ANNALES DE L'I. H. P., SECTION A

JAMES S. HOWLAND

Floquet operators with singular spectrum. I

Annales de l'I. H. P., section A, tome 50, nº 3 (1989), p. 309-323
<http://www.numdam.org/item?id=AIHPA_1989__50_3_309_0>

Ann. Inst. Henri Poincaré,

Vol. 49, n° 3, 1989, p. 309-323. *Physique théorique*

Floquet Operators with Singular Spectrum. I

by

James S. HOWLAND [1]

Department of Mathematics,
University of Virginia,
Charlottesville, VA 22903, U.S.A.

ABSTRACT. — A positive, discrete Hamiltonian H is perturbed by a time-periodic perturbation $V(t)$. If the gap between successive eigenvalues of H grows sufficiently rapidly, then generically (in a probabilistic sense) $H+\beta V(t)$ has dense pure point Floquet spectrum.

RÉSUMÉ. — Nous perturbons un opérateur Hamiltonien discret par un opérateur $V(t)$ périodique en temps. Si la distance entre les valeurs successives de H croît vite, alors génériquement (en un sens probabiliste) $H+\beta V(t)$ a un spectre de Floquet purement ponctuel dense.

1. INTRODUCTION

Let H be a discrete Hamiltonian operator on $\mathscr{H}$ with eigenvalues λ_k, and $V(t)$ a periodic, time-dependent perturbation of H:

$$V(t+a)=V(t).$$

[1] Supported by N.S.F. Contracts DMS-8500516 and 8001548.

The natural object to consider for a periodic Hamiltonian

$$H(t, \beta) = H + \beta V(t)$$

is the *Floquet Hamiltonian*:

$$K(\beta) = i\frac{d}{dt} + H(t, \beta)$$

with periodic boundary condition $u(a) = u(0)$, acting on the space $\mathcal{K} = L_2[0, a] \otimes \mathcal{H}$. (*See*, e. g., [7], [13] and a vast physics literature.)

If $\beta = 0$, and the period is normalized to $a = 2\pi$, then the operator $K(0)$ has pure point spectrum with eigenvalues

$$\Lambda(n, k) = n + \lambda k$$

($n = 0, \pm 1, \ldots$). Except in rare cases, this spectrum will be dense in the line.

The question that we wish to consider is this: *When does the perturbed operator* $K(\beta)$ *also have dense pure point spectrum?* The question has generated considerable recent interest ([1], [2], [3], [6], [7], [11]), and we refer particularly the reader to article [1] of Bellisard. Most of this work has resulted in operator-theoretic versions of the Kolmogorov-Arnold-Moser theorem, which asserts pure point spectrum for generic values of certain parameters in $H(t)$ and for small coupling β. The essential idea, though, is that K *will be pure point if there is no resonance*.

The present paper takes a different approach to the problem, based on the author's generalization [8] of the Simon-Wolff-Kotani [12] method from the theory of localization. We shall show under certain conditions that $K(\beta)$ *is pure point for "almost every* H". Thus, the method yields generic results on a *probabilistic*, rather than in a metric sense. The essential technical condition, which we believe can be weakened substantially, is that the *gap between eigenvalues of* H

$$\Delta\lambda_n = \lambda_{n+1} - \lambda_n$$

grows like $n^{2+\varepsilon}$, $\varepsilon > 0$.

After recalling some results from [8], we consider in paragraph 3 as an easy consequence, certain *compact* (actually, trace class) perturbations of H, generalizing a result of [7]. We then state the Main Theorem, and show in paragraph 5 how an adiabatic analysis of $H(t, \beta)$ reduces the problem to one like that of paragraph 3. We close with some remarks and conjunctures.

2. NOTATION AND PREVIOUS RESULTS

Let H be a *positive definite, discrete* selfadjoint operator of *simple multiplicity* on a separable Hilbert space $\mathscr{H}$. Let φ_n be a complete orthonormal set of eigenvectors of H:

$$\mathrm{H}\,\varphi_n = \lambda_n\,\varphi_n$$

with $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ Let $\mathrm{V}(t)$ be a uniformly bounded measurable family of bounded operators, which is 2π-periodic in t:

$$\mathrm{V}(t+2\pi) = \mathrm{V}(t)$$

and define the *Floquet operator*

$$\mathrm{K}(\beta) = i\frac{d}{dt} + \mathrm{H} + \beta\,\mathrm{V}(t)$$

on $\mathscr{K}\mathrm{L}^2[0, 2\pi] \otimes \mathscr{H}$, with *periodic boundary condition:*

$$u(0) = u(2\pi).$$

We shall sometimes write $\mathrm{K}_0 = \mathrm{K}(0)$ and $\mathrm{K} = \mathrm{K}(1)$.

