Preface

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PREFACE

The aim of the present work is to study some interesting phenomena in statistical physics connected with a continuous symmetry property of the model. We restrict ourselves to a special example, because we are unable to overcome the technical difficulties which arise in the study of more general models. But we believe that analogous results hold in the general case, and even the method of proof may be similar.

In this work we investigate the large-scale limit of the equilibrium state of Dyson's vector-valued hierarchical model at low temperatures. This model has a Hamiltonian function invariant with respect to rotations. We are going to show that this invariance (in the literature generally called symmetry), has very deep consequences. In the language of probability theory large-scale limit problems mean questions about multi-dimensional limit theorems for appropriately normalized partial sums of random variables. In problems of statistical physics these random variables have a Gibbs distribution which depends on the Hamiltonian function and a physical parameter, the temperature. We are mainly interested in the dependence of the large-scale limit on this parameter. In interesting models there is a particular value of the parameter, the critical temperature, at which large-scale limit has a non-typical, « critical » behaviour. In this case an unusual normalization has to be applied, and the limit may be non-Gaussian. We are going to show that our model, due to its continuous symmetry, has an even more complex behaviour. It shows a behaviour similar to the critical one for all low temperatures. We explain this in some more detail.

The Hamiltonian function of our model depends on a parameter c which measures the strength of the interaction. It plays a role similar to that of dimension in translation invariant models; it strongly influences the behaviour of the model. At low temperatures a so-called spontaneous magnetization occurs, and it is natural to investigate the large-scale limit in the direction of the magnetization and in the direction orthogonal to it, separately. In our paper [5] we have investigated this problem in the case $\sqrt{2} < c < 2$, *and now we consider the case* $1 < c < \sqrt{2}$, *which is essentially different from the previous one. In the direction orthogonal to the magnetization an unusual normalization has to be applied in both cases, and a Gaussian limit is obtained, but the behaviour in the direction of the magnetization is different* *in the two cases. For* $\sqrt{2} < c < 2$, *a classical normalization has to be applied, and the limit is Gaussian, but for* $1 < c < \sqrt{2}$ *neither the normalization is classical, nor the limit is Gaussian.*

The large-scale limit of Dyson's model can be described by solving two analytical problems. Since they play a most important role in our investigation, and similar problems may occur in other cases we are going to discuss them in more detail.

The first problem arises when we want to determine the asymptotic distribution of the average spin in a large volume with a Gibbs distribution in this volume without boundary conditions. The second one appears in the investigation of the Radon-Nikodym derivative of the limit Gibbs distribution with respect to the Gibbs measure defined in the first problem. Both problems lead to the investigation of products of certain integral operators, more precisely they are equivalent to describing the asymptotic behaviour of this product applied to a function which depends on the temperature. The solution of these two problems enables us to construct the limit Gibbs state and then to investigate its large-scale limit.

Since the investigation of these two problems require essentially different arguments, we have divided our work to two parts. In Part I we solve the problem connected with the behaviour of the average spin, in Part II we consider the problem about the Radon-Nikodym derivative, and carry out the limiting procedure leading to the description of the large-scale limit. Finally, the proofs of some results are presented in an Appendix.

The unusual normalization in the direction orthogonal to the magnetization is connected with the problem about the behaviour of the Radon-Nikodym derivative discussed in Part II. The solution of this problem is similar in cases $1 < c < \sqrt{2}$ *and* $\sqrt{2} < c < 2$. *The situation is quite different with the analytical problem discussed in Part I. Here one has to find the right scaling under which a sequence of real functions converges, and also to describe the limit. A different scaling has to be applied in the cases* $1 < c < \sqrt{2}$ *and* $\sqrt{2} < c < 2$, *and also the limit is different in these two cases. In this problem one has to study the powers of a d-dimensional integral operator whose one-dimensional version has been discussed for instance in [8]. In that paper it has been shown that the « critical » behaviour of our model at a certain parameter value is closely connected with the following instability property of this operator: the behaviour of the function obtained by applying a large power of this operator to a starting function heavily depends on this starting function. The unexpected phenomenon observed in Part I of our work is also closely connected with this instability. In classical probability theory the case is quite different. In that case, a large power of the convolution operator (i. e. of the convolution of a function with itself), which is the natural counterpart of our operator in probability theory, turns all nice* *functions to an almost Gaussian density function, according to the central limit theorem. This difference is crucial, since the above mentioned instability is not a peculiarity of our model, but it is quite common in statistical physics. By our opinion, this is the most essential difference between problems in classical probability theory and probabilistic problems in statistical physics.*

After formulating the main results of Part I in Section 1, we translate the problem into an analytical problem in Section 2. The first important step in its solution consists of finding the right scaling, under which our sequence has a non-trivial limit. Then a problