JAMES S. HOWLAND
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Floquet Operators with Singular Spectrum. I

by

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ABSTRACT. — A positive, discrete Hamiltonian $H$ is perturbed by a time-periodic perturbation $V(t)$. If the gap between successive eigenvalues of $H$ grows sufficiently rapidly, then generically (in a probabilistic sense) $H + \beta V(t)$ has dense pure point Floquet spectrum.

RÉSUMÉ. — Nous perturbons un opérateur Hamiltonien discret par un opérateur $V(t)$ périodique en temps. Si la distance entre les valeurs successives de $H$ croît vite, alors génériquement (en un sens probabiliste) $H + \beta V(t)$ a un spectre de Floquet purement ponctuel dense.

1. INTRODUCTION

Let $H$ be a discrete Hamiltonian operator on $\mathcal{H}$ with eigenvalues $\lambda_k$, and $V(t)$ a periodic, time-dependent perturbation of $H$:

$$V(t + a) = V(t).$$

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The natural object to consider for a periodic Hamiltonian

\[ H(t, \beta) = H + \beta V(t) \]

is the Floquet Hamiltonian:

\[ K(\beta) = i \frac{d}{dt} + H(t, \beta) \]

with periodic boundary condition \( u(a) = u(0) \), acting on the space \( \mathcal{H} = L^2[0, a] \otimes \mathcal{H} \). (See, e.g., [7], [13] and a vast physics literature.)

If \( \beta = 0 \), and the period is normalized to \( a = 2\pi \), then the operator \( K(0) \) has pure point spectrum with eigenvalues

\[ \Lambda(n, k) = n + \lambda k \]

\((n = 0, \pm 1, \ldots)\). Except in rare cases, this spectrum will be dense in the line.

The question that we wish to consider is this: When does the perturbed operator \( K(\beta) \) also have dense pure point spectrum? The question has generated considerable recent interest ([1], [2], [3], [6], [7], [11]), and we refer particularly the reader to article [1] of Bellisard. Most of this work has resulted in operator-theoretic versions of the Kolmogorov-Arnold-Moser theorem, which asserts pure point spectrum for generic values of certain parameters in \( H(t) \) and for small coupling \( \beta \). The essential idea, though, is that \( K \) will be pure point if there is no resonance.

The present paper takes a different approach to the problem, based on the author's generalization [8] of the Simon-Wolff-Kotani [12] method from the theory of localization. We shall show under certain conditions that \( K(\beta) \) is pure point for "almost every \( H \)". Thus, the method yields generic results on a probabilistic, rather than in a metric sense. The essential technical condition, which we believe can be weakened substantially, is that the gap between eigenvalues of \( H \)

\[ \Delta \lambda_n = \lambda_{n+1} - \lambda_n \]

grows like \( n^{2+\varepsilon}, \varepsilon > 0 \).

After recalling some results from [8], we consider in paragraph 3 as an easy consequence, certain compact (actually, trace class) perturbations of \( H \), generalizing a result of [7]. We then state the Main Theorem, and show in paragraph 5 how an adiabatic analysis of \( H(t, \beta) \) reduces the problem to one like that of paragraph 3. We close with some remarks and conjectures.
2. NOTATION AND PREVIOUS RESULTS

Let $H$ be a positive definite, discrete selfadjoint operator of simple multiplicity on a separable Hilbert space $\mathcal{H}$. Let $\varphi_n$ be a complete orthonormal set of eigenvectors of $H$:

$$H \varphi_n = \lambda_n \varphi_n$$

with $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$. Let $V(t)$ be a uniformly bounded measurable family of bounded operators, which is $2\pi$-periodic in $t$:

$$V(t + 2\pi) = V(t)$$

and define the Floquet operator

$$K(\beta) = i \frac{d}{dt} + H + \beta V(t)$$

on $\mathcal{H} L^2[0, 2\pi] \otimes \mathcal{H}$, with periodic boundary condition:

$$u(0) = u(2\pi).$$

We shall sometimes write $K_0 = K(0)$ and $K = K(1)$.

