda_0<\lambda_1<\dots$ of simple multiplicity, satisfying

$$\Delta\lambda_n=\lambda_n-\lambda_{n-1}\geqq cn^\alpha. \tag{2.1}$$

Let $V(t)$ be a strongly continuous family of bounded operators, 2π-periodic in t, with

$$\int_0^{2\pi} V(t)\,dt=0. \tag{2.2}$$

Define the *Floquet Hamiltonian*

$$K=i\frac{d}{dt}+H+V(t) \tag{2.3}$$

on $L^2[0, 2\pi]\otimes\mathscr{H}$, with periodic boundary condition $u(0)=u(2\pi)$.

2.1. THEOREM. – *Let $\mathscr{H}$ satisfy* (2.1) *for $\alpha>0$. Let $V(t)$ be strongly C^r and satisfy* (2.3). *If $r=[\alpha^{-1}]+1$, then the Floquet operator K has no absolutely continuous spectrum.*

Here, $[\alpha]$ denotes the greatest integer in α. From now on, we shall write

$$D=i\frac{d}{dt}$$

with periodic boundary condition.

Let $\mathscr{B}\mathscr{B}(\mathscr{H})$ be the space of all bounded operators on $\mathscr{H}$.

Let Λ be a self-adjoint operator on $\mathscr{H}$ with $0<\Lambda\leqq I$. For $\gamma, \gamma'>0$, define

$$\mathscr{I}(\gamma, \gamma'; \Lambda)=\Lambda^{\gamma}\mathscr{B}\Lambda^{\gamma'}$$

to be the space of all bounded operators B such that $\Lambda^{-\gamma}B\Lambda^{-\gamma'}$ is bounded. Let

$$\mathscr{L}(\alpha; \Lambda)=\bigcap_{\gamma+\gamma'<2\alpha}\mathscr{I}(\gamma, \gamma'; \Lambda)$$

be the space of all operators B such that $B\in\mathscr{I}(\gamma, \gamma', \Lambda)$ whenever $\gamma, \gamma'>0$, and $\gamma+\gamma'<2\alpha$.

Note that if $\alpha>\beta$, then

$$\mathscr{L}(\alpha)\subset\mathscr{L}(\beta).$$

If Λ is invertible, all these spaces collapse to $\mathscr{B}$. However, the example we are interested in is the case where Λ is diagonal (*i.e.*, commutes with H), and has entries $\Lambda_{nn}=\dfrac{1}{n}$ on the diagonal:

$$\Lambda=\langle\,.\,,\varphi_0\rangle\varphi_0+\sum_{n=1}^{\infty}\frac{1}{n}\langle\,.\,,\varphi_n\rangle\varphi_n.$$

In this case

$$\mathscr{I}(\gamma, \gamma'; \Lambda)\subset\mathscr{I}_p$$

(the trace ideal) whenever $\gamma+\gamma'>p^{-1}$ and $p\geqq 1$. All $\mathscr{J}$ classes therefore consist of compact operators.

Henceforth, Λ *will denote this operator,* and will usually be supressed in the notation.

We shall say that $V(t)$ is in $C(r, \alpha; \Lambda)$ if for $\gamma, \gamma'>0$ and $\gamma+\gamma'<2\alpha$, the operator

$$\Lambda^{-\gamma} V(t) \Lambda^{-\gamma'}$$

is strongly C^r.

2.3. THEOREM. — *Let*

$$K_1 = D + H_1 + V_1(t)$$

where H_1 *satisfies* (2.1) *and* $V_1 \in C(r+1, \delta)$. *If* $0<\eta<\alpha/2$, *then* K_1 *is unitarily equivalent to an operator*

$$K_2 = D + H_2 + V_2(t)$$

where H_2 *also satisfies* (2.1) (*possibly with a different c*) *and commutes with* H_1, *and where* $V_2(t) \in C(r, \delta+\eta)$.

