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On stability for time-periodic perturbations of harmonic oscillators

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ABSTRACT. — For two classes of harmonic oscillators with time-periodic perturbations, it is shown that the kinetic and potential energy remain bounded and the monodromy operator has pure point spectrum. An example is also given for which these conclusions fail.

RÉSUMÉ. — Pour deux classes d'oscillateurs harmoniques avec perturbations périodiques en temps, il est démontré que l'énergie potentielle et cinétique restent bornées et l'opérateur de monodromie a un spectre purement ponctuel. On donne aussi un exemple pour lequel ces conclusions ne tiennent pas.

1. INTRODUCTION

Let $H(t)$, $t \in \mathbb{R}$, be a family of self-adjoint operators acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. The operator $H(t)$ is viewed as the Hamiltonian at time t for the physical system whose states are the unit vectors in $\mathcal{H}$. We assume that $H(t)$ generates a unitary propagator $U(t, s)$, $t, s \in \mathbb{R}$, in $\mathcal{H}$ [9] solving the Schrödinger equation

$$(1.1) \qquad i \frac{d}{dt} \varphi(t) = H(t)\varphi(t).$$

That is, the solution of (1.1) with the initial condition $\varphi(s) = \varphi_s$ is given by $\varphi(t) = U(t, s)\varphi_s$.

We call $\varphi \in \mathcal{H}$ a bound state if it is localized in a bounded region for all time in the sense that

$$(1.2) \qquad \sup_{-\infty < t < \infty} \| F(|x| > R)U(t, 0)\varphi \| \to 0 \quad \text{as} \quad R \to \infty$$

where F(M) denotes the operator multiplication by the characteristic function of M, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $p_j = -i\partial/\partial x_j$ be the jth component of the momentum operator $p = -i\nabla$. The following theorem gives a criterion for the quantum trapping (1.2).

THEOREM 1. — Suppose that there is a fundamental set $\mathcal{S} \subset \mathcal{H}$ so that both $\| |x| U(t, 0)\varphi \|$ and $\| |p| U(t, 0)\varphi \|$ are uniformly bounded in t for each $\varphi \in \mathcal{S}$. Then the trajectory $\{ U(t, 0)\varphi / t \in \mathbb{R} \}$ is a precompact subset of $\mathcal{H}$ and (1.2) holds for all $\varphi \in \mathcal{H}$. If H(t) is periodic with period d, then the monodromy operator U(d, 0) has pure point spectrum.

Proof. — Let $\varphi \in \mathcal{S}$. By Rellich's criterion ([*10*], p. 247), we see that $\{ U(t, 0)\varphi / t \in \mathbb{R} \}$ is contained in a compact subset of $\mathcal{H}$, so it is precompact. Since $\mathcal{S}$ is fundamental, it follows that every trajectory is precompact. As a result, (1.2) holds for all $\varphi \in \mathcal{H}$. In particular, the monodromy operator has pure point spectrum for the periodic case (cf. Enss-Veselić [*5*]). □

Recently, the quantum stability for time-periodic perturbations has been the subject of many investigations ([*1*]-[*4*] and references therein). In [*2*], Combescure studied a charged particle in a three-dimensional quadrupole radio-frequency trap with an AC plus DC electric field. The Hamiltonian in question is

$$(1.3) \qquad H(t) = \frac{p^2}{2m} + \frac{e}{r_0^2}\left(x_1^2 - \frac{x_2^2 + x_3^2}{2}\right)(V_{dc} - V_{ac} \cos \Omega t)$$

on $L^2(\mathbb{R}^3)$, which has period $d = 2\pi/\Omega$. She reduced her analysis to a one dimensional problem, then based on the exact solvability of the associated Mathieu equation—the classical equation of motion associated to (1.3)—established a trapping result (1.2) for Ω in a certain interval. It should be pointed out here that the trapping result is a direct consequence of the uniform boundedness of $\| |x| U(t, 0)\varphi \|$ and $\| |p| U(t, 0)\varphi \|$ in time, and that this criterion could be used to handle more general cases where the exact solutions are not completely known.