We shall also consider families $H(\omega)$ of operators satisfying these conditions, which are measurable on a probability space $(P, \Omega)$. We shall refer to these briefly as "random Hamiltonians", and will write

$$H(\omega) \varphi_n(\omega) = \lambda_n(\omega) \varphi_n(\omega).$$

If

$$K(\omega) = i \frac{d}{dt} + H(\omega) + V(t)$$

is the corresponding Floquet operator, we define $K$ to be the multiplication operator

$$K u(\omega) = K(\omega) u(\omega)$$

on $L^2(P, \Omega) \otimes \mathcal{H}$. If the coupling constant $\beta$ is included, we obtain $K(\beta)$, so that $K = K(1)$.

We shall next summarize some results of [8]. Let $H$ be pure point and $A$ bounded. We say [8], p. 64, that $A$ is strongly $H$-finite on an open interval $J$ iff

$$\sum \{|A \varphi_n|: \lambda_n \in J\} < \infty.$$
(i) $N$ has Lebesgue measure zero, and  
(ii) $N$ supports the continuous spectrum of $H$ in $J$.

Finally, we have the following version of "Kotani's trick".

2.2. Proposition [8], p. 59. — Let $K(\omega)$ be a random self-adjoint operator such that:

(i) there exists a set $N$ of Lebesgue measure zero, independent of $\omega$, which supports the continuous part $K(\omega)$ a.s.,

(ii) $K$ has absolutely continuous spectral measure. Then $K(\omega)$ is pure point a.s.

3. COMPACT PERTURBATIONS OF FLOQUET OPERATORS

Let $H$ be positive, discrete and of simple multiplicity, and $A$ strongly $H$-finite. Let $K$ be the Floquet operator

$$K = \frac{d}{dt} + H.$$ 

3.1. Proposition. — $1 \otimes A$ is strongly $K$-finite on any finite interval $J$.

Proof. — The eigenvalue of $K$ are $\Lambda(n, k) = n + \lambda_k$ with eigenvectors $\Phi(n, k)(t) = e^{\lambda_k t} \varphi_k$. If we assume that $J$ has length less than 1, then $\Lambda(n, k)$ is in $J$ for at most one value of $n$, which we call $n_k$. Thus, for each $k$,

$$\sum \{|(1 \otimes A) \Phi(n, k)| : \Lambda(n, k) \in J\} = \sum_k |(1 \otimes A) \Phi(n_k, k)| \leq \sum_k |A \varphi_k| < \infty. \quad \Box$$

Let $W(t)$ be a uniformly bounded, $2\pi$-periodic measurable family of self-adjoint operators, and

$$V(t) = A^* W(t) A.$$ 

Note that the sum of two terms of this form can be written in the same form:

$$[A_1^* W_1(t) A_1 + A_2^* W_2(t) A_2] \varphi = A^* W(t) A \varphi$$

where $A : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ (cf. [7]).

3.2. Theorem. — If $W(t) > 0$, then the Floquet operator for

$$H(t, \beta) = H + \beta V(t)$$

is pure point for a.e. $\beta$. 

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Proof. — It suffices to consider $|\beta| < 1$. From Proposition 2.1, $K(\beta)$ has its continuous part is concentrated on a set $N(H, A)$ of measure zero, independent of $\beta$. To be able to apply Proposition 2.2, we write $\beta = \tanh x$, where $-\infty < x < \infty$. The operator $K$ of multiplication by $K(x)$ then has a positive commutator with the bounded operator

$$B = i \arctan(p/2)$$

where $p = -i \frac{d}{dx}$, and is therefore absolutely continuous by the Putnam-Kato Theorem (cf. [8], p. 60).

We shall next show that for "almost every $H"$, $K(\beta)$ is pure point for every $\beta$. To be precise, let $X_j(\omega)$ be i.i.d, and uniform on $[-1, 1]$, and $\varepsilon_j > 0$ with

$$\sum_{j=0}^{\infty} \varepsilon_j < \infty.$$  

3.3. Theorem. — The Floquet operator for

$$H(t, \omega) = H + \sum_{j=0}^{\infty} \varepsilon_j^2 X_j(\omega) \langle \varphi_j, \varphi_j \rangle + V(t)$$

is pure point a.s.

Thus, if the eigenvalues of $H$ are all wiggled independently by a tiny amount, $K$ will have pure point spectrum.

Proof. — The second term of $H(t, \omega)$ can be written as $E X(\omega) E$ where $E = \sum_j \varepsilon_j \langle \varphi_j, \varphi_j \rangle \varphi_j$ and $X(\omega) = \sum_j X_j(\omega) \langle \cdot, \varphi_j \rangle \varphi_j$. Since $E$ is strongly $H$-finite, we find from (3.1) and Proposition 2.1, that $K(\omega)$ has continuous spectrum concentrated on a null set $N = N(H, A, E)$ independent of $\omega$. Absolute continuity of $K$ is obtained as in [8], pp. 67-69.

Remarks. — Several improvements in the results are easily made. The distribution of $X_j$ need not be uniform [8], nor is simple multiplicity necessary. The randomness in $H(\omega)$ could be more simply taken as

$$H + \alpha A_1^2$$

with $A_1$ strongly $H$-finite. The reader may formulate such results for himself.

4. MAIN THEOREM

As above, let $H$ be positive, discrete and of simple multiplicity, and $V(t)$ be bounded and $2\pi$-periodic satisfying

$$\int_0^{2\pi} V(t) dt = 0. \quad (4.1)$$

Let $X_n(\omega)$ be i.i.d. and uniform on $[-1, 1]$, $\epsilon_n > 0$ with

$$\sum_{n=0}^{\infty} \epsilon_n < \infty$$

and define

$$H(t, \omega) = H + \sum_{n=0}^{\infty} \epsilon_n X_n(\omega) \langle \cdot, \varphi_n \rangle \varphi_n + V(t). \quad (4.2)$$

Let $\Delta \lambda_n$ be the gap between eigenvalues:

$$\Delta \lambda_n = \lambda_n - \lambda_{n-1}.$$ 

4.1. THEOREM. - Let $V(t)$ be strongly $C^1$, and satisfy for some $c > 0$

$$\Delta \lambda_n \geq cn^\alpha \quad (4.3)$$

and $\alpha > 2$. Then $K(\omega)$ is pure point a.s.

Proof. — Absolute continuity of $K$ follows before. By the adiabatic analysis of $H(t, \omega)$ carried out in the next section, the operator $K(\omega)$ is unitarily equivalent to an operator

$$i \frac{d}{dt} + H + AW(t, \omega) A.$$

The operator $A$ is strongly $H$-finite since $\gamma > 1$, and $W(t, \omega)$ bounded in norm uniformly in $t$ and $\omega$. Existence of a null set $N(H, A)$, independent of $\omega$ and supporting the continuous part of $K(\omega)$, now follows from Proposition 2.1. \Box

5. ADIABATIC ANALYSIS OF $H(t, \beta)$

Let $A$ be a diagonal operator

$$A = \sum_{n=0}^{\infty} a_n \langle \cdot, \varphi_n \rangle \varphi_n$$

with $a_n > 0$. We shall prove the following theorem:

5.1. THEOREM. — Let $H$ be positive, discrete and of simple multiplicity, and $V(t)$ strongly $C^{r+1}$, satisfying (4.1). Assume that for some $c > 0$ and
\[ \alpha > 0, \]
\[ \Delta \lambda_n \geq cn^\alpha, \quad \alpha > 0 \]  
\[ (5.1) \]

and let \( a_n = n^{-\gamma} \) for \( n \geq 1 \) with \( 0 < 2\gamma < \alpha \).

Then \( K(t) \) is unitarily equivalent to

\[ i \frac{d}{dt} + H + A W(t, \beta) A \]

where \( W(t, \beta) \) is strongly \( C^r \) in \( t \) and uniformly bounded.

**Remark.** — Actually, \( \lambda_n = \lambda_n(\omega) \) is random, but we will suppress \( \omega \) and assume that (5.1) holds uniformly in \( \omega \). In fact, we shall assume that for some \( \eta \), \( |\lambda_n(\omega) - \lambda_n| \leq \eta \) for all \( \omega \). Then if (5.1) holds for \( \lambda_n \), it will hold uniformly for \( \lambda_n(\omega) \), with a smaller \( c \).

Let \( |V(t)| \leq M, |\dot{V}(t)| \leq M \). Let \( \lambda_n(t, \beta) \) be the \( n \)-th eigenvalue of

\[ H(t, \beta) = H + \beta V(t) \]

and

\[ R(z; t, \beta) = (H(t, \beta) - z)^{-1} \]

its resolvent.

The reason for including \( \beta \) will be apparent in the proof of Lemma 5.3 below.

Note that if \( n > k \), then

\[ \lambda_n - \lambda_k = \Delta \lambda_n + \ldots + \Delta \lambda_{k+1} \geq c(n^\alpha + \ldots + (k+1)^\alpha) \geq c \int_k^n x^\alpha \, dx \]

so that for \( n > k \),

\[ \lambda_n - \lambda_k \geq c(\alpha + 1)^{-1} (n^{\alpha+1} - k^{\alpha+1}). \]  
\[ (5.2) \]

In particular, (since \( \lambda_0 > 0 \))

\[ \lambda_n \geq c(\alpha + 1)^{-1} n^{\alpha+1}. \]  
\[ (5.3) \]

Let

\[ r_n = \frac{c}{2} n^\alpha \leq \frac{1}{2} \min \{ \lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1} \} \]

and let \( \Gamma_n (= \Gamma_n(\omega)) \) be the positively oriented contour \( |z - \lambda_n| = r_n \).

We now choose and fix \( N \) such that \( r_n \leq 2M \) for \( n \geq N \).

5.2. **Lemma.** — (a) For \( n \geq N \) and \( |\beta| \leq 1 \),

\[ |\lambda_n(\beta, t) - \lambda_n| \leq M \leq r_n/2 \]

and hence

\[ \text{dist}(\lambda_n(\beta, t), \Gamma_n) \geq r_n - M \geq r_n/2. \]
Moreover, $\lambda_n(\beta, t)$ is the only point of $\sigma(H(t, \beta))$ inside $\Gamma_n$.

Proof. – This follows by upper semicontinuity of the spectrum, since the norm of the perturbation does not exceed $|\beta| M$. \[\square\]

Note that this gives

$$|R(z, t, \beta)| \leq 2 r_n^{-1}$$

for $z \in \Gamma_n$, and

$$|\lambda_n(t, \beta) - \lambda_k(t, \beta)| \geq 1/2 |\lambda_n - \lambda_k|.$$

For $n \geq N$, let the spectral projection for $\lambda_n(t, \beta)$ is

$$P_n(t, \beta) = \frac{1}{2\pi i} \int_{\Gamma_n} R(z; t, \beta) dz = \langle ., \varphi_n(t, \beta) \rangle \varphi_n(t, \beta).$$

The phase of $\varphi_n(t, \beta)$ is fixed by the choice

$$\varphi_n(t, \beta) = |P_n(t, \beta) \varphi_n|^{-1} P_n(t, \beta) \varphi_n$$

which makes $\varphi_n(t, \beta)$ smooth and $2\pi$-periodic in $t$. Note that the norm of $P_n(t, \beta) \varphi_n$ is never zero; for we have

$$P_n(t, \beta) \varphi_n - \varphi_n = (2\pi i)^{-1} \int_{\Gamma_n} R(z; t, \beta) VR(z, t, 0) dz$$

which yields, by Lemma 3.2, the estimate

$$|P_n(t, \beta) \varphi_n - \varphi_n| \leq M r_n^{-1} \leq 1/2.$$

We now need to separate off the first $N$ eigenvalues in a group. Let

$$Q(t, \beta) = I - \sum_{j=N+1}^{\infty} P_j(t, \beta)$$

be the spectral projection onto the first $N$ eigenvectors of $H(t, \beta)$.

We can write

$$Q(t, \beta) = (2\pi i)^{-1} \int_{\Gamma_0} R(z, t, \beta) dz$$

where $\Gamma_0$ is a suitable contour encircling $\lambda_j(t, \beta)$ for $0 \leq j \leq N$. From this representation, we obtain immediately the uniform boundedness and continuity of such operators as

$$\frac{\partial Q}{\partial \beta}, \frac{\partial Q}{\partial t} \quad \text{and} \quad \frac{\partial^2 Q}{\partial t \partial \beta}.$$
5.3. LEMMA. — There exists a bounded operator-valued function $Z(t, \beta)$, defined and $2\pi$-periodic in $t$ for $|\beta| \leq 1$, and satisfying the following:

(a) $Z(t, \beta)$ is strongly $C^{r+1}$ in $t$, and analytic in $\beta$, and $|Z(t, \beta)| \leq 1$.