We shall also consider families $\mathrm{H}(\omega)$ of operators satisfying these conditions, which are measurable on a probability space (P, Ω). We shall refer to these briefly as "*random Hamiltonians*", and will write

$$\mathrm{H}(\omega)\,\varphi_n(\omega) = \lambda_n(\omega)\,\varphi_n(\omega).$$

If

$$\mathrm{K}(\omega) = i\frac{d}{dt} + \mathrm{H}(\omega) + \mathrm{V}(t)$$

is the corresponding Floquet operator, we define $\mathbf{K}$ to be the *multiplication operator*

$$\mathbf{K}\,u(\omega) = \mathrm{K}(\omega)\,u(\omega)$$

on $\mathrm{L}^2(\mathrm{P}, \Omega) \otimes \mathscr{K}$. If the coupling constant β is included, we obtain $\mathbf{K}(\beta)$, so that $\mathbf{K} = \mathbf{K}(1)$.

We shall next summarize some results of [8]. Let H be pure point and A bounded. We say [8], p. 64, that A is *strongly* H-*finite on an open interval* J iff

$$\Sigma\{|\mathrm{A}\,\varphi_n| : \lambda_n \in \mathrm{J}\} < \infty.$$

2.1. PROPOSITION [8], pp. 56-58. – *Let* H *be pure point,* A *strongly* H-*finite on* J, and W *bounded and self-adjoint. Let*

$$\mathrm{H}_1 = \mathrm{H} + \mathrm{A}^*\,\mathrm{W}\mathrm{A}.$$

Then there exists a set $\mathrm{N} = \mathrm{N}(\mathrm{H}, \mathrm{A})$ *not depending on* W *such that*

(i) N *has Lebesque measure zero, and*

(ii) N *supports the continuous spectrum of* H *in* J.

Finally, we have the following version of "Kotani's trick".

2.2. PROPOSITION [8], p. 59. — *Let* $K(\omega)$ *be a random self-adjoint operator such that:*

(i) *there exists a set* N *of Lebesque measure zero, independent* ω, *which supports the continous part* $K(\omega)$ *a. s.,*

(ii) **K** *has absolutely continuous spectral measure. Then* $K(\omega)$ *is pure point a. s.*

3. COMPACT PERTURBATIONS OF FLOQUET OPERATORS

Let H be positive, discrete and of simple multiplicity, and A strongly H-finite. Let K be the Floquet operator

$$K=i\frac{d}{dt}+H.$$

3.1. PROPOSITION. — $1\otimes A$ *is strongly* K-*finite on any finite interval* J.

Proof. — The eigenvalue of K are $\Lambda(n, k)=n+\lambda_k$ with eigenvectors $\Phi(n, k)(t)=e^{int}\varphi_k$. If we assume that J has length less than 1, then $\Lambda(n, k)$ is in J for at most one value of n, which we call n_k. Thus, for each k,

$$\Sigma\{|(1\otimes A)\Phi(n, k)|: \Lambda(n, k)\in J\} =\sum_k |(1\otimes A)\Phi(n_k, k)| \leqq \sum_k |A\varphi_k| < \infty. \quad \blacksquare$$

Let $W(t)$ be a uniformly bounded, 2π-periodic measurable family of self-adjoint operators, and

$$V(t)=A^* W(t) A.$$

Note that the sum of two terms of this form can be written in the same form:

$$[A_1^* W_1(t) A_1+A_2^* W_2(t) A_2]\varphi=A^* W(t) A\varphi =(A_1^*, A_2^*)\begin{pmatrix} W_1(t) & 0 \\ 0 & W_2(t)\end{pmatrix}\begin{pmatrix} A_1\varphi \\ A_2\varphi\end{pmatrix}$$

where $A:\mathscr{H}\to\mathscr{H}\oplus\mathscr{H}$ (*cf.* [7]).

3.2. THEOREM. — *If* $W(t)>0$, *then the Floquet operator for*

$$H(t, \beta)=H+\beta V(t)$$

is pure point for a. e. β.

Proof. — It suffices to consider $|\beta| < 1$. From Proposition 2.1, $K(\beta)$ has its continous part is concentrated on a set $N(H, A)$ of measure zero, independent of β. To be able to apply Proposition 2.2, we write $\beta = \tanh x$, where $-\infty < x < \infty$. The operator $\mathbf{K}$ of multiplication by $K(x)$ then has a positive commutator with the bounded operator

$$B = i \arctan(p/2)$$

$\left(\text{where } p = -i \dfrac{d}{dx}\right)$, and is therefore absolutely continous by the Putnam-Kato Theorem (*cf.* [8], p. 60).

We shall next show that for "almost every H", $K(\beta)$ is pure point for *every* β. To be precise, let $X_j(\omega)$ be i. i. d, and uniform on $[-1, 1]$, and $\varepsilon_j > 0$ with

$$\sum_{j=0}^{\infty} \varepsilon_j < \infty.$$

3.3. THEOREM. — *The Floquet operator for*

$$H(t, \omega) = H + \sum_{j=0}^{\infty} \varepsilon_j^2 X_j(\omega) \langle \,.\,, \varphi_j \rangle \varphi_j + V(t)$$

is pure point a. s.