EXAMPLE (One dimensional oscillators). — Consider the time-periodic Hamiltonians

$$(1.4) \qquad H(t) = -\frac{d^2}{dx^2} + f(t)x^2$$

on $\mathcal{H} = L^2(\mathbb{R})$, where $f \not\equiv 0$ is real, d-periodic and allowed to be piecewise-continuous. We shall reduce the problem of quantum trapping to a stability problem of the classical Hill equation. Since f is piecewise-continuous, the associated propagator is expected to have possibly different one-side derivatives at the discontinuities of f. Let $\mathscr{D} = \mathscr{D}(p^2) \cap \mathscr{D}(x^2)$ ($\mathscr{D}(\mathrm{T})$ denotes the domain of an operator T), and for each $\varphi \in \mathscr{D}$, set

$$\| \varphi \|_{\mathscr{D}} = \| \varphi \| + \| p^2 \varphi \| + \| x^2 \varphi \| .$$

Then $(\mathscr{D}, \| \cdot \|_{\mathscr{D}})$ forms a Banach space which is continuously and densely embedded in $\mathscr{H}$. The following existence theorem is based on a modification of a result of Kato ([7], Theorem 6.1) with a glueing process.

THEOREM 2. — There exists a unique unitary propagator $\mathrm{U}(t, s)$ in $\mathscr{H}$ with the following properties

a) $\mathrm{U}(t, s)\mathrm{U}(s, r) = \mathrm{U}(t, r)$ and $\mathrm{U}(t, t) = \mathrm{I}$ for all t, s, r.

b) $(t, s) \mapsto \mathrm{U}(t, s) \in \mathscr{B}(\mathscr{H})$ is strongly continuous.

c) $\mathrm{U}(t, s)\mathscr{D} \subseteq \mathscr{D}$, and $(t, s) \mapsto \mathrm{U}(t, s) \in \mathscr{B}(\mathscr{D})$ is strongly continuous.

d) $$\frac{d}{dt} \mathrm{U}(t, s)\varphi = - i\mathrm{H}(t)\mathrm{U}(t, s)\varphi \quad \text{for all } \varphi \in \mathscr{D}.$$

e) $$\frac{d}{ds} \mathrm{U}(t, s)\varphi = i\mathrm{U}(t, s)\mathrm{H}(s)\varphi \quad \text{for all } \varphi \in \mathscr{D}$$

where the derivatives are taken in the sense of strong derivatives in $\mathscr{H}$, whose values at the discontinuities of f are understood in the following sense

$$\frac{d^+}{dt} \mathrm{U}(t, s)\varphi \,|_{t=a} = - i\mathrm{H}(a^+)\mathrm{U}(a, s)\varphi$$

$$\frac{d^-}{dt} \mathrm{U}(t, s)\varphi \,|_{t=a} = - i\mathrm{H}(a^-)\mathrm{U}(a, s)\varphi$$

and similarly for $\dfrac{d}{ds}$.

Now, let $\varphi, \psi \in \mathscr{D}$ be fixed, and set

$$x(t ; \psi, \varphi) = \langle \mathrm{U}(t, 0)\psi, x\mathrm{U}(t, 0)\varphi \rangle$$
$$p(t ; \psi, \varphi) = \langle \mathrm{U}(t, 0)\psi, p\mathrm{U}(t, 0)\varphi \rangle$$

Using the properties of $\mathrm{U}(t, 0)$, one easily shows that $x(t ; \psi, \varphi)$ is governed almost everywhere by the Hill equation

$$\ddot{x}(t) + 4f(t)x(t) = 0 \tag{1.5}$$

with the initial conditions

$$x(0) = \langle \psi, x\varphi \rangle, \qquad \dot{x}(0) = 2 \langle \psi, p\varphi \rangle .$$

Let us assume that

$$\int_0^d f(t)dt \geq 0 \quad \text{and} \quad \int_0^d f_+(t)dt \leq \frac{1}{d} \tag{1.6}$$

where

$$f_+(t) = \begin{cases} f(t) & \text{if} \quad f(t) \geq 0 \\ 0 & \text{if} \quad f(t) < 0. \end{cases}$$

