SHU NAKAMURA

Scattering theory for the shape resonance model. II. Resonance scattering


<http://www.numdam.org/item?id=AIHPA_1989__50_2_133_0>
Scattering Theory
for the Shape Resonance Model II.
Resonance Scattering

by

Shu NAKAMURA

Department of Pure and Applied Sciences, University of Tokyo,
Komaba, Meguro-ku, Tokyo 153, Japan

ABSTRACT. — Continuing the analysis of the part I, we consider the asymptotic behavior of the scattering matrix for the shape resonance model. We give an asymptotic formula for the scattering matrix near the (complex) resonance eigenvalues under the assumptions similar to those in Combes-Duclos-Klein-Seiler [2] (exterior scaling analyticity, non-trapping condition, etc.).

RÉSUMÉ. — Nous poursuivons l’analyse de la première partie et nous considérons le comportement asymptotique de la matrice de diffusion pour le modèle de résonance de forme.

Nous donnons une forme asymptotique pour la matrice de diffusion près des valeurs propres résonantes (complexes) sous des hypothèses analogues à celles de Combes-Duclos-Klein-Seiler (2) (analyticité par dilatation, non piégeage, etc.).

§ 1. INTRODUCTION

We consider

\[ H = H(h) = H_0(h) + V(x), \quad H_0 = H_0(h) = -\hbar^2 \Delta \]

on \( L^2(\mathbb{R}^n) \), \( D(H_0) = H^2(\mathbb{R}^n) \) where \( \hbar > 0 \) is the Planck constant. Instead of \( (A)_x \) in (I) (we refer to the part I of this series as (I), we assume for \( x > 1 \),
ASSUMPTION (A)′. — V is a real-valued continuously differentiable function satisfying
\[ |V(x)| \leq C(1 + |x|)^{-\delta}, \quad (x \in \mathbb{R}^n) \] (1.1)
and there is \( \delta > 0 \) such that
\[ |x \cdot \nabla V(x)| \leq C |x| (1 + |x|)^{-\delta} \quad (x \in \mathbb{R}^n) \] (1.2)

We set \( \Omega_{\text{int}} = \{ x \in \mathbb{R}^n | |x| < R \} \), \( K = \mathbb{R} \cdot S^{n-1} \) for some \( R > 0 \), and suppose that \( \inf_{x \in \mathbb{R}^n} V(x) = \lambda_0 > 0 \). As in (I), \( \Omega_{\text{ext}} = \mathbb{R}^n \setminus \overline{\Omega}_{\text{int}} \), \( \mathcal{J}(\lambda) = \{ x \in \mathbb{R}^n | V(x) > \lambda \} \), \( \Delta^D \) is the Laplacian with Dirichlet boundary condition on \( K \), and \( H^D(h) = -h^2 \Delta^D + V(x) = H_{\text{int}} \oplus H_{\text{ext}} \) on \( L^2(\mathbb{R}^n) = L^2(\Omega_{\text{int}}) \oplus L^2(\Omega_{\text{ext}}) \).

Next we assume for \( \lambda \in (0, \lambda_0) \).

ASSUMPTION (B)′. — There is \( \delta > 0 \) such that for any \( x \in \Omega_{\text{ext}} \setminus \mathcal{J}(\lambda) \),
\[ V(x) + \frac{1}{2} x \cdot \nabla V(x) \leq \lambda - \delta. \] (1.3)

(B)′ is a sufficient condition of (B) in (I) for some neighborhood \( I \) of \( \lambda \) provided \( (\alpha > 0) \) (see Proposition A.1 of (I)).

The exterior scaling \( U(\theta) \) is defined as follows (cf. Sect. 2 of [2]): for \( \theta \in \mathbb{R} \) and \( x \in \mathbb{R}^n \), \( T(\theta)x = (e^{\theta}(|x| - R) + R) \frac{x}{|x|} \) if \( |x| \geq R \), \( = x \) if \( |x| < R \). For \( f \in L^2(\mathbb{R}^n) \), \( U(\theta)f \) is defined by \( (U(\theta)f)(x) = e^{\theta \chi(|x|)/2} \cdot f(T(x)) \) where \( \chi(r) = 1 \) if \( r \geq R \), \( = 0 \) if \( r < R \). \( U(\theta) \) is the 1-parameter unitary group generated by \( A = 0 \oplus (2i)^{-1} \left\{ \frac{|x| - R}{|x|} \cdot \nabla + \nabla \cdot \frac{x}{|x|} (|x| - R) \right\} \).

ASSUMPTION (C). — \( V(\theta) = U(\theta)VU(\theta)^{-1} (\theta \in \mathbb{R}) \) can be extended to a bounded operator valued analytic function on the strip \( S_{\gamma} = \{ \theta \in \mathbb{C} | |\text{Im} \theta| < \gamma \} \) for some \( \gamma > 0 \).