Proof of Theorem 2.1. — From [4], Theorem 5.1, we know that K is unitarily equivalent to

$$K_1 = D + H_1 + V_1(t)$$

with $V_1(t) \in C(r-1, \delta)$ if $0<\delta<\alpha/2$. If we iterate $(N-1)$ times, using Theorem 2.3, we improve δ by an amount η arbitrarily close to $\alpha/2$ each time. Thus, if $N\alpha/2>1/2$, or

$$N = [\alpha^{-1}] + 1,$$

then $V_1(t) \in C(r-N, \beta)$ for $\beta>1/2$, and is therefore trace class. By [3], Theorem 5, the absolutely continuous part of K_1 is unitarily equivalent to the absolutely continuous part of $D+H_1$. But the latter operator has pure point spectrum. ■

3. PROOF OF THEOREM 2.3

We shall regard operators T on $\mathscr{H}$ as infinite matrices $T=(T_{nl})$ in the basis φ_n. It will be assumed throughout that H satisfies (2.1):

$$\Delta\lambda_n \geqq c n^{\alpha}$$

from which it follows [4], § 5, that

$$\lambda_n - \lambda_l \geqq c\,(n^{\alpha+1} - l^{\alpha+1}). \tag{3.2}$$

Let V be an *off-diagonal* operator:

$$V_{nn} = 0.$$

We shall need to solve the *commutator equation*

$$[\mathrm{H}, \mathrm{X}]=\mathrm{V} \tag{3.3}$$

for X. In components, this is

$$(\lambda_n-\lambda_l)\,\mathrm{X}_{nl}=\mathrm{V}_{nl},$$

so that the solution is

$$\mathrm{X}_{nl}=(\lambda_n-\lambda_l)^{-1}\,\mathrm{V}_{nl}. \tag{3.4}$$

We shall write this as

$$\mathrm{X}=\Gamma(\mathrm{V}).$$

3.1. PROPOSITION. – *If* $\mathrm{V}\in\mathscr{L}(\delta)$ *and* $0<\eta<\alpha/2$, *then* $\Gamma(\mathrm{V})\in\mathscr{L}(\delta+\eta)$.

Prrof. – Let $\mathrm{V}\in\mathscr{J}(\gamma, \gamma')$, so that

$$\mathrm{V}_{nl}=n^{-\gamma}\,l^{-\gamma'}\,\mathrm{W}_{nl} \tag{3.5}$$

with $|\mathrm{W}_{nl}|\leqq 1$. We need to show that $\mathrm{T}=\Lambda^{-\beta}\,\Gamma(\mathrm{V})\,\Lambda^{-\beta'}$ is bounded for appropriate values of β, β'. Now

$$\mathrm{T}_{nl}=(\lambda_n-\lambda_l)^{-1}\,n^{-\gamma+\beta}\,l^{-\gamma'+\beta'}\,\mathrm{W}_{nl} \tag{3.6}$$

is majorized by

$$\left|n^{-\alpha+1}-l^{\alpha+1}\right|^{-1}n^{-\gamma+\beta}\,l^{-\gamma'+\beta'}.$$

By Theorem A.1 of the Appendix with $p=1$, this operator is bounded if

$$\beta<\alpha+\gamma, \qquad \beta'<\alpha+\gamma' \tag{3.7}$$

and

$$\beta+\beta'<\alpha+\gamma+\gamma'. \tag{3.8}$$

It remains to show that given any positive β and β' with $\beta+\beta'<2\,\delta+2\,\eta$, one can find positive γ and γ' with

$$\gamma+\gamma'<2\,\delta \tag{3.9}$$

such that (3.7) and (3.8) hold. Let β, β' be given. If we choose ε, $\varepsilon'>0$ and define γ, γ' by

$$\beta+\varepsilon=\alpha+\gamma, \qquad \beta'+\varepsilon'=\alpha+\gamma',$$

then (3.7) holds. Choose

$$\alpha<\varepsilon+\varepsilon'<2\,\alpha-2\,\eta,$$

then (3.9) holds:

$$\gamma+\gamma'=(\beta+\beta')-2\,\alpha+(\varepsilon+\varepsilon')<2\,\delta+2\,\eta-2\,\alpha+2\,\alpha-2\,\eta=2\,\delta,$$

and (3.8) as well:

$$\beta+\beta'=\gamma+\gamma'+2\,\alpha-(\varepsilon+\varepsilon')<\gamma+\gamma'+\alpha. \quad \blacksquare$$