Then by a classical result of M. G. Krein ([8] or [11], p. 729), all solutions of (1.5) are bounded on $(-\infty, \infty)$ and are almost-periodic functions. Using these facts, one can prove by a standard argument that both $\| x\mathrm{U}(t,0)\varphi \|$ and $\| p\mathrm{U}(t,0)\varphi \|$ are uniformly bounded in t for all $\varphi \in \mathscr{D}$. Thus, by Theorem 1, the monodromy operator $\mathrm{U}(d, 0)$ has pure point spectrum and (1.2) holds for all $\varphi \in \mathrm{L}^2(\mathbb{R})$. □

In the next section, we shall give a counterexample showing that the trapping result for (1.4) may be false without the Krein condition (1.6). In that example we also prove that the monodromy operator has purely absolutely continuous spectrum, and that the kinetic energy $\langle \mathrm{U}(t,0)\varphi, p^2\mathrm{U}(t,0)\varphi \rangle$ is unbounded in time. Finally, in section 3 we consider the stability problem for the n-dimensional harmonic oscillator

$$\mathrm{H} = p^2 + \sum_{j=1}^{n} \omega_j^2 x_j^2, \quad \omega_j > 0$$

with a time-periodic, space-quadratic perturbation

$$\mathrm{H}_\varepsilon(t) = \mathrm{H} + \varepsilon \sum_{1 \leq i \leq j \leq n} a_{ij}(t) x_i x_j$$

where $\varepsilon \in \mathbb{R}$, and the a_{ij}'s are continuous, d-periodic real functions. We shall prove that under suitable conditions on the ω_j's, the conclusions of Theorem 1 hold for small coupling constant ε. The proof relies on Lyapunov's stability theory for reciprocally differential systems [6].

2. AN EXAMPLE EXHIBITING INSTABILITY

Let f be d-periodic with

$$f(t) = \begin{cases} \omega^2 & \text{if} \quad 0 \leq t < c \\ 0 & \text{if} \quad c \leq t < d. \end{cases}$$

Then the monodromy operator associated to (1.4) takes the simple form

$$\mathrm{U} \equiv \mathrm{U}(d, 0) = e^{-i(d-c)p^2} e^{-ic\mathrm{H}_\omega}$$

where $H_\omega = p^2 + \omega^2 x^2$. Recall that the spectrum of $H_\omega(\omega > 0)$ is discrete consisting of eigenvalues

$$\lambda_n = \omega(2n+1), \qquad n = 0, 1, 2, \ldots$$

and the corresponding eigenfunctions (normalized) are

$$\varphi_n = c_n(p + i\omega x)^n e^{-\omega x^2/2}, \qquad n = 0, 1, 2, \ldots$$

where the constants c_n's are chosen to make $\| \varphi_n \| = 1$.

Let $\mathscr{H}_e$ be the closed subspace of $\mathscr{H}$ spanned by the even eigenfunctions $\{ \varphi_0, \varphi_2, \varphi_4, \ldots \}$, and let $\mathscr{H}_0$ be the closed subspace spanned by the odd eigenfunctions $\{ \varphi_1, \varphi_3, \varphi_5, \ldots \}$. Then $\mathscr{H}_e$ and $\mathscr{H}_0$ are orthogonal complements to each other and $\mathscr{H} = \mathscr{H}_e \oplus \mathscr{H}_0$. Moreover,

$$e^{-i(\pi/2\omega)H_\omega}|_{\mathscr{H}_e} = -i, \qquad e^{-i(\pi/2\omega)H_\omega}|_{\mathscr{H}_0} = i.$$

Note that the free Laplacian p^2 is reduced by both $\mathscr{H}_e$ and $\mathscr{H}_0$, hence e^{itp^2} leaves both $\mathscr{H}_e$ and $\mathscr{H}_0$ invariant for any t. Thus, if we take $c = \pi/2\omega < d$, then

$$U_e \equiv U|_{\mathscr{H}_e} = -ie^{i[(\pi/2\omega)-d]p^2}|_{\mathscr{H}_e} \tag{2.1}$$

$$U_0 \equiv U|_{\mathscr{H}_0} = ie^{i[(\pi/2\omega)-d]p^2}|_{\mathscr{H}_0} \tag{2.2}$$

Since U_e and U_0 are both spectrally absolutely continuous, so is their direct sum $U = U_e \oplus U_0$.