It follows from (C) that \( H(h, \theta) = U(\theta)H(h)U(\theta)^{-1} \) can be extended to an operator valued function on \( S_{\gamma} \) and it is analytic in the resolvent sense (cf. Appendix of [2]).

Let \( \lambda \in (0, \lambda_0) \) and let \( h_i \downarrow 0 \) (\( i \to \infty \)) be a sequence of positive numbers. Suppose that \( E_i^D \) is an isolated simple eigenvalue of \( H_{\text{int}}(h_i) \) such that \( E_i^D \to \lambda \) (\( i \to \infty \)). We suppose furthermore that

ASSUMPTION (D). — For some \( b > 0 \) and \( q > 0 \),
\[ \text{dist} (\sigma(H_{\text{int}}(h_i)) \setminus \{ E_i^D \}, E_i^D) \geq b(h_i)^q \] (1.4)
for any \( i \).

Then it is known that there exists a (resonance) eigenvalue \( E_i \) of \( H(h_i, \theta) \) (\( \text{Im} \theta > 0 \)) such that \( E_i \) is exponentially close to \( E_i^D \) in \( h_i^{-1} \):
\[ |E_i - E_i^D| \leq C_{\varepsilon} \exp \left\{ -d_0(K, \Omega_{\text{int}} \setminus \mathcal{J}(\lambda)) - \varepsilon/h_i \right\} \] (1.5)

Annales de l'Institut Henri Poincaré - Physique théorique
for any $\varepsilon > 0$, where $d_\lambda$ is the pseudo-distance associated to the Agmon metric $ds^2 = \max (V(x) - \lambda, 0) dx^2$ (cf. Proposition 2.5, or Theorem V-2 of [2]).

Let $\eta_i = \eta_i(\theta; x)$ be an eigenfunction of $H(h_i, \theta)$ corresponding to $E_i$: $H(h_i, \theta)\eta_i = E_i\eta_i$. $T_{\text{int/ext}}$ denotes the trace operator to $K$:

$$
(T_{\text{int/ext}}f)(x) = f(x) \quad (x \in K, f \in C^\infty(\Omega_{\text{int/ext}}));
$$

$$
T_{\text{int/ext}} \in B(H^1(\Omega_{\text{int/ext}}), L^2(K)).
$$

$V_n = \frac{\nabla R}{R}$ is the unit normal derivative on $K$. Let $\Phi_\pm(h_i; \mu) \in L^2(-x^2/2, \infty)$ be the generalized eigenfunction of $H_{\text{ext}}(h_i)$ corresponding to $\varphi \in L^2(S^{n-1})$ defined by (3.3) of (I). We define $\pi_i(\mu) \in B(L^2(S^{n-1}))$ by

$$(\varphi, \pi_i(\mu)\psi) = 2\pi h_i^4(\eta_i(\theta), \eta_i(\theta))^{-1} \times
\left( T_{\text{int}}\eta_i(\theta), T_{\text{ext}} V_n \Phi_+(h_i; \mu) \varphi \right) \left( T_{\text{int}}\eta_i(\theta), T_{\text{ext}} V_n \Phi_-(h_i; \mu) \psi \right)$$

for $\mu \in (0, \infty)$ and Im $\theta > 0$. $\eta_i(\theta)$ is an eigenfunction of $H(h_i, \theta) = H(h_i, \theta)^*$ with eigenvalue $E_i$. It will be shown that $\pi_i(\mu)$ is independent of $\theta$ (Proposition 3.4 and Remark 3.5). We set $\Lambda_i = \{ \mu \in \mathbb{R} \mid |\mu - E_i^P| \leq b(h_i)^{\gamma}/2 \}$ where $b$ and $q$ are the constants in (D). On the definition of scattering matrices $S(H, H_0; \lambda, \epsilon)$, etc., we refer to Sect. I of (I). Then our main result is:

**Theorem 1.** — Suppose (A)$_\gamma$ ($\alpha > 1$), (B)$_\gamma$ ($\alpha \in (0, \alpha_0)$) and (C). Let $h_i \downarrow 0$ ($i \to \infty$) and let $E_i^P \to \lambda$ ($i \to \infty$) be a sequence of isolated simple eigenvalues of $H_{\text{int}}(h_i)$ such that (D) holds. Then for any $\varepsilon > 0$, there is $C > 0$ such that for $\mu \in \Lambda_i$

$$
\| S(H(h_i), H_0(h_i); \mu) - S(H^P(h_i), H_0(h_i); \mu) - (E_i - \mu)^{-1} \pi_i(\mu) \| 
\leq C \exp \{ -2(d_\lambda(K, \Omega_{\text{ext}} \setminus \mathcal{S}(\lambda)) - \varepsilon)/h_i \}; \quad (1.7)
$$

$$
\| \pi_i(\mu) \| \leq C \exp \{ -2(d_\lambda(K, \Omega_{\text{ext}} \setminus \mathcal{S}(\lambda)) + d_\lambda(K, \Omega_{\text{int}} \setminus \mathcal{S}(\lambda)) - \varepsilon)/h_i \}, \quad (1.8)
$$

where $d_\lambda$ is the pseudo-distance associated to the Agmon metric $ds^2 = \max (V(x) - \lambda, 0) dx^2$.