3.2. PROPOSITION. — *(a) If* $A \in \mathscr{L}(\alpha)$, *and* $B \in \mathscr{L}(\beta)$, *and* $\gamma+\gamma'<2\alpha+2\beta$, *then* $AB \in \mathscr{J}(\gamma, \gamma')$;

(b) if $A, B \in \mathscr{L}(\alpha)$, *then* $AB \in \mathscr{L}(2\alpha)$.

Proof. — Let $2\alpha=\gamma+2\varepsilon$, and $2\beta=\gamma'+2\varepsilon'$. Then

$$\Lambda^{-\gamma} A B \Lambda^{-\gamma'}=(\Lambda^{-\gamma} A \Lambda^{-\varepsilon}) \Lambda^{\varepsilon+\varepsilon'}(\Lambda^{-\varepsilon'} B \Lambda^{-\gamma'})$$

is bounded. This proves *(a)*, and *(b)* follows immediately.

3.3. PROPOSITION. — *Let* $A \in \mathscr{L}(\alpha)$, *and let* $f(z)$ *be entire with* $f(0)=0$. *Then* $f(A) \in \mathscr{L}(\alpha)$.

Proof. — Let $\gamma+\gamma'<2\alpha$, $\gamma+2\varepsilon=2\alpha$, and $\gamma'+2\varepsilon'=2\alpha$. Then, for $n \geqq 2$,

$$\Lambda^{-\gamma} A^n \Lambda^{-\gamma'}=(\Lambda^{-\gamma} A \Lambda^{-\varepsilon})(\Lambda^{\varepsilon} A^{n-2} \Lambda^{\varepsilon'})(\Lambda^{-\varepsilon'} A \Lambda^{-\gamma'}) \tag{3.10}$$

is bounded with norm not exceeding a constant times $\|A\|^{n-2}$. If $f(z)=\sum_{n=1}^{\infty} c_n A^n$, it follows that

$$\Lambda^{-\gamma} f(A) \Lambda^{-\gamma'}=c_1 \Lambda^{-\gamma} A \Lambda^{-\gamma'}+\sum_{n=2}^{\infty} c_n \Lambda^{-\gamma} A^n \Lambda^{-\gamma'}$$

converges and is bounded. ■

It would be interesting to know for what class of functions f this result holds, but the functions we have in mind are all entire.

Proof of Theorem 2.3. — Let $H_1=\operatorname{diag}\{\lambda_n\}$ and $H_2=\operatorname{diag}\{\lambda_n+\mu_n\}$ where

$$\mu_n=\frac{1}{2\pi} \int_0^{2\pi}(V_1(t) \varphi_n, \varphi_n)\, dt. \tag{3.11}$$

Since $V_1 \in \mathscr{L}(\delta)$, we have $|\mu_n| \leqq c n^{-2\delta+\varepsilon}$, and so H_2 also satisfies condition (α). By replacing H_1 by H_2, we can assume that $V_1(t)$ vanishes on the diagonal. In fact, write

$$(V_1(t) \varphi_n, \varphi_n)=\mu_n+g_n'(t)$$

with $g_n(t)$ in C^{r+2} and 2π-periodic, and let

$$U(t)=\operatorname{diag}\{e^{-i g_n(t)}\}.$$

Then if U is the operator on $\mathscr{H}$ of multiplication by $U(t)$, we have

$$U(D+H_1+V_1(t)) U^*=D+H_2+\hat{V}(t)$$

where $\hat{V}_{ll}(t)=0$ and

$$\hat{V}_{kl}(t)=V_{lk}(t) e^{+i[g_l(t)-g_l(t)]} \qquad (k \neq l).$$

Thus, $\hat{V}(t)$ is off-diagonal and still $C(r+1, \delta)$.