It should be noted that with this choice of c

$$\int_0^d f_+(t)dt = \frac{\omega\pi}{2} > \frac{\pi^2}{4d} > \frac{1}{d}$$

violates the second condition in (1.6).

From now on, we fix $c = \pi/2\omega$. We shall argue that the kinetic energy $\| pU(t,0)\varphi \|^2$ is unbounded in time for $\varphi \in \mathscr{D}$, $\varphi \neq 0$. Let $n \in \mathbb{Z}^+$. A straightforward calculation shows that

$$\| pU(nd+s,0)\varphi \|^2 = \begin{cases} \| p\varphi \|^2 \quad \text{for} \quad \pi/2\omega \le s < d \\ \omega^2 \sin^2(2\omega s) \| xU^n\varphi \|^2 + \cos^2(2\omega s) \| p\varphi \|^2 \\ \qquad - (\omega/2)\sin(4\omega s) \langle U^n\varphi, (px+xp)U^n\varphi \rangle \quad \text{for} \quad 0 \le s < \pi/2\omega \end{cases} \tag{2.3}$$

$$\| xU(nd+s,0)\varphi \|^2 = \begin{cases} \| xU^n\varphi \|^2 + 4[s-(\pi/2\omega)]^2 \| p\varphi \|^2 \\ \qquad + 2[s-(\pi/2\omega)] \langle U^n\varphi, (px+xp)U^n\varphi \rangle \quad \text{for} \quad \pi/2\omega \le s < d \\ \cos^2(2\omega s) \| xU^n\varphi \|^2 + (1/\omega^2)\sin^2(2\omega s) \| p\varphi \|^2 \\ \qquad + (1/2\omega)\sin(4\omega s) \langle U^n\varphi, (px+xp)U^n\varphi \rangle \quad \text{for} \quad 0 \le s < \pi/2\omega. \end{cases} \tag{2.4}$$

where we have used the following basic identities:

$$e^{itp^2}xe^{-itp^2} = x + 2pt$$
$$e^{itH_\omega}xe^{-itH_\omega} = x\cos(2\omega t) + (p/\omega)\sin(2\omega t)$$
$$e^{itH_\omega}pe^{-itH_\omega} = -\omega x\sin(2\omega t) + p\cos(2\omega t).$$

We claim that $\| xU^n\varphi \|$ is not uniformly bounded for $n \in \mathbb{Z}^+$. For, if it were bounded, then both $\| pU(t, 0)\varphi \|$ and $\| xU(t, 0)\varphi \|$ would be uniformly bounded for $t > 0$ by (2.3) and (2.4). As in the proof of Theorem 1, it then follows that φ would belong to the pure point subspace of U. But, U is spectrally absolutely continuous, so we must have $\varphi = 0$. Since $\left\| pU\left(nd + \frac{\pi}{4\omega}, 0\right)\varphi \right\| = \omega \| xU^n\varphi \|$, we conclude that the kinetic energy is unbounded.

Finally, we show that (1.2) fails. Let $\varphi = \varphi_e + \varphi_0 \in \mathcal{H}$, where $\varphi_e \in \mathcal{H}_e$, and $\varphi_0 \in \mathcal{H}_0$. By (2.1) and (2.2), we have

$$\| F(|x| \leq R)U^n\varphi \| = \| F(|x| \leq R)e^{in[(\pi/2\omega)-d]p^2}[\varphi_e + (-1)^n\varphi_0] \|$$

which converges to zero for any finite R as $n \to \pm\infty$. So, (1.2) is impossible unless $\varphi = 0$.