If (A)$_\gamma$ holds for $\alpha > (n + 1)/2$, Theorem 1 can be improved analogously to Theorem 2 in (I). Let $\Psi_\pm(h_i; \lambda, \omega)$ be the solution of the Lippman-Schwinger equation with respect to $H_{\text{ext}}(h_i)$ defined by (3.9) of (I). We define $\Xi(\mu, \omega, \omega')$ ($\omega, \omega' \in S^{n-1}$) by

$$
\Xi(\mu, \omega, \omega') = 2\pi h_i^4(\eta_i(\theta), \eta_i(\theta))^{-1} \times
\left( T_{\text{int}}\eta_i(\theta), T_{\text{ext}} V_n \Psi_+(h_i; \mu, \omega) \right) \left( T_{\text{int}}\eta_i(\theta), T_{\text{ext}} V_n \Psi_-(h_i; \mu, \omega') \right).
$$
Then we have

**THEOREM 2.** — **Let the assumptions of Theorem 1 be satisfied with**

\( x > (n + 1)/2 \). **Then for any** \( \epsilon > 0 \), there is \( C > 0 \) such that for \( \mu \in \Lambda_i \)

\[
|S(H(h_i), H_0(h_i); \mu, \omega, \omega') - S(H^0(h_i), H_0(h_i); \mu, \omega, \omega')| \leq C \exp \left\{ -2(d_s(K, \Omega_{\text{ext}} \setminus \mathcal{I}(\lambda)) - \epsilon)/h_i \right\} \tag{1.10}
\]

\[
|\Xi_i(\mu, \omega, \omega')| \leq C \exp \left\{ -2(d_s(K, \Omega_{\text{ext}} \setminus \mathcal{I}(\lambda)) + d_s(K, \Omega_{\text{int}} \setminus \mathcal{I}(\lambda)) - \epsilon)/h_i \right\} \tag{1.11}
\]

uniformly in \( \omega, \omega' \in S^{n-1} \).

**REMARK 1.1.** — **It is expected** that \( \text{Im} E_i \) and \( \| \pi_i \| \) are of order \( h_i^a \exp \left\{ -d_s(\Omega_{\text{ext}} \setminus \mathcal{I}(\lambda), \Omega_{\text{int}} \setminus \mathcal{I}(\lambda))/h_i \right\} \) and

\[
h_i^c \exp \left\{ -d_s(\Omega_{\text{ext}} \setminus \mathcal{I}(\lambda), \Omega_{\text{int}} \setminus \mathcal{I}(\lambda))/h_i \right\}
\]

respectively for some \( a \) and \( c \). If it is true, the order of the influence of resonances to the scattering matrix is of order \( h_i^{(c-a)} \).

In Sect. 2 we review the results of Combes-Duclos-Klein-Seiler [2], and prove some related results in Sect. 3. Finally we prove Theorems 1 and 2 using the results of Sect. 3 and (I).

In this paper, we will use freely the symbols and results of (I), and we refer to (I) for historical remarks and further references.

---

**§ 2. PRELIMINARIES: RESONANCE EIGENVALUES**

In this section, we review the results of [2] in a slightly extended form, since we are interested not only in low lying eigenvalues but also in highly excited ones. Throughout this paper we suppose \( (\alpha)_{\lambda} (\alpha > 0), (B)_{\lambda} (\lambda \in (0, \lambda_0)) \) and \( (C) \).

The next proposition asserts the existence of an analytic continuation of the resolvent for \( H_{\text{ext}} \):

**PROPOSITION 2.1** ([2] Lemma II-3 or [3] Theorem 3.1). — **There are** \( \Theta = \{ \ z \in \mathbb{C} \mid |\text{Re} z - \lambda| < \text{const}, \ \text{Im} z > -\text{const.} \} \) : a complex neighborhood of \( \lambda \) such that

\[
\| (H_{\text{ext}}(h, \theta_0) - z)^{-1} \| \leq C \tag{2.1}
\]

for \( 0 < h < h_0 \) and \( z \in \Theta \).