Thus, if the eigenvalues of H are all wiggled independently by a tiny amount, K will have pure point spectrum.

Proof. — The second term of $H(t, \omega)$ can be written as $EX(\omega)E$ where $E = \sum_j \varepsilon_j \langle \,.\,, \varphi_j \rangle \varphi_j$ and $X(\omega) = \sum_j X_j(\omega) \langle \,.\,, \varphi_j \rangle \varphi_j$. Since E is strongly H-finite, we find from (3.1) and Proposition 2.1, that $K(\omega)$ has continous spectrum concentrated on a null set $N = N(H, A, E)$ indepenent of ω. Absolute continuity of $\mathbf{K}$ is obtained as in [8], pp. 67-69. ■

Remarks. — Several improvements in the results are easily made. The distribution of X_j need not be uniform [8], nor is simple multiplicity necessary. The randomness in $H(\omega)$ could be more simply taken as

$$H + \alpha A_1^2$$

with A_1 strongly H-finite. The reader may formulate such results for himself.

4. MAIN THEOREM

As above, let H be positive, discrete and of simple multiplicity, and $V(t)$ be bounded and 2π-periodic satisfying

$$\int_0^{2\pi} V(t)\,dt=0. \tag{4.1}$$

Let $X_n(\omega)$ be i. i. d. and uniform on $[-1, 1]$, $\varepsilon_n>0$ with

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty$$

and define

$$H(t, \omega)=H+\sum_{n=0}^{\infty} \varepsilon_n X_n(\omega) \langle \cdot, \varphi_n \rangle \varphi_n+V(t). \tag{4.2}$$

Let $\Delta\lambda_n$ be the *gap* between eigenvalues:

$$\Delta\lambda_n=\lambda_n-\lambda_{n-1}.$$

4.1. THEOREM. — *Let* $V(t)$ *be strongly* C^1, *and satisfy for some* $c>0$

$$\Delta\lambda_n \geqq cn^{\alpha} \tag{4.3}$$

and $\alpha>2$. *Then* $K(\omega)$ *is pure point a. s.*

Proof. — Absolute continuity of **K** follows before. By the adiabatic analysis of $H(t, \omega)$ carried out in the next section, the operator $K(\omega)$ is *unitarily equivalent* to an operator

$$i\frac{d}{dt}+H+AW(t, \omega)A.$$

The operator A is strongly H-finite since $\gamma>1$, and $W(t, \omega)$ bounded in norm uniformly in t and ω. Existence of a null set $N(H, A)$, independent of ω and supporting the continuous part of $K(\omega)$, now follows from Proposition 2.1. ■

5. ADIABATIC ANALYSIS OF $H(t, \beta)$

Let A be a diagonal operator

$$A=\sum_{n=0}^{\infty} a_n \langle \cdot, \varphi_n \rangle \varphi_n$$

with $a_n>0$. We shall prove the following theorem:

5.1. THEOREM. — *Let* H *be positive, discrete and of simple multiplicity, and* $V(t)$ *strongly* C^{r+1}, *satisfying* (4.1). *Assume that for some* $c>0$ *and*

$\alpha > 0$,

$$\Delta\lambda_n \geqq c n^\alpha, \qquad \alpha > 0 \tag{5.1}$$

and let $a_n = n^{-\gamma}$ for $n \geqq 1$ *with* $0 < 2\gamma < \alpha$.

Then $K(\beta)$ *is unitarily equivalent to*

$$i\frac{d}{dt} + H + AW(t, \beta)A$$

where $W(t, \beta)$ *is strongly* C^r *in t and uniformly bounded.*

Remark. — Actually, $\lambda_n = \lambda_n(\omega)$ is *random*, but we will surpress ω and assume that (5.1) holds uniformly in ω. In fact, *we shall assume that for some* η, $|\lambda_n(\omega) - \lambda_n| \leqq \eta$ *for all* ω. Then if (5.1) holds for λ_n, it will hold uniformly for $\lambda_n(\omega)$, with a smaller c.

Let $|V(t)| \leqq M$, $|\dot{V}(t)| \leqq \dot{M}$. Let $\lambda_n(t, \beta)$ be the n-th eigenvalue of

$$H(t, \beta) = H + \beta V(t)$$

and

$$R(z; t, \beta) = (H(t, \beta) - z)^{-1}$$

its resolvent.

The reason for including β will be apparent in the proof of Lemma 5.3 below.