3. N-DIMENSIONAL OSCILLATORS

It is well-known, in time independent theory, that a Hamiltonian operator $H = -\Delta + V$ with potential $V \in L^1_{loc}(\mathbb{R}^n)$, $V \geq -c$ and $V(x) \to +\infty$ as $|x| \to +\infty$ has purely discrete spectrum and a complete set of eigenfunctions [*10*]. A typical example is the n-dimensional harmonic oscillator

$$H = -\Delta + \sum_{j=1}^{n} \omega_j^2 x_j^2, \qquad \omega_j > 0.$$

Suppose now we perturb H by a time-periodic, space-quadratic perturbation, for instance, consider

$$H_\varepsilon(t) = H + \varepsilon \sum_{1 \leq i \leq j \leq n} a_{ij}(t)x_i x_j \tag{3.1}$$

where $\varepsilon \in \mathbb{R}$, and the a_{ij}'s are real continuous d-periodic functions. We may ask whether the perturbed system (3.1) remains stable in the sense that the associated monodromy operator $U_\varepsilon(d, 0)$ has pure point spectrum. The results presented in this section concern only the small coupling constant ε.

We first notice that $H_\varepsilon(t)$ is self-adjoint with a common core $C_0^\infty(\mathbb{R}^n)$ by the Faris-Lavine theorem [*9*]. The proof of the existence of propagator

$U_\varepsilon(t, s)$ runs parallel with that of Theorem 2. In fact, the set $\mathscr{D} = \mathscr{D}(\boldsymbol{p}^2) \cap \mathscr{D}(\boldsymbol{x}^2)$ equipped with the norm

$$\| \varphi \|_{\mathscr{D}} = \| \varphi \| + \| \boldsymbol{p}^2 \varphi \| + \| \boldsymbol{x}^2 \varphi \|, \qquad \varphi \in \mathscr{D}$$

forms a Banach space which is continuously and densely embedded in $\mathscr{H} = L^2(\mathbb{R}^n)$. Obviously, $\mathscr{D} \subseteq \mathscr{D}(H_\varepsilon(t))$ for all t and $t \mapsto H_\varepsilon(t) \in \mathscr{B}(\mathscr{D}, \mathscr{H})$ is continuous everywhere in the operator norm. Also, if we take $S = \boldsymbol{p}^2 + \boldsymbol{x}^2 + i$, then $S \in \mathscr{B}(\mathscr{D}, \mathscr{H})$ is an isomorphism with

$$SH_\varepsilon(t)S^{-1} = H_\varepsilon(t) + G_\varepsilon(t)$$

where

$$G_\varepsilon(t) = 2i \left\{ \sum_{j=1}^{n} (1-\omega_j^2)(p_j x_j + x_j p_j) - \varepsilon \sum_{1 \le i \le j \le n} a_{ij}(t)(p_i x_j + x_i p_j) \right\} S^{-1}$$

is in $\mathscr{B}(\mathscr{H})$ and is continuous everywhere in the operator norm. Therefore, Kato's theorem [*7*] is applicable and the conclusions *a*)-*e*) in Theorem 2 hold for the couple $(H_\varepsilon(t), U_\varepsilon(t, s))$ without any exception for t and s in the derivatives.

We can now turn to our main concerns. As we shall see, Lyapunov's theory for reciprocal systems plays a key role in our discussions below. A convenient reference for this theory is [*6*].

THEOREM 3. — Suppose that $2\omega_j d \not\equiv 0 \pmod{\pi}$ for all $1 \le j \le n$, and $(\omega_j \pm \omega_k)d \not\equiv 0 \pmod{\pi}$ for all $j \neq k$. Then there exists an $\varepsilon_0 > 0$ such that if $|\varepsilon| \le \varepsilon_0$, the monodromy operator $U_\varepsilon(d, 0)$ associated to (3.1) has pure point spectrum and (1.2) holds for all $\varphi \in \mathscr{H}$.

Proof. — Let $\varphi, \psi \in \mathscr{D}$ and set

$$x_k(t; \psi, \varphi) = \langle U_\varepsilon(t, 0)\psi, x_k U_\varepsilon(t, 0)\varphi \rangle$$
$$p_k(t; \psi, \varphi) = \langle U_\varepsilon(t, 0)\psi, p_k U_\varepsilon(t, 0)\varphi \rangle .$$

Then we have

$$\dot{x}_k(t; \psi, \varphi) = 2p_k(t; \psi, \varphi)$$
$$\dot{p}_k(t; \psi, \varphi) = -2\omega_k^2 x_k(t; \psi, \varphi) - \varepsilon \sum_{1 \le i \le j \le n} a_{ij}(t) \{ \delta_{ki} x_j(t; \psi, \varphi) + \delta_{kj} x_i(t; \psi, \varphi) \}.$$