\( T_{\text{int/\text{ext}}} \) denotes the trace operator from \( H^1(\Omega_{\text{int/\text{ext}}}) \) to \( L^2(K) \) and if

Annales de l'Institut Henri Poincaré - Physique théorique
\[ T_{\text{int}} \varphi = T_{\text{ext}} \varphi \] for \( \varphi \in H^1(\Omega_{\text{int}}) \oplus H^1(\Omega_{\text{ext}}) \), we write \( T \varphi = T_{\text{int}} \varphi \). Then \( A(0, a) \) and \( B(0, a) (0 \in \mathbb{R}, a \in \mathbb{C}, \text{Im } a > 0) \) are defined by

\[
A(0, a) = T_{\text{int}}(H(0) - a)^{-1}: L^2(\mathbb{R}^*) \to L^2(K),
\]

\[
B(0, a) = B_{\text{ext}}(0, a) + B_{\text{ext}}(\theta, a): L^2(\Omega_{\text{int}}) \oplus L^2(\Omega_{\text{ext}}) \to \Omega^2(K),
\]

\[
A_{\text{int}}(a) = -T_{\text{int}} V_n(H_{\text{int}} - a)^{-1}: L^2(\Omega_{\text{int}}) \to L^2(K);
\]

\[
B_{\text{ext}}(\theta, a) = e^{-3\theta/2} T_{\text{ext}} V_n(\Omega_{\text{ext}}(\theta) - a)^{-1}: L^2(\Omega_{\text{ext}}) \to L^2(K),
\]

where we have written \( H^P(\theta) = U(\theta) H^P U(\theta) = H_{\text{int}} \oplus H_{\text{ext}}(\theta) \). The above definitions can be extended to \( \theta \in \mathbb{S}_\gamma \).

**Proposition 2.2** (Krein's formula, [2] Appendix II). — For \( a : 0 \),

\[
W(\theta_0, a) = (H(\theta_0) - a)^{-1} - (H^P(\theta_0) - a)^{-1}
= h^2 A(\theta_0, \bar{a}) B(\theta_0, a)
= h^4 B(\theta_0, a) T(H - a)^{-1} T B(\theta_0, a).
\]

**Proposition 2.3** ([2] Theorem II-3). — Let \( a(h) \) be a complex-valued function of \( h \) such that \( \text{Im } a(h) > 0; \text{Re } a(h) \to \lambda \) (\( h \downarrow 0 \)); \( (\text{Im } a(h))^{-1} = O(h^{-p}) \) for some \( p > 0 \). Then \( B(\theta_0, a) = O(h^{-3/2}); A(\theta_0, a) = O(h^{-1/2}) \) and \( W(\theta_0, a) = O(1) \) \( (h \downarrow 0) \).

The proof of Proposition 2.3 is the same as that of [2] Theorem II-3 except for the choice of the cut off function \( \chi \) (cf. (I) Proposition 2.2).

Let \( h_i \downarrow 0 (i \to \infty) \) be a sequence of positive numbers, and let \( J_i = \{ E_i^{1, \ldots, E_i^{P, N} \ldots} \} \) be a set of eigenvalues for \( H_{\text{int}}(h_i) \) such that \( E_i^{k, l} \to \lambda \) \( (i \to \infty) \) for each \( k \in \{ 1, \ldots, N \} \). We set \( \bar{J}_i := \text{inf } J_i, \sup J_i, |J_i| = \text{sup } J_i - \text{inf } J_i \) and \( \Delta_i = \text{dist } (J_i, \sigma(H_{\text{int}}(h_i))) \setminus J_i \). We suppose

**Assumption (E).** — \( \bar{J}_i \cap \sigma(H_{\text{int}}(h_i)) = J_i ; b(h_i)^q \leq \Delta_i + |J_i|^2/\Delta_i = 0(1) \) for some \( b > 0 \) and \( q > 0 \).

Let \( J_i = (\Sigma_{k=1}^N E_i^{k, l}/N \) and let \( a_i = J_i + i \Delta_i. \) \( \Gamma_i = \partial G_i \) is defined as the boundary of the region:

\[
G_i = \{ z \in \mathbb{C} | \text{dist } (\text{Re } z, J_i) < 3\Delta_i/4, | \text{Im } z | < \Delta_i/2 \}.
\]

**Proposition 2.4** ([2] Sect. IV). — Let \( a_i, G_i \text{ and } \Gamma_i \) be as above. Then

1) \[(H^P(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1} = 0(1) ; \]

2) \[(H(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1} = 0(1) \]

uniformly in \( z \in \Gamma_i \) if \( i \to \infty \).