Note that if $n > k$, then

$$\lambda_n - \lambda_k = \Delta\lambda_n + \ldots + \Delta\lambda_{k+1} \geqq c(n^\alpha + \ldots + (k+1)^\alpha) \geqq c\int_k^n x^\alpha\, dx$$

so that for $n > k$,

$$\lambda_n - \lambda_k \geqq c(\alpha+1)^{-1}(n^{\alpha+1} - k^{\alpha+1}). \tag{5.2}$$

In particular, (since $\lambda_0 > 0$)

$$\lambda_n \geqq c(\alpha+1)^{-1} n^{\alpha+1}. \tag{5.3}$$

Let

$$r_n = \frac{c}{2} n^\alpha \leqq \frac{1}{2}\min\{\lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1}\}$$

and let $\Gamma_n (= \Gamma_n(\omega))$ be the positively oriented contour $|z - \lambda_n| = r_n$.

We now *choose and fix* N such that $r_n \geqq 2M$ for $n \geqq N$.

5.2. Lemma. — *(a) For* $n \geqq N$ *and* $|\beta| \leqq 1$,

$$|\lambda_n(\beta, t) - \lambda_n| \leqq M \leqq r_n/2$$

and hence

$$\operatorname{dist}(\lambda_n(\beta, t), \Gamma_n) \geqq r_n - M \geqq r_n/2.$$

Moreover, $\lambda_n(\beta, t)$ *is the only point of* $\sigma(\mathrm{H}(t, \beta))$ *inside* Γ_n.

Proof. — This follows by upper semicontinuity of the spectrum, since the norm of the perturbation does not exceed $|\beta|\mathrm{M}$. ■

Note that this gives

$$|\mathrm{R}(z, t, \beta)| \leqq 2 r_n^{-1}$$

for $z \in \Gamma_n$, and

$$|\lambda_n(t, \beta) - \lambda_k(t, \beta)| \geqq 1/2\, |\lambda_n - \lambda_k|.$$

For $n \geqq \mathrm{N}$, let the spectral projection for $\lambda_n(t, \beta)$ is

$$\mathrm{P}_n(t, \beta) = \frac{1}{2\pi i} \int_{\Gamma_n} \mathrm{R}(z; t, \beta)\, dz = \langle \,.\,, \varphi_n(t, \beta) \rangle\, \varphi_n(t, \beta).$$

The phase of $\varphi_n(t, \beta)$ is fixed by the choice

$$\varphi_n(t, \beta) = |\mathrm{P}_n(t, \beta)\, \varphi_n|^{-1} . \mathrm{P}_n(t, \beta)\, \varphi_n \tag{5.5}$$

which makes $\varphi_n(t, \beta)$ smooth and 2π-periodic in t. Note that the norm of $\mathrm{P}_n(t, \beta)\, \varphi_n$ is never zero; for we have

$$\mathrm{P}_n(t, \beta)\, \varphi_n - \varphi_n = (2\pi i)^{-1} \int_{\Gamma_n} \mathrm{R}(z; t, \beta)\, \mathrm{VR}(z, t, 0)\, dz$$

which yields, by Lemma 3.2, the estimate

$$|\mathrm{P}_n(t, \beta)\, \varphi_n - \varphi_n| \leqq \mathrm{M}\, r_n^{-1} \leqq 1/2.$$

We now need to separate off the first N eigenvalues in a group. Let

$$\mathrm{Q}(t, \beta) = \mathrm{I} - \sum_{j=\mathrm{N}+1}^{\infty} \mathrm{P}_j(t, \beta)$$

be the spectral projection onto the first N eigenvectors of $\mathrm{H}(t, \beta)$.

We can write

$$\mathrm{Q}(t, \beta) = (2\pi i)^{-1} \int_{\Gamma_0} \mathrm{R}(z, t, \beta)\, dz \tag{5.6}$$

where Γ_0 is a suitable contour encircling $\lambda_j(t, \beta)$ for $0 \leqq j \leqq \mathrm{N}$. From this representation, we obtain immediately the uniform boundedness and continuity of such operators as

$$\frac{\partial \mathrm{Q}}{\partial \beta}, \quad \frac{\partial \mathrm{Q}}{\partial t} \quad \text{and} \quad \frac{\partial^2 \mathrm{Q}}{\partial t\, \partial \beta}.$$

5.3. LEMMA. — *There exists a bounded operator-valued function* $Z(t, \beta)$, *defined and* 2π-*periodic in* t *for* $|\beta| \leqq 1$, *and satisfying the following:*

(*a*) $Z(t, \beta)$ *is strongly* C^{r+1} *in* t, *and analytic in* β, *and* $|Z(t, \beta)| \leqq 1$.

(*b*) $Z(t, \beta)$ *maps* $Q(t, \beta)\mathscr{H}$ *isometricly onto* $Q(0, 0)\mathscr{H}$ *and anihilates the complement of* $Q(t, \beta)\mathscr{H}$.