From this, we see that the $x_k(t; \psi, \varphi)$ are governed by the system of differential equations

$$(3.2) \quad \ddot{x}_k(t) = -4\omega_k^2 x_k(t) - 2\varepsilon \sum_{1 \le i \le j \le n} a_{ij}(t) \{ \delta_{ki} x_j(t) + \delta_{kj} x_i(t) \} \quad (1 \le k \le n)$$

with initial conditions

$$(3.3) \quad x_k(0) = \langle \psi, x_k \varphi \rangle, \qquad \dot{x}_k(0) = 2 \langle \psi, p_k \varphi \rangle \qquad (1 \le k \le n)$$

Let $A(t)$ be the upper-triangular matrix whose (i, j)-entry is $a_{ij}(t)$, and let

$$F_\varepsilon(t) = -4 \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2) - 2\varepsilon\{A(t) + A(t)^T\}$$

(A^T denotes the transpose of a matrix A). Then system (3.2) is equivalent to the d-periodic system of order $2n$:

$$\dot{y}(t) = B_\varepsilon(t)y(t) \tag{3.4}$$

where

$$y(t) = (x_1(t), \ldots, x_n(t), \dot{x}_1(t), \ldots, \dot{x}_n(t))^T, \quad B_\varepsilon(t) = \begin{bmatrix} 0 & I_n \\ F_\varepsilon(t) & 0 \end{bmatrix}.$$

Note that system (3.4) is reciprocal because $F_\varepsilon(t)$ is real symmetric. On the other hand, the unperturbed ($\varepsilon = 0$) system

$$\dot{y}(t) = B_0(t)y(t) \equiv B_0 y(t) \tag{3.5}$$

may be regarded as d-periodic so that the associated characteristic multipliers are given by

$$\rho_{2j-1} = e^{i2\omega_j d}, \qquad \rho_{2j} = \overline{\rho_{2j-1}} \quad (j = 1, 2, \ldots, n).$$

Thus, all of the characteristic multipliers of system (3.5) have unit modulii. Furthermore, the hypotheses imply that they are distinct. Therefore, according to Lyapunov's stability theory, there exists an $\varepsilon_0 > 0$ such that all solutions of system (3.4) are bounded on $(-\infty, \infty)$ for all $|\varepsilon| \leq \varepsilon_0$.

Now, let $Z_\varepsilon(t)$ be the principal matrix solution of (3.4) at $t = 0$. Since each column $(z_{1j}(t), \ldots, z_{nj}(t), \dot{z}_{1j}(t), \ldots, \dot{z}_{nj}(t))^T$, $1 \leq j \leq 2n$, of $Z_\varepsilon(t)$ satisfies (3.4), and since $(x(t; \psi, \varphi), \dot{x}(t; \psi, \varphi))^T$ is a solution of (3.4), we may express $x(t; \psi, \varphi)$ and $\dot{x}(t; \psi, \varphi)$ in terms of the $z_{ij}(t)$ and $\dot{z}_{ij}(t)$. Using the initial conditions (3.3) and $Z_\varepsilon(0) = I_{2n}$, we obtain for all k, $1 \leq k \leq n$, the following:

$$x_k(t; \psi, \varphi) = \sum_{j=1}^{n} \langle \psi, x_j\varphi \rangle z_{kj}(t) + 2 \sum_{j=1}^{n} \langle \psi, p_j\varphi \rangle z_{k,n+j}(t) \tag{3.6}$$

$$p_k(t; \psi, \varphi) = \frac{1}{2} \sum_{j=1}^{n} \langle \psi, x_j\varphi \rangle \dot{z}_{kj}(t) + \sum_{j=1}^{n} \langle \psi, p_j\varphi \rangle \dot{z}_{k,n+j}(t). \tag{3.7}$$