2) For sufficiently large \( i \),

\[
P_i = - (2 \pi i)^{-1} \int_{\Gamma_i} (H(h_i, \theta_0) - z)^{-1} dz
\]
is well-defined, and has the same dimension as
\[ P_i^D = -(2\pi i)^{-1} \int_{\Gamma_i} (H^D(h_i, \theta_0) - z)^{-1} \, dz. \]

3) Let \( Q_i^D = 1 - P_i^D \). Then for sufficiently large \( i \) and for any \( z \in \Gamma_i \), \( Q_i^D((H(h_i, \theta_0) - a_i)^{-1} - (z - a_i)^{-1})Q_i^D \) is invertible on \( \text{Ran} \, Q_i^D \). Furthermore, the inverse is \( O(1) \) uniformly in \( z \in \Gamma_i \) if \( i \to \infty \).

We next consider the case when \( J = \{ E_i^D \} \) and \( E_i^D \) is simple. Then \( (E) \) is equivalent to \((D)\).

**Proposition 2.5 ([2] Theorem V-2).** — Let \( E_i \) be the eigenvalue of \( H(\theta_0, h_i) \) corresponding to \( P_i \), then for any \( \varepsilon > 0 \) there is \( C \) such that
\[ |E_i - E_i^D| \leq C \exp \{- (d_i(K, \Omega_{\text{int}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_i \}. \] (2.8)

§ 3. RESONANCE EIGENFUNCTION AND EIGENPROJECTION

In this section we suppose \((A)_{\varepsilon} (\varepsilon > 0), (B)_{\varepsilon} (\varepsilon \in (0, \lambda_0)), (C) \) and \((D)\).

Let \( \eta_i = \eta_i(\theta_0) \) be an \((E_i)\)-eigenfunction of \( H(h_i, \theta_0) \) such that \( \eta_i = \eta_i^P + \zeta_i \) where \( \eta_i^P \) is a normalized \( \text{int}(h_i) \) and \( (\zeta_i) \perp (\eta_i^P) \). Then

**Proposition 3.1.** — For any \( \varepsilon > 0 \), there is \( C \) such that
\[ \| \zeta_i \| \leq C \exp \{- (d_i(K, \Omega_{\text{int}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_i \}. \] (3.1)

**Proof.** — We write \( R = (H(h_i, \theta_0) - a_i)^{-1} \), \( R^D = (H^D(h_i, \theta_0) - a_i)^{-1} \), \( F = (E_i - a_i)^{-1} \), \( F^D = (E_i^P - a_i)^{-1} \) and \( W = R - R^D \). Then since \( W\eta_i = F\eta_i \), we have
\[ (F - R)\zeta_i = W\eta_i + (F^D - F)\eta_i^P \] (3.2)
by easy computations. Applying \( Q_i^D = 1 - P_i^D \) to the both sides of (3.2), and noting that \( (R - F) \) is invertible on \( \text{Ran} \, Q_i^D \) by Proposition 2.4-(3), we see
\[ \zeta_i = (Q_i^D(F - R)Q_i^D)^{-1}Q_i^DW\eta_i^P. \] (3.3)

By Proposition 2.2 and a trace estimate (see e.g. Lemma III-4 of [2]),
\[ \| W\eta_i^P \| \leq \| A(\theta_0, a_i) \| \cdot \| B_{\text{int}}(a_i)\eta_i^P \| \]
\[ \leq \| A(\theta_0, a_i) \| \cdot (\text{Im} a_i)^{-1} \cdot C \cdot \left( \| \nabla_{\text{int}}\eta_i^P \| : \| \nabla_{\text{int}}\nabla^P\eta_i^P \| \right)^{1/2} \] (3.4)
for a smooth cut off function such that \( \chi = 1 \) on \( K \). For any \( \varepsilon > 0 \), if \( \chi \) is supported in a sufficiently small neighborhood of \( K \), the Agmon estimate gives ([5], [7], see also Proposition 3.3 of (I))
\[ \| \chi\nabla_{\text{int}}\eta_i^P \| + \| \nabla_{\text{int}}\chi\nabla_{\text{int}}\eta_i^P \| \leq C \exp \{- (d_i(K, \Omega_{\text{int}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_i \}. \] (3.5)
Combining (3.4) with Proposition 2.3, Assumption (D) and (3.5), we have
\[ \| W\eta_l^D \| \leq C e h_i^{-1/2} \cdot h_i^{-q} \exp \left\{ - (d_\lambda(K, \Omega_{\text{int}} \setminus \mathcal{I}(\lambda)) - \varepsilon)/h_i \right\} \] (3.6)
for any \( \varepsilon > 0 \). (3.1) follows from (3.3), (3.6) and Proposition 2.4-(3).

**COROLLARY 3.2.** — For any \( \varepsilon > 0 \), there is \( C \) such that
\[ \| T_{\text{int}} \eta_l \| = \| T_{\text{int}} \xi_l \| \leq C \exp \left\{ - (d_\lambda(K, \Omega_{\text{int}} \setminus \mathcal{I}(\lambda)) - \varepsilon)/h_i \right\} . \] (3.7)

**Proof.** — Since \( \eta_l = \eta_l^D + \xi_l \) is an \( E_l \)-eigenfunction of \(- h_i^2 \Delta + \mathcal{V}\) in the distribution sense, we have
\[ (- h_i^2 \Delta + \mathcal{V}) \xi_l = (E_l + E_l^D) \eta_l^D + E_l \xi_l \]
and hence
\[ \| \xi_l \|_{H^2(\Omega_{\text{int}})} \leq C \exp \left\{ - (d_\lambda(K, \Omega_{\text{int}} \setminus \mathcal{I}(\lambda)) - \varepsilon)/h_i \right\} . \] (3.9)
for any \( \varepsilon > 0 \) by (2.7) and (3.1). (3.7) follows immediately from (3.9) and the trace estimate.

**PROPOSITION 3.3.** — Let \( \eta_l(\theta_0) \) be an \( (E_l) \)-eigenfunction of \( H(h_i, \theta_0) \) and let \( \eta_l(\theta_0) \) be an \( (E_l) \)-eigenfunction of \( H(h_i, \theta_0^*) \). Then
\[ P_l \varphi = (\eta_l(\theta_0), \eta_l(\theta_0))^{-1} \eta_l(\theta_0) \cdot (\eta_l(\theta_0), \varphi) . \] (3.10)

**Proof.** — Since \( P_l \) is a rank one operator, \( P_l \) can be written as \( P_l \varphi = f \cdot (g, \varphi) \) for some \( f, g \in L^2(\mathbb{R}^n) \) by the Riesz representation theorem. But since \( P_l \) is an eigenprojection of \( H(h_i, \theta_0) \), \( f = \text{const.} \eta_l(\theta_0) \). On the other hand, \( P_l^* \) is an eigenprojection of \( H(h_i, \theta_0^*) = H(h_i, \theta_0^*) \), and hence \( g = \text{const.} \eta_l(\theta_0) \). Because \( P_l \) is projective, the constant is easily determined to conclude (3.10).

Now, \( \pi_l(\mu) = \pi_l(\theta_0; \mu) \) is defined by
\[ (\varphi, \pi_l(\theta_0; \mu) \psi) = 2\pi i h_i^4 (\eta_l(\theta_0), \eta_l(\theta_0))^{-1} \times \]
\[ \times \left( T_{\text{ext}} \nabla_\theta \Phi(h_i, \mu \varphi) \cdot (T_{\text{ext}} \eta_l(\theta_0), T_{\text{ext}} \nabla_\theta \Phi) - (h_i, \mu \psi) \right) \] (3.11)
for \( \varphi, \psi \in L^2(\mathbb{R}^n) \) and \( \mu \in (0, \infty) \). In fact, \( E_l \) and \( \pi_l \) is independent of \( \theta \). We remark that Proposition 2.1, and hence all the results of Sect. 2.1 hold for \( \theta \) in a neighborhood of \( \theta_0 \), and one can define \( E_l(\theta) \) and \( \pi_l(\theta, \mu) \) similarly (cf. [3], Theorem 3.1).

**PROPOSITION 3.4.** — \( E_l(\theta) \) and \( \pi_l(\theta; \mu) \) is independent of \( \theta \) in a neighborhood of \( \theta_0 \).

**Proof.** — Let \( \theta \) be in the neighborhood of \( \theta_0 \) and let \( \eta_l(\theta_0) \) be an \( (E_l) \)-eigenvector of \( H(h_i, \theta_0) \). If \( \text{Im} \theta = \text{Im} \theta_0 \), we see
\[ (H(h_i, \theta) - E_l) U(\theta - \theta_0) \eta_l(\theta_0) = U(\theta - \theta_0)(H(h_i, \theta_0) - E_l) \eta_l(\theta_0) = 0 \]
and hence \(U(\theta - \theta_0)\eta_i(\theta_0)\) is an \((E_i)\)-eigenvector of \(H(h_i, \theta)\). By the analytic perturbation theory, \(H(h_i, \theta)\) has only one eigenvalue \(E_i(\theta)\) near \(E_i\) if \(|\theta - \theta_0|\) is sufficiently small, and \(E_i(\theta)\) is analytic in \(\theta\). But if \(\text{Im} \theta = \text{Im} \theta_0\) then \(E_i(\theta) = E_i(\theta_0)\), and hence \(E_i(\theta) = E_i(\theta_0)\) in the neighborhood of \(\theta_0\) (cf. [4], Theorem XIII-36). Similarly, if \(\text{Im} \theta = \text{Im} \theta_0\),

\[
(\varphi, \pi_i(\theta, \mu)\psi) = 2\pi i h_i^4(U(\theta - \theta_0)\eta_i(\theta_0), U(\theta - \theta_0)\cdot \eta_i(\theta_0))^{-1} \times \nabla_n \Phi_+(h_i, \mu) \varphi \times (T_{\text{int}} U(\theta - \theta_0)\eta_i(\theta_0), T_{\text{ext}} \nabla_n \Phi_-(h_i, \mu) \psi)
\]

since \(U(\theta - \theta_0) = (U(\theta - \theta_0)^{-1})^*\) and \(T_{\text{int}} U(\theta - \theta_0) = T_{\text{int}}\). The eigenfunction \(\eta_i(\theta)\) can be taken to be analytic in \(\theta\), and hence \(\pi_i(\theta; \mu)\) is also analytic in \(\theta\). Thus \(\pi_i(\theta, \mu) = \pi_i(\theta_0, \mu)\) in the neighborhood of \(\theta_0\).

**Remark 3.5.** — For each \(\theta : 0 < \text{Im} \theta < \text{Im} \theta_0\), \(\pi_i(\theta; \mu)\) is well-defined if \(i\) is sufficiently large since Proposition 2.1 can be considerably improved ([3]). Then the above argument shows that \(E_i\) and \(\pi_i\) is independent of such \(\theta\). On the other hand, the similar result holds for \(\tilde{\pi}_i\), defined by

\[
(\varphi, \tilde{\pi}_i(\mu)\psi) = (T_{\text{ext}} \nabla_n \Phi_+(h_i, \mu) \varphi, (T_{\text{int}} P_i T_{\text{int}}^*) T_{\text{ext}} \nabla_n \Phi_-(h_i, \mu) \psi)
\]

\((\varphi, \psi \in L^2(\mathbb{R}^n))\), instead of \(\pi_i\) under Assumption (E).

### § 4. PROOF OF THEOREMS 1 AND 2

In this section we assume (A)\(\prime\) (\(\alpha > 1\)), (B)\(\prime\) (\(\lambda \in (0, \lambda_0)\)) and (C).

**Proposition 4.1.** — (*A representation of the scattering matrix*)

\[
(\varphi, \{ S(H, H_0; \mu) - S(H^D_0, H_0; \mu) \} \psi) = 2\pi i h_i^4(T_{\text{ext}} \nabla_n \Phi_+(h_i, \mu) \varphi, \{ T_{\text{int}}(H(h_i, \theta) - \mu)^{-1} T_{\text{int}}^* \} T_{\text{ext}} \nabla_n \Phi_-(h_i, \mu) \psi)
\]

for \(\mu\) in a neighborhood of \(\lambda\) and \(\varphi, \psi \in L^2(S^{n-1})\).

**Proof.** — \(T_{\text{int}}(H(h_i, \theta) - \mu)^{-1} T_{\text{int}}^*\) is analytic in \(\theta \in \mathfrak{S} \cap \mathfrak{C}_+\) (\(\mathfrak{C}_+ = \{ z \in \mathbb{C} | \text{Im} z > 0 \}\)) and if \(\text{Im} \theta = \text{Im} \theta'\),

\[
T_{\text{int}}(H(h_i, \theta) - (\mu + i\varepsilon)^{-1} T_{\text{int}}^* = T_{\text{int}}(H(h_i, \theta) - (\mu + i\varepsilon)^{-1} U(\theta - \theta') T_{\text{int}}^* = T_{\text{int}}(H(h_i, \theta) - (\mu + i\varepsilon)^{-1} T_{\text{int}}^* \nabla_n \Phi_-(h_i, \mu) \psi)\]

Hence \(T_{\text{int}}(H(h_i, \theta) - (\mu + i\varepsilon)^{-1} T_{\text{int}}^*\) is independent of \(\theta\). Letting \(\varepsilon \to 0\), \(\theta = 0\) and \(\theta' = \theta_0\), we have

\[
T(H(h_i) - (\mu + i0)^{-1} T_{\text{int}}^* = T_{\text{int}}(H(h_i, \theta_0) - \mu)^{-1} T_{\text{int}}^* \nabla_n \Phi_-(h_i, \mu) \psi)\]

(4.3) and Proposition 5.1 in (I) conclude (4.1).
PROPOSITION 4.2. — Suppose (E), then for any \( \varepsilon > 0 \) there is \( C \) such that for \( \mu \in \mathcal{M}_t = \{ \mu \in \mathbb{R} | \text{dist} (\mathcal{I}_t, \mu) \leq b \cdot g_t / 2 \} \) (\( b \) and \( q \) are the constants in (E)) and \( \varphi, \psi \in L^2(S^{n-1}) \),

\[
| (\varphi, \{ S(H(h_t), H_0(h_t)) ; \mu \} - S(H^0(h_t), H_0(h_t)) ; \mu \} \psi) \\
- 2\pi i h^4 \{ T_{\text{ext}} V_n \Phi^+(h_t, \mu) \varphi, T_{\text{int}}(H(h_t, \theta_0) - \mu)^{-1} P_1 T_{\text{int}} * \} T_{\text{ext}} V_n \Phi^-(h_t, \mu) \psi | \\
\leq C \exp \{ - 2(\mathbf{d}_\lambda(K, \Omega_{\text{ext}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_t \} || \varphi || || \psi || .
\]  