(b) $Z(t, \beta)$ maps $Q(t, \beta)\mathcal{H}$ isometrically onto $Q(0, 0)\mathcal{H}$ and anihilates the complement of $Q(t, \beta)\mathcal{H}$.

(c) $\partial Z(t, \beta)/\partial t$ is uniformly bounded.

Proof. — Given the projection valued function $Q(\beta, t)$ defined for $|\beta| \leq 1$ and $t \in \mathbb{R}$, we proceed as in Kato’s proof of the adiabatic theorem [5], p. 99 (see also [14]), to define an operator $Z_1(\beta, t)$ as the solution of the linear intial value problem:

$$\frac{\partial Z_1}{\partial \beta} = \left[ \frac{\partial Q}{\partial \beta}, Q \right] Z_1 \quad Z_1(0, t) = 1. \quad (5.7)$$

Since $Z_1$ is the sum of a uniformly convergent Volterra series, $Z_1$ will be analytic in $\beta$ and $C^1$ and $2\pi$-periodic in $t$.

We will have, as in [5],

$$Q(\beta, t) Z_1(\beta, t) = Z_1(\beta, t) Q(0, t) = Z_1(\beta, t) Q(0, 0).$$

For part (c), we have the equation

$$\frac{\partial Z_1}{\partial \beta} = \left[ \frac{\partial Q}{\partial \beta}, Q \right] Z_1 + \left[ \frac{\partial Q}{\partial \beta}, \dot{Q} \right] Z_1 + \left[ \frac{\partial Q}{\partial \beta}, Q \right] \dot{Z}_1. \quad (5.8)$$

By using Gronwall’s inequality, we can obtain a bound on $\dot{Z}_1$ depending only on bounds on the coefficients. In particular, we can get bounds independent of a parameter $\omega$ in $H(\omega)$.

For $n, k \geq N$, define

$$a_{n,k}(t, \beta) = \langle \varphi_k(t, \beta), \varphi_n(t, \beta) \rangle$$

where the dot denotes differentiation with respect to $t$.

5.4. LEMMA. — For $n, k \geq N$ and $|\beta| \leq 1$, we have

$$|a_{n,k}(t, \beta)| \leq 8 \beta M |\lambda_n - \lambda_k|^{-1} \quad (5.9)$$

for $n \neq k$

$$|a_{mn}(t, \beta)| \leq 8 \beta M r_n^{-1} \quad (5.10)$$

and

$$|Q(t, \beta) \dot{\varphi}_n(t, \beta)| \leq \beta M C(N) \lambda_n^{-1}. \quad (5.11)$$

Remark. — If $H(t, \beta)$ commutes with an antiunitary $C$, such as complex conjugation, one can choose $\varphi_n(t)$ with $C \varphi_n(t) = \varphi_n(t)$, which implies that
\( \langle \varphi_n(t), \varphi_n(t) \rangle \) is real. In this case,

\[
a_{nn}(t) = 1/2 \frac{d}{dt} |\varphi_n(t)|^2 = 0.
\]

**Proof.** — For simplicity, we suppress \( \beta \) throughout most of the proof. Differentiate

\[
P_n(t) = \langle \varphi_n(t), \varphi_n(t) \rangle \varphi_n(t)
\]
to obtain

\[
P_n(t) = \langle \varphi_n(t), \varphi_n(t) \rangle \varphi_n(t) + \langle \varphi_n(t), \varphi_n(t) \rangle \dot{\varphi}_n(t)
\]
and hence

\[
\dot{P}_n(t) P_k(t) = a_{n,k}(t) \langle \varphi_k(t), \varphi_n(t) \rangle.
\]

Thus,

\[
|a_{n,k}(t)| = |\dot{P}_n(t) P_k(t)|. \tag{5.12}
\]

Differentiate

\[
P_n(t) = (2 \pi i)^{-1} \int_{\Gamma_n} R(z, t) \, dz
\]
to obtain

\[
\dot{P}_n(t) = -(2 \pi i)^{-1} \int_{\Gamma_n} R(z, t) \dot{V}(t) R(z, t) \, dz.
\]

Hence

\[
\dot{P}_n(t) P_k(t) = -(2 \pi i)^{-2} \int_{\Gamma_n} \int_{\Gamma_k} R(z, t) \dot{V}(t) R(z, t) R(z', t) \, dz \, dz'
\]

\[
= -(2 \pi i)^{-2} \int_{\Gamma_n} dz \, R(z, t)
\]

\[
\times \int_{\Gamma_k} \dot{V}(t)(z-z')^{-1} [R(z, t) - R(z', t)] \, dz'. \tag{5.13}
\]

Now (for \( k \neq n \)), \( z \in \Gamma_n \) is fixed, so \( (z-z')^{-1} R(z, t) \) is analytic inside \( \Gamma_k \) as a function of \( z' \). So the first term drops out and we obtain

\[
\dot{P}_n(t) P_k(t) = (2 \pi i)^{-2} \int_{\Gamma_n} \int_{\Gamma_k} (z'-z)^{-1} R(z, t) \dot{V}(t) R(z', t) \, dz' \, dz. \tag{5.14}
\]

Estimating gives

\[
|\dot{P}_n(t) P_k(t)| \leq 8 \beta \bar{M} |\lambda_n - \lambda_k|^{-1}
\]

which is (5.9).
For (5.10), estimate (5.13) directly with \( k = n \). For (5.11), write

\[
Q(t) \dot{\varphi}_n(t) = \sum_{k=0}^{N} P_k(t) \dot{P}_n(t) \varphi_n = -\sum_{k=0}^{N} \dot{P}_k(t) P_n(t) \varphi_n
\]  

(5.15)

where the second step results from the identity

\[
0 = \frac{d}{dt} [P_n, P_k] = \dot{P}_n P_k + P_n \dot{P}_k \quad (n \neq k).
\]

But (5.15) is equal to the right side of (5.14) with \( \Gamma_n \) replaced by the contour \( \Gamma_0 \), encircling the first \((N+1)\) eigenvalues. Since \( \Gamma_0 \) can be chosen with

\[
\text{dist}(\Gamma_0, \sigma(H(t, \beta))) \geq r_N
\]

we obtain

\[
|Q(t) \dot{\varphi}_n(t)| \leq \left( \frac{8 \beta ML}{2 \pi r_N} \right) (\lambda_n - \lambda_N)^{-1}
\]

(5.16)

where \( L \) is the length of \( \Gamma_N \).

But

\[
\frac{\lambda_n - \lambda_N}{\lambda_n} = 1 - \frac{\lambda_N}{\lambda_n} \geq 1 - \frac{\lambda_N}{\lambda_{N+1}} = \frac{\lambda_{N+1} - \lambda_N}{\lambda_{N+1}} \geq \frac{1}{N+1}
\]

by (5.1). Thus (5.16) does not exceed

\[
8 \beta ML (N+1) (2 \pi r_N)^{-1} \lambda_n^{-1} = \beta ML (N) \lambda_n^{-1}.
\]

If we now define

\[
U_1(t, \beta) = \sum_{k=N+1}^{\infty} \langle \cdot, \varphi_k(t, \beta) \rangle \varphi_n
\]

(5.17)

then the operator

\[
U(t, \beta) = Z(t, \beta) + U_1(t, \beta)
\]

is unitary, and maps \( Q(t, \beta) \mathcal{H} \) onto \( Q(0, 0) \mathcal{H} \) and \( \varphi_n(t, \beta) \) to \( \varphi_n \) for \( n > N \). Let \( U(\beta) \) be the operator-valued multiplication operator on \( \mathcal{H} \) defined by

\[
(U(\beta) u)(t) = U(t, \beta) u(t)
\]

and compute that

\[
U(\beta) K(\beta) U^*(\beta) = i \frac{d}{dt} + \sum_{k=N+1}^{\infty} \lambda_k(t, \beta) P_k + \Delta(t, \beta)
\]

(5.18)

where

\[
\Delta = ZHQZ^* + i \{ U_1 \dot{U}_1^* + Z \dot{Z}^* + U_1 \dot{Z}^* + Z \dot{U}_1^* \}.
\]

(5.19)
We wish to choose \( a_n > 0 \) so that if
\[
A = \sum_{n=0}^{\infty} a_n P_n
\]
then \( A^{-1} \Delta(t, \beta) A^{-1} \) is uniformly bounded. Observe first that
\[
A^{-1} Q(p) = \sum_{k=0}^{N} a_k^{-1} P_k
\]
is bounded. Second, note that
\[
Z(t) = Q(0) Z(t) = Z(t) Q(t) = Q(0) Z(t) Q(t)
\]
and hence that
\[
\dot{Z}(t) = Q(0) \dot{Z}(t).
\]
Third, differentiate
\[
U_1(t) Z^*(t) = 0
\]
to obtain
\[
\dot{Z}(t) U_1^*(t) = - Z(t) \dot{U}_1^*(t).
\]
Consider now the five terms of \( A^{-1} \Delta(t, \beta) A^{-1} \). The two terms
\[
A^{-1} Z(t) H(t) Q(t) Z^*(t) A^{-1}
\]
\[
= (A^{-1} Q(0)) (Z(t) H(t) Q(t) Z^*(t)) (A^{-1} Q(0))^*
\]
and
\[
A^{-1} Z(t) \dot{Z}(t) A^{-1} = (A^{-1} Q(0)) (Z(t) \dot{Z}(t)) (A^{-1} Q(0))^*
\]
are uniformly bounded [Lemma 5.2 (c) and \( H(t) Q(t) \leq \lambda_N(t) \)]. The two terms
\[
A^{-1} Z(t) \dot{U}_1^*(t) A^{-1} = (A^{-1} Q(0)) Z(t) (Q(t) \dot{U}_1^*(t) A^{-1})
\]
and
\[
A^{-1} U_1(t) \dot{Z}^*(t) A^{-1} = (\dot{Z}(t) U_1^*(t) A^{-1})* (A^{-1} Q(0))^*
\]
\[
= -(Z(t) \dot{U}_1^*(t) A^{-1} Q(0))^*
\]
\[
= (Q(t) \dot{U}_1^*(t) A^{-1})* Z^*(t) (A^{-1} Q(0))^*
\]
will both be uniformly bounded if
\[
Q(t) \dot{U}_1^*(t) A^{-1}
\]
is bounded. Thus we need only estimate this operator and
\[
A^{-1} U_1(t) \dot{U}_1^*(t) A^{-1}.
\]