Define $G(t)$ by

$$i[H_2, G(t)]=\hat{V}(t) \tag{3.12}$$

so that $G(t)\in C(r+1, \delta+\eta)$, $0<\eta<\alpha/2$, and transform $D+H_2+\hat{V}(t)$ by $e^{iG(t)}$. We obtain

$$e^{iG(t)}De^{-iG(t)}$$
$$=D-i\dot{G}+\sum_{n=2}^{\infty}\frac{(-i)^n}{n!}\{\dot{G}G^{n-1}+G\dot{G}G^{n-2}+\ldots+G^{n-1}\dot{G}\}. \tag{3.13}$$

Both $\dot{G}$ and the terms of the sum are all $C(r, \delta+\eta)$. The sum can be dominated, as in Proposition 3.3, by $n\|G\|^{n-3}$. Also, writing

$$ad\,G[H]=[G, H],$$

we have

$$e^{iG(t)}H_2e^{-iG(t)}=e^{iad\,G(t)}[H_2]$$
$$=H_2+i[G(t), H_2]+\sum_{n=2}^{\infty}\frac{i^n}{n!}[ad\,G(t)]^{n-1}[G(t), H_2]$$
$$=H_2-\hat{V}(t)+\sum_{n=2}\frac{i^{n+1}}{n!}[ad\,G(t)]^{n-1}[\hat{V}(t)]. \tag{3.14}$$

In the expansion of the $n+1$-term, there are 2^{n-1} terms, each a product of $(n-1)$ G's and one $\hat{V}$ in some order. Thus, again estimating as in Proposition 3.3, the terms of the series are in $C(r, \delta+\eta)$ (at least) and dominated by $2^n\|G(t)\|^{n-3}$.

Finally, if $f_\pm(z)=e^{\pm iz}-1$, we have

$$e^{iG(t)}\hat{V}(t)e^{iG(t)}$$
$$=\hat{V}(t)+f_+(G)\hat{V}(t)+\hat{V}(t)f_-(G)+f_+(G)\hat{V}(t)f_-(G) \tag{3.15}$$

where the last three terms are $C(r+1; \delta+\eta)$ by Proposition 3.3.

Combining these estimates, we find that, due to cancellation of the $\hat{V}(t)$ terms,

$$e^{iG(t)}(D+H_2+\hat{V}(t))e^{-iG(t)}=D+H_2+V_2(t)$$

with $V_2(t)\in C(r, \delta+\eta)$.

4. APPLICATIONS AND REMARKS

(1) Let us briefly indicate the application to one-dimensional Schrödinger operators, with no attempt at all at completeness. Let

$$H=-\frac{d^2}{dx^2}+q(x)$$

on $0 \leqq x \leqq \pi$ with Dirichlet boundary conditions, where $|q(x)| \leqq M$. By perturbation theory,

$$|\lambda_n - n^2| \leqq M,$$

so that H satisfies condition (2.1) with $\alpha = 1$. Hence, $N = [\alpha^{-1}] + 1 = 2$, and so if $v(x, t)$ is C^2 and periodic in t, then the Floquet spectrum of

$$H(t) = -\frac{d^2}{dx^2} + q(x) + v(x, t)$$

is purely singular.

(2) One would expect the same result the *pulsed rotor*, discussed by Bellissard [1]:

$$H(t) = -\frac{d^2}{d\theta^2} + v(x, t)$$

on $0 < \theta < 2\pi$, with *periodic* boundary conditions, and $v(x, t)$ in C^2. However, $H = -d^2/d\theta^2$ has multiplicity *two*, and so our theory doesn't apply. The author supposes the problem of extending the theory to this case to be only technical, but he has not attempted to do so.

(3) Some smoothness of $v(x, t)$ seems required, since the *kicked rotor*, for which $v(x, t)$ contains a delta function in t, has very different behavior [1].

(4) The case $\alpha = 0$ of bounded, but non-increasing gap, includes, of course, the *harmonic oscillator*. Hagadorn, Loss, and Slawny [2] solve explicitly the problem of an oscillatory Stark perturbation of the harmonic oscillator, and find absolutely continuous spectrum in one resonant case.