(4.4)

Proof. — By virtue of Proposition 4.1, it is sufficient to prove

\[
| (T_{\text{ext}} V_n \Phi^+(h_t, \mu) \varphi, T_{\text{int}}(H(h_t, \theta_0) - \mu)^{-1}(1 - P_1) T_{\text{int}} * \} T_{\text{ext}} V_n \Phi^-(h_t, \mu) \psi | \\
\leq C \exp \{ - 2(\mathbf{d}_\lambda(K, \Omega_{\text{ext}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_t \} || \varphi || || \psi || .
\]  

(4.5)

By easy computations, we have

\[
(H(h_t, \theta_0) - z)^{-1} \\
= -(H(h_t, \theta_0) - a_t)^{-1}(z - a_t)^{-1} \{ (H(h_t, \theta_0) - a_t)^{-1} - (z - a_t)^{-1} \}^{-1}
\]  

(4.6)

for \( z \in \Gamma_t \) and hence

\[
||(H(h_t, \theta_0) - z)^{-1}|| \\
\leq \|[H(h_t, \theta_0) - a_t]^{-1}|| \cdot |z - a_t|^{-1} \cdot \|[H(h_t, \theta_0) - a_t]^{-1} - (z - a_t)^{-1}\|^{-1}|| \\
\leq C \cdot h_t^{-q}
\]  

(4.7)

uniformly in \( z \in \Gamma_t \) by Propositions 2.1, 2.3 and 2.4-(1). Since \( (H(h_t, \theta_0) - z)^{-1}(1 - P_1) \) is analytic in \( \Gamma_t (= \text{the interior of } \Gamma_t) \),

\[
(H(h_t, \theta_0) - \mu)^{-1}(1 - P_1) = -(2\pi i)^{-1} \int_{\Gamma_t} \frac{(H(h_t, \theta_0) - z)^{-1}(1 - P_1)}{\mu - z} \, dz 
\]  

(4.8)

by Cauchy's integral formula for \( \mu \in \Gamma_t \), (4.7) and (4.8) implies

\[
|| (H(h_t, \theta_0) - \mu)^{-1}(1 - P_1) || \leq C \cdot h_t^{-q} \cdot h_t^{-q} = C \cdot h_t^{-2q}
\]  

(4.9)

uniformly in \( \mu \in \mathcal{M}_t \). The trace estimate and the standard argument for elliptic operators give

\[
|| T_{\text{int}}(H(h_t, \theta_0) - \mu)^{-1}(1 - P_1) T_{\text{int}} * || \leq C \cdot h_t^{-2-2q}
\]  

(4.10) and Corollary 3.4 of (I) conclude (4.5). \( \Box \)

Proof of Theorem 1. — (1.7) follows immediately from Propositions 3.3 and 4.2. By Corollary 3.2 and Corollary 3.4 of (I), we see

\[
| (T_{\text{int}} \eta(\theta_0), T_{\text{ext}} V_n \Phi^+(h_t, \mu) \varphi) | \leq || T_{\text{int}} \eta(\theta_0) || || T_{\text{ext}} V_n \Phi^+(h_t, \mu) \varphi || \\
\leq C \exp \{ - (\mathbf{d}_\lambda(K, \Omega_{\text{int}} \setminus \mathcal{J}(\lambda)) + \mathbf{d}_\lambda(K, \Omega_{\text{ext}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_t \} || \varphi || ;
\]  

(11.11)

\[
| (T_{\text{int}} \eta(\theta_0), T_{\text{ext}} V_n \Phi^-(h_t, \mu) \psi) | \\
\leq C \exp \{ - (\mathbf{d}_\lambda(K, \Omega_{\text{int}} \setminus \mathcal{J}(\lambda)) + \mathbf{d}_\lambda(K, \Omega_{\text{ext}} \setminus \mathcal{J}(\lambda)) - \varepsilon)/h_t \} || \psi || .
\]  

(11.12)

On the other hand, \( (\eta(\theta_0), \eta(\theta_0)) = 1 + o(1) \) by Proposition 3.1, (1.8) follows from (11.11) and (11.12). \( \Box \)

Vol. 50, n° 2-1989.
Theorem 2 can be proved quite analogously by using Corollary 3.8 (instead of Corollary 3.4) of (I) and Corollary 5.2 (instead of Proposition 5.1) of (I), so we omit it.

REFERENCES


(Manuscrit reçu le 3 mars 1988)

(Version révisée reçue le 17 mai 1988)