5.5. **Lemma.** — Let \( a_n = n^{-\gamma} \) where \( 0 < 2 \gamma < \alpha \). Then (5.28) and (5.29) are norm bounded uniformly in \( t \) and \( \beta, |\beta| \leq 1 \).
Proof. — We compute that
\[ Q(t) \hat{U}_t^*(t) A^{-1} = \sum_{j=N+1}^{\infty} a_j^{-1} \langle ., \phi_j \rangle Q(t) \phi_j(t). \]
Hence by (5.11), its norm does not exceed,
\[ \beta \mathcal{M} C(N) \sum_{j=N+1}^{\infty} j^{r} \lambda_j^{-1} \]
which is finite by (5.1).
Similarly, (5.29) is equal to
\[ \sum_{j,l=N+1}^{\infty} a_j^{-1} \bar{a}_l(t) a_l^{-1} \langle ., \phi_j \rangle \phi_l. \]
By (5.2) and (5.9), it therefore suffices to show boundedness of the infinite matrix \((b_{jl})\) with
\[
b_{jl} = \begin{cases} j^{\gamma} & j \neq l, \\ j^{2\gamma-a} & j = l. \end{cases} \tag{5.30}
\]
Since \(b_{jl}\) is symmetric, the Schur-Holmgren condition for boundedness is simply
\[ \sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty. \]
The diagonal is bounded, and so causes no problem; thus we require finiteness of
\[
\sup_n n^{\gamma} \sum_{k=n+1}^{\infty} \frac{k^{\gamma}}{(k^{\alpha+1} - n^{\alpha+1})} + \sup_n n^{\gamma} \sum_{k=1}^{n-1} \frac{k^{\gamma}}{(n^{\alpha+1} - k^{\alpha+1})}. \tag{5.31}
\]
For the first term, we have
\[
n^{\gamma} \sum_{k=n+1}^{\infty} \frac{k^{\gamma}}{(k^{\alpha+1} - n^{\alpha+1})} \leq n^{\gamma} \int_{n+1}^{\infty} \frac{x^{\gamma}}{x^{\alpha+1} - n^{\alpha+1}} dx = n^{2\gamma-a} \int_{1+1/n}^{\infty} \frac{s^{\gamma}}{s^{\alpha+1} - 1} ds \leq c(\alpha, \gamma) n^{2\gamma-a} \log n
\]
which goes to zero if \(0 < 2\gamma < \alpha\). Similarly,
\[
n^{\gamma} \sum_{k=1}^{n-1} \frac{k^{\gamma}}{n^{\alpha+1} - k^{\alpha+1}} = n^{2\gamma-a} \sum_{k=1}^{n-1} \frac{1}{(k/n)^{\alpha+1}} \left( \frac{k}{n} \right)^{\gamma} \frac{1}{n} \leq n^{2\gamma-a} \left\{ \int_0^{1-1/n} \frac{s^{\gamma}}{s^{\alpha+1} - 1} ds + c(\alpha) \right\} \leq c(\sigma, \gamma) n^{2\gamma-a} \log n
\]
We have now shown that $K(\beta)$ is uniformly equivalent to an operator of the form (4.3), with $H$ replaced by the diagonal operator

$$\sum_{n=0}^{\infty} \lambda_n(t, \beta) P_n.$$

From perturbation theory [5], p. 88, we have

$$\lambda_n(t, \beta) = \lambda_n + \beta (V(t) \varphi_n, \varphi_n) + E_n(t, \beta)$$

where the error term satisfies

$$|E_n(t, \beta)| \leq 2 \beta^2 M^2 r_n^{-1}.$$

Since $n^2 r_n^{-1}$ is bounded, the term

$$\sum_{n=0}^{\infty} E_n(t, \beta) P_n$$

can be absorbed into the $AW(t, \beta)A$ term in (4.3). To eliminate the remaining term, note that by (4.1),

$$(V(t) \varphi_n, \varphi_n) = g_n(t)$$

where $g_n(t)$ is $2\pi$-periodic. Let $G(t)$ be the unitary transformation

$$G(t) = \sum_{n=0}^{\infty} e^{i \beta g_n(t)} P_n$$

and $Gu(t) = G(t)u(t)$. If we now transform by the "gauge transformation" $G$, the term

$$\sum_{n=0}^{\infty} \beta (V(t) \varphi_n, \varphi_n) P_n$$

disappears, while the form of $AW(t, \beta)A$ is preserved, since $G(t)$ commutes with $A$. This completes the proof. ■

6. CONCLUDING REMARKS

(1) Our theorem is unsatisfactory in several ways. In the first place, one would like to reduce the value of $\alpha$. Hamiltonians like the one-dimensional rotor considered by Bellisard ([1], [2]) corresponds to $\Delta \lambda_n \approx n$, or $\alpha=1$. (It also has multiplicity two.) The harmonic oscillator ([3], [6]) has $\alpha=0$, and is doubtless more delicate.

(2) One would also like to be able to randomize $H(\omega)$ within a natural class. For example, if $H$ is a Schroedinger operator, we would like to have
The chief problem here is the difficulty of proving that operator multiplications like $K$ are absolutely continuous. Theorems of this type would be very useful, both here and in localization theory [10].

(3) The theorem here seems essentially one-dimensional in its assumption of increasing gap $\Delta \lambda_n$. For example, if $H$ represents the particle in a box in $d$ dimensions, then for $d \geq 3$, the density of eigenvalues becomes larger as energy increases, rather than smaller. Does this result in a different spectral type for $K$? An answer to this question would be very interesting.

REFERENCES