(5) Finally, what about *pure point spectrum?* In order to apply the method of [4] in the present case, we need to have $V(t) \in \mathscr{L}(\beta)$ with $\beta > 1$. This requires

$$N = [2\alpha^{-1}] + 1$$

iterations, and so for the Schrödinger case, we would need $v(x, t) \in C^3$ for the *method* to work. We do not know if this is essential or not.

We could now easily prove an "a. e. H" result analogous to [4], Theorem 4.1. To be more precise, we could prove that if H is perturbed by a very small *random*, but *trace class*, perturbation, then the resulting K would be pure point *almost surely*. However, in the Schrödinger case, which is the one of greatest interest, we want $H(\omega)$ to be a *Schrödinger* operator, a. s., not just some abstract random perturbation of H. What is needed here is a better technology for proving multiplication operators to be absolutely continuous. We hope to discuss this in a future publication.

APPENDIX

Let T be the infinite matrix defined by $T_{nn}=0$, and

$$T_{nl}=n^{\gamma} l^{\gamma'} \left| n^{\alpha+1}-l^{\alpha+1} \right|^{-p}, \qquad n, l \geqq 0.$$

A.1. THEOREM. — *T is bounded on l_2, provided that one of the conditions* (A.2) *below holds.*

Proof. — The proof follows as in [4], § 5, by applying the Schur-Holmgren condition: we need to show that

$$\sum_{l=n+1}^{\infty} \frac{n^{\gamma} l^{\gamma'}}{(l^{\alpha+1}-n^{\alpha+1})^p} + \sum_{l=1}^{n-1} \frac{n^{\gamma} l^{\gamma'}}{(n^{\alpha+1}-l^{\alpha+1})^p} \tag{A.1}$$

is bounded uniformly in n; and similarly with γ and γ' interchanged. By the *integral test*, the first term does not exceed

$$\int_{n+1}^{\infty} \frac{n^{\gamma} y^{\gamma'}}{(y^{\alpha+1}-n^{\alpha+1})^p}\, dy = \frac{n^{\gamma+\gamma'+1}}{n^{(\alpha+1)p}} \int_{1+1/n}^{\infty} s^{\gamma'} (s^{\alpha+1}-1)^{-p}\, ds.$$

If $p(\alpha+1)-\gamma'>1$ for convergence of the integral at infinity, then the integral is of order n^{p-1} if $p>1$, of order $\log n$ if $p=1$ and bounded if $p<1$.

The second term is equal to the *Riemann sum*

$$n^{\gamma+\gamma'-(\alpha+1)p+1} \sum_{k=1}^{n-1} \left[1-\left(\frac{k}{n}\right)^{\alpha+1}\right]^{-1} \left(\frac{k}{n}\right)^{\gamma'} \cdot \frac{1}{n}.$$

Estimate the sum from $k=1$ to $n-2$ by an integral and the $k=n-1$ term separately: the term is of order n^{p-1} and the integral is

$$\int_{0}^{1-1/n} [1-s^{\alpha+1}]^{-p} s^{\gamma'}\, ds = \mathrm{I}(p, n)$$

which is of the same order of magnitude as the previous integral. Combining, we find that (A.1) is of order

$$n^{\gamma+\gamma'-(\alpha+1)p+1}\, \mathrm{I}(p, n).$$

This is uniformly bounded if

$$\alpha > \gamma+\gamma' \qquad (p=1) \tag{A.2a}$$

or

$$\alpha p \geqq \gamma+\gamma' \qquad (p>1) \tag{A.2b}$$

or

$$\gamma+\gamma' \leqq p(\alpha+1)-1 \qquad (p<1).$$

Since these conditions are symmetric in γ and γ', the other half of Schur-Holmgren is also satisfied, provided that the integrability condition $p(\alpha+1)-\gamma>1$ also holds.

This proves the result.

Note added in proof. The trick of applying scattering theory to operators with no absolutely continuous spectrum is also used, independently, by B. Simon and T. Spencer in a paper to appear in *Comm. Math. Phys.*

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