(*c*) $\partial Z(t, \beta)/\partial t$ *is uniformly bounded.*

Proof. — Given the projection valued function $Q(\beta, t)$ defined for $|\beta| \leqq 1$ and $t \in \mathbb{R}$, we proceed as in Kato's proof of the adiabatic theorem [5], p. 99 (*see* also [14]), to define an operator $Z_1(\beta, t)$ as the solution of the linear intial value problem:

$$\begin{gathered}\frac{\partial Z_1}{\partial \beta} = \left[\frac{\partial Q}{\partial \beta}, Q\right] Z_1 \\ Z_1(0, t) = I.\end{gathered} \tag{5.7}$$

Since Z_1 is the sum of a uniformly convergent Volterra series, Z_1 will be analytic in β and C^1 and 2π-periodic in t.

We will have, as in [5],

$$Q(\beta, t) Z_1(\beta, t) = Z_1(\beta, t) Q(0, t) = Z_1(\beta, t) Q(0, 0).$$

For part (*c*), we have the equation

$$\frac{\partial \dot{Z}_1}{\partial \beta} = \left[\frac{\partial \dot{Q}}{\partial \beta}, Q\right] Z_1 + \left[\frac{\partial Q}{\partial \beta}, \dot{Q}\right] Z_1 + \left[\frac{\partial Q}{\partial \beta}, Q\right] \dot{Z}_1. \tag{5.8}$$

By using Gronwall's inequality, we can obtain a bound on $\dot{Z}_1$ depending only on bounds on the coefficients. In particular, we can get bounds *independent of a parameter* ω in $H(\omega)$. ■

For $n, k \geqq N$, define

$$a_{n,k}(t, \beta) = \langle \varphi_k(t, \beta), \varphi_n(t, \beta) \rangle$$

where the dot denotes differentiation with respect to t.

5.4. LEMMA. — *For* $n, k \geqq N$ *and* $|\beta| \leqq 1$, *we have*

$$|a_{n,k}(t, \beta)| \leqq 8 \beta \dot{M} |\lambda_n - \lambda_k|^{-1} \tag{5.9}$$

for $n \neq k$

$$|a_{nn}(t, \beta)| \leqq 8 \beta \dot{M} r_n^{-1} \tag{5.10}$$

and

$$|Q(t, \beta) \dot{\varphi}_n(t, \beta)| \leqq \beta \dot{M} C(N) \lambda_n^{-1}. \tag{5.11}$$

Remark. — If $H(t, \beta)$ commutes with an *antiunitary* C, such as complex conjugation, one can choose $\varphi_n(t)$ with $C\varphi_n(t) = \varphi_n(t)$, which implies that

$\langle \varphi_n(t), \varphi_n(t) \rangle$ is real. In this case,

$$a_{nn}(t) = 1/2 \frac{d}{dt} |\varphi_n(t)|^2 = 0.$$

Proof. — For simplicity, we surpress β throughout most of the proof. Differentiate

$$P_n(t) = \langle \cdot, \varphi_n(t) \rangle \varphi_n(t)$$

to obtain

$$P_n(t) = \langle \cdot, \dot{\varphi}_n(t) \rangle \varphi_n(t) + \langle \cdot, \varphi_n(t) \rangle \dot{\varphi}_n(t)$$

and hence

$$\dot{P}_n(t) P_k(t) = a_{n,k}(t) \langle \cdot, \varphi_k(t) \rangle \varphi_n(t).$$

Thus,

$$|a_{n,k}(t)| = |\dot{P}_n(t) P_k(t)|. \tag{5.12}$$

Differentiate

$$P_n(t) = (2\pi i)^{-1} \int_{\Gamma_n} R(z, t)\, dz$$

to obtain

$$\dot{P}_n(t) = -(2\pi i)^{-1} \int_{\Gamma_n} R(z, t) \dot{V}(t) R(z, t)\, dz.$$

Hence

$$\begin{aligned} \dot{P}_n(t) P_k(t) &= -(2\pi i)^{-2} \int_{\Gamma_k} \int_{\Gamma_n} R(z, t) \dot{V}(t) R(z, t) R(z', t)\, dz\, dz' \\ &= -(2\pi i)^{-2} \int_{\Gamma_n} dz\, R(z, t) \\ &\quad \times \int_{\Gamma_n} \dot{V}(t) (z - z')^{-1} [R(z, t) - R(z', t)]\, dz'. \end{aligned} \tag{5.13}$$

Now (for $k \neq n$), $z \in \Gamma_n$ is fixed, so $(z - z')^{-1} R(z, t)$ is analytic inside Γ_k as a function of z'. So the first term drops out and we obtain

$$\dot{P}_n(t) P_k(t) = (2\pi i)^{-2} \int_{\Gamma_n} \int_{\Gamma_k} (z' - z)^{-1} R(z, t) \dot{V}(t) R(z', t)\, dz'\, dz. \tag{5.14}$$

Estimating gives

$$|\dot{P}_n(t) P_k(t)| \leqq 8 \beta \dot{M} |\lambda_n - \lambda_k|^{-1}$$

which is (5.9).