Let $|\varepsilon| \leq \varepsilon_0$. We have seen that there is a constant M such that $|Z_\varepsilon(t)| \leq M$ for all $t \in (-\infty, \infty)$. So, by (3.6) and (3.7), we have

$$\| x_k U_\varepsilon(t, 0)\varphi \| = \sup_{\psi \in \mathcal{D}, \|\psi\| \leq 1} |x_k(t; \psi, \varphi)| \leq M \sum_{j=1}^{n} (\| x_j\varphi \| + 2 \| p_j\varphi \|)$$

$$\| p_k U_\varepsilon(t, 0)\varphi \| = \sup_{\psi \in \mathcal{D}, \|\psi\| \leq 1} |p_k(t; \psi, \varphi)| \leq M \sum_{j=1}^{n} \left(\frac{1}{2} \| x_j\varphi \| + \| p_j\varphi \|\right).$$

It follows that

$$\| \, |\boldsymbol{x}| \, \mathrm{U}_\varepsilon(t,0)\varphi \, \| + \| \, |\boldsymbol{p}| \mathrm{U}_\varepsilon(t,0)\varphi \, \| \leq 3n\mathrm{M} \sum_{j=1}^{n} (\| x_j \varphi \| + \| p_j \varphi \|) < \infty .$$

Since this is true for all $\varphi \in \mathscr{D}$, the assertions of the theorem now follow from Theorem 1. □

For the diagonal case: $a_{ij}(t) = 0$ for $i < j$, the assumptions of Theorem 3 can be weakened so as not to involve any connection among those ω_j's.

THEOREM 4. — Suppose that $2\omega_j d \not\equiv 0 \pmod{\pi}$ for all $1 \leq j \leq n$. Then there is an $\varepsilon_0 > 0$ such that the conclusions of Theorem 3 hold for the system

$$\mathrm{H}_\varepsilon(t) = -\Delta + \sum_{j=1}^{n} \omega_j^2 x_j^2 + \varepsilon \sum_{j=1}^{n} a_j(t) x_j^2$$

provided that $|\varepsilon| \leq \varepsilon_0$.

Proof. — Let's borrow those terms used in the proof of Theorem 3. This time we are led to deal with the system of differential equations

$$\ddot{x}_k(t) = -4(\omega_k^2 + \varepsilon a_k(t)) x_k(t) \qquad (1 \leq k \leq n) \tag{3.8}$$

with initial values $x_k(0) = \langle \psi, x_k \varphi \rangle$, $\dot{x}_k(0) = 2 \langle \psi, p_k \varphi \rangle$. But we shall treat each equation in (3.8) individually. Note that the kth equation in (3.8) is equivalent to the d-periodic reciprocal system of order 2:

$$\dot{y}_k(t) = \mathrm{B}_{k,\varepsilon}(t) y_k(t) \tag{3.9}$$

where

$$y_k(t) = \begin{bmatrix} x_k(t) \\ \dot{x}_k(t) \end{bmatrix}, \qquad \mathrm{B}_{k,\varepsilon}(t) = \begin{bmatrix} 0 & 1 \\ -4(\omega_k^2 + \varepsilon a_k(t)) & 0 \end{bmatrix}.$$

On the other hand, the kth unperturbed ($\varepsilon = 0$) system may be regarded as d-periodic whose characteristic multipliers are given by

$$\rho_1 = e^{i2\omega_k d}, \qquad \rho_2 = e^{-i2\omega_k d}$$

which have unit modulii and are distinct by hypothesis. So, by Lyapunov's theory, there exists an $\varepsilon_k > 0$ such that all solutions of (3.9) are bounded on $(-\infty, \infty)$ provided that $|\varepsilon| \leq \varepsilon_k$.

Thus, following the idea of the proof of Theorem 3, we can show that if $|\varepsilon| \leq \varepsilon_0 \equiv \min \{ \varepsilon_k / 1 \leq k \leq n \}$, then the following estimate holds

$$\| \, |\boldsymbol{x}| \, \mathrm{U}_\varepsilon(t,0)\varphi \, \| + \| \, |\mathbf{p}| \, \mathrm{U}_\varepsilon(t,0)\varphi \, \| \leq \text{const.} \sum_{k=1}^{n} (\| x_k \varphi \| + \| p_k \varphi \|) < \infty$$

for all $\varphi \in \mathscr{D}$ and all $t \in (-\infty, \infty)$. This completes the proof. □

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