For (5.10), estimate (5.13) directly with $k=n$. For (5.11), write

$$Q(t)\dot{\varphi}_n(t)=\sum_{k=0}^{N} P_k(t)\dot{P}_n(t)\varphi_n=-\sum_{k=0}^{N}\dot{P}_k(t)P_n(t)\varphi_n \tag{5.15}$$

where the second step results from the identity

$$0=\frac{d}{dt}[P_n, P_k]=\dot{P}_n P_k+P_n\dot{P}_k \qquad (n\neq k).$$

But (5.15) is equal to the right side of (5.14) *with* Γ_n *replaced by the contour* Γ_0, encircling the first $(N+1)$ eigenvalues. Since Γ_0 can be chosen with

$$\operatorname{dist}(\Gamma_0, \sigma(H(t, \beta))\geqq r_N$$

we obtain

$$|Q(t)\dot{\varphi}_n(t)|\leqq\left(\frac{8\beta\dot{M}L}{2\pi r_N}\right)(\lambda_n-\lambda_N)^{-1} \tag{5.16}$$

where L is the length of Γ_N.

But

$$\frac{\lambda_n-\lambda_N}{\lambda_n}=1-\frac{\lambda_N}{\lambda_n}\geqq 1-\frac{\lambda_N}{\lambda_{N+1}}=\frac{\lambda_{N+1}-\lambda_N}{\lambda_{N+1}}\geqq\frac{1}{N+1}$$

by (5.1). Thus (5.16) does not exceed

$$8\beta\dot{M}L(N+1)(2\pi r_N)^{-1}\lambda_n^{-1}=\beta\dot{M}C(N)\lambda_n^{-1}.$$

If we now define

$$U_1(t, \beta)=\sum_{k=N+1}^{\infty}\langle\cdot, \varphi_k(t, \beta)\rangle\varphi_n \tag{5.17}$$

then the operator

$$U(t, \beta)=Z(t, \beta)+U_1(t, \beta)$$

is unitary, and maps $Q(t, \beta)\mathscr{H}$ onto $Q(0, 0)\mathscr{H}$ and $\varphi_n(t, \beta)$ to φ_n for $n>N$. Let $U(\beta)$ be the operator-valued multiplication operator on $\mathscr{K}$ defined by

$$(U(\beta)u)(t)=U(t, \beta)u(t)$$

and compute that

$$U(\beta)K(\beta)U^*(\beta)=i\frac{d}{dt}+\sum_{k=N+1}^{\infty}\lambda_k(t, \beta)P_k+\Delta(t, \beta) \tag{5.18}$$

where

$$\Delta=ZHQZ^*+i\{U_1\dot{U}_1^*+Z\dot{Z}^*+U_1\dot{Z}^*+Z\dot{U}_1^*\}. \tag{5.19}$$

We wish to choose $a_n > 0$ so that if

$$A = \sum_{n=0}^{\infty} a_n P_n$$

then $A^{-1} \Delta(t, \beta) A^{-1}$ is uniformly bounded. Observe first that

$$A^{-1} Q(p) = \sum_{k=0}^{N} a_k^{-1} P_k$$

is bounded. Second, note that

$$Z(t) = Q(0) Z(t) = Z(t) Q(t) = Q(0) Z(t) Q(t) \tag{5.20}$$

and hence that

$$\dot{Z}(t) = Q(0) \dot{Z}(t). \tag{5.21}$$

Third, differentiate

$$U_1(t) Z^*(t) = 0 \tag{5.22}$$

to obtain

$$\dot{Z}(t) U_1^*(t) = -Z(t) \dot{U}_1^*(t). \tag{5.23}$$

Consider now the five terms of $A^{-1} \Delta(t, \beta) A^{-1}$. The two terms

$$\begin{aligned} &A^{-1} Z(t) H(t) Q(t) Z^*(t) A^{-1} \\ &\qquad = (A^{-1} Q(0))(Z(t) H(t) Q(t) Z^*(t))(A^{-1} Q(0))^* \end{aligned} \tag{5.24}$$

and

$$A^{-1} Z(t) \dot{Z}(t)^* A^{-1} = (A^{-1} Q(0))(Z(t) \dot{Z}^*(t))(A^{-1} Q(0))^* \tag{5.25}$$

are uniformly bounded [Lemma 5.2 (c) and $H(t) Q(t) \leqq \lambda_N(t)$]. The two terms

$$A^{-1} Z(t) \dot{U}_1^*(t) A^{-1} = (A^{-1} Q(0)) Z(t) (Q(t) \dot{U}_1^*(t) A^{-1}) \tag{5.26}$$

and

$$\begin{aligned} A^{-1} U_1(t) \dot{Z}^*(t) A^{-1} &= (\dot{Z}(t) U_1^*(t) A^{-1})^* (A^{-1} Q(0))^* \\ &= -(Z(t) \dot{U}_1^*(t) A^{-1} Q(0))^* \\ &= (Q(t) \dot{U}_1^*(t) A^{-1})^* Z^*(t) (A^{-1} Q(0))^* \end{aligned} \tag{5.27}$$

will both be uniformly bounded if

$$Q(t) \dot{U}_1^*(t) A^{-1} \tag{5.28}$$

is bounded. Thus we need only estimate this operator and

$$A^{-1} U_1(t) \dot{U}_1^*(t) A^{-1}. \tag{5.29}$$

5.5. LEMMA. — *Let* $a_n = n^{-\gamma}$ *where* $0 < 2\gamma < \alpha$. *Then* (5.28) *and* (5.29) *are norm bounded uniformly in* t *and* β, $|\beta| \leqq 1$.

Proof. — We compute that

$$Q(t)\dot{U}_1^*(t)A^{-1}=\sum_{j=N+1}^{\infty} a_j^{-1}\langle\,.,\,\varphi_j\rangle\,Q(t)\,\varphi_j(t).$$

Hence by (5.11), its norm does not exceed,

$$\beta\dot{M}C(N)\sum_{j=N+1}^{\infty} j^{\gamma}\lambda_j^{-1}$$

which is finite by (5.1).

Similarly, (5.29) is equal to

$$\sum_{j,\,l=N+1}^{\infty} a_j^{-1}\,\bar{a}_{jl}(t)\,a_l^{-1}\langle\,.,\,\varphi_j\rangle\,\varphi_l.$$

By (5.2) and (5.9), it therefore suffices to show boundedness of the infinite matrix (b_{jl}) with

$$\begin{aligned} b_{jl}&=\frac{j^{\gamma}\,l^{\gamma}}{j^{\alpha+1}-l^{\alpha+1}}, \qquad j\neq l\\ b_{jj}&=j^{2\gamma-\alpha}. \end{aligned} \tag{5.30}$$

Since b_{kl} is symmetric, the Schur-Holmgren condition for boundedness is simply

$$\sup_n \sum_{k=0}^{\infty} |b_{nk}|<\infty.$$

The diagonal is bounded, and so causes no problem; thus we require finiteness of

$$\sup_n n^{\gamma}\sum_{k=n+1}^{\infty}\frac{k^{\gamma}}{(k^{\alpha+1}-n^{\alpha+1})}+\sup_n n^{\gamma}\sum_{k=1}^{n-1}\frac{k^{\gamma}}{(n^{\alpha+1}-k^{\alpha+1})}. \tag{5.31}$$

For the first term, we have

$$\begin{aligned} n^{\gamma}\sum_{k=n+1}^{\infty}\frac{k^{\gamma}}{(n^{\alpha+1}-k^{\alpha+1})}&\leqq n^{\gamma}\int_{n+1}^{\infty}\frac{x^{\gamma}}{x^{\alpha+1}-n^{\alpha+1}}dx\\ &=n^{2\gamma-\alpha}\int_{1+1/n}^{\infty}\frac{s^{\gamma}}{s^{\alpha+1}-1}ds\leqq c(\alpha,\,\gamma)\,n^{2\gamma-\alpha}\log n \end{aligned}$$

which goes to zero if $0<2\gamma<\alpha$. Similarly,

$$\begin{aligned} n^{\gamma}\sum_{k=1}^{n-1}\frac{k^{\gamma}}{n^{\alpha+1}-k^{\alpha+1}}&=n^{2\gamma-\alpha}\sum_{k=1}^{n-1}\frac{1}{1\,(k/n)^{\alpha+1}}\left(\frac{k}{n}\right)^{\gamma}\frac{1}{n}\\ &\leqq n^{2\gamma-\alpha}\left\{\int_0^{1-1/n}\frac{s^{\gamma}}{s^{\alpha+1}-1}ds+c(\alpha)\right\}\leqq c(\sigma,\,\gamma)\,n^{2\gamma-\alpha}\log n \end{aligned}$$

[we have estimated the $k=n-1$ term by $c(\alpha)=2^{\gamma}(\alpha+1)^{-1}$]. ■

We have now shown that $K(\beta)$ is uniformly equivalent to an operator of the form (4.3), *with* H *replaced by the diagonal operator*

$$\sum_{n=0}^{\infty} \lambda_n(t, \beta) P_n.$$

From perturbation theory [5], p. 88, we have

$$\lambda_n(t, \beta)=\lambda_n+\beta(V(t)\varphi_n, \varphi_n)+E_n(t, \beta)$$

where the error term satisfies

$$|E_n(t, \beta)| \leqq 2\beta^2 M^2 r_n^{-1}.$$

Since $n^{2\gamma} r_n^{-1}$ is bounded, the term

$$\sum_{n=0}^{\infty} E_n(t, \beta) P_n$$

can be absorbed into the $AW(t, \beta)A$ term in (4.3). To eliminate the remaining term, note that by (4.1),

$$(V(t)\varphi_n, \varphi_n)=\dot{g}_n(t)$$

where $g_n(t)$ is 2π-periodic. Let $G(t)$ be the unitary transformation

$$G(t)=\sum_{n=0}^{\infty} e^{i\beta g_n(t)} P_n$$

and $\mathbf{G}u(t)\equiv G(t)u(t)$. If we now transform by the "gauge transformation" $\mathbf{G}$, the term

$$\sum_{n=0}^{\infty} \beta\langle V(t)\varphi_n, \varphi_n\rangle P_n$$

disappears, while the form of $AW(t, \beta)A$ is preserved, since $G(t)$ commutes with A. This completes the proof. ■

6. CONCLUDING REMARKS

(1) Our theorem is unsatisfactory in several ways. In the first place, one would like to reduce the value of α. Hamiltonians like the one-dimensional rotor considered by Bellisard ([1], [2]) corresponds to $\Delta\lambda_n \simeq n$, or $\alpha=1$. (It also has multiplicity two.) The harmonic oscillator ([3], [6]) has $\alpha=0$, and is doubtless more delicate.

(2) One would also like to be able to randomize $H(\omega)$ within a natural class. For example, if H is a Schroedinger operator, we would like to have

H(ω) a Schroedinger operator as well. The chief problem here is the *difficulty of proving that operator multiplications like* **K** *are absolutely continuous.* Theorems of this type would be very useful, both here and in localization theory [10].

(3) The theorem here seems essentially one-dimensional in its assumption of increasing gap $\Delta\lambda_n$. For example, if H represents the particle in a box in d dimensions, then for $d \geqq 3$, the density of eigenvalues becomes *larger* as energy increase, rather than smaller. *Does this result in a different spectral type for* K? An answer to this question would be very interesting.

REFERENCES

[1] J. BELLISSARD, *Stability and Instability in Quantum Mechanics* in *Trends and Developments in the Eighties*, ALBEVARIO and BLANCHARD Eds., World Scientific, Singapore, 1985.

[2] J. BELLISSARD, *Stability and Chaotic Behavior of Quantum Rotors* in *Stochastic Processes in Classical and Quantum Systems*, S. ALBEVARIO, G. CASATI and D. MERLINI Eds., *Lec. Notes in Phys.*, Vol. **262**, 1986, pp. 24-38, Springer-Verlag, New York.

[3] M. COMBESCURE, The Quantum Stability Problem for Time-Periodic Perturbations of the Harmonic Oscillator, *Ann. Inst. H. Poincaré*, vol. **47**, 1987, pp. 63-84; 451-4.

[4] T. KATO, Wave Operators and Similarity for Sone Non-Self-Adjoint Operators, *Math. Ann.*, Vol. **162**, 1966, pp. 258-279.

[5] T. KATO, *Perturbation Theory for Linear Operators*, 1st edition, Springer-Verlag, New York, 1966.

[6] G. HAGADORN, M. LOSS and J. SLAWNY, Non-Stochasticity of Time-Dependent Quadratic Hamiltonians and the Spectra of Transformations, *J. Phys. A.*, Vol. **19**, 1986, pp. 521-531.

[7] J. S. HOWLAND, Scattering Theory for Hamiltonians Periodic in Time, *Indiana J. Math.*, Vol. **28**, 1979, pp. 471-494.

[8] J. S. HOWLAND, Perturbation Theory of Dense Point Spectrum. *J. Func. Anal.*, Vol. **74**, 1987, pp. 52-80.

[9] J. S. HOWLAND, *Random Perturbation Theory and Quantum Chaos* in *Differential Equations and Mathematical Physics*, pp. 197-204, I. W. KNOWLES and Y. SAITO Eds., *Lec. Notes in Math.*, Vol. **1285**, Springer-Verlag, New York, 1987.

[10] J. S. HOWLAND, *A Localization Theorem for One-Dimensional Schroedinger Operators* (to appear).

[11] M. SAMUELS, R. FLECKINGER, L. TOUZILLIER and J. BELLISSARD, The Rise of Chaotic Behavior in Quantum Systems and Spectral Transition, *Europhys. Lett.*, Vol. **1**, 1986, pp. 203-208.

[12] B. SIMON and T. WOLFF, Singular Continuous Spectrum Under Rank Are Perturbations and Localization for Random Hamiltonians, *Comm. Pure Appl. Math.*, Vol. **39**, 1986, pp. 75-90.

[13] K. YAJIMA, Scattering Theory for Schroedinger Equation with Potential Periodic in Time, *J. Math Soc. Japan*, vol. **29**, 1977, pp. 729-743.

[14] J. AVRON, R. SEILER and L. G. YAFFE, Adiabatic Theorems and Applications to the Quantum Hall Effect, *Comm. Math. Phys.*, Vol. **110**, 1987, pp. 33-49.

(*Manuscrit reçu le* 16 *août* 1988)
(*version révisé* 1^{er} *février* 1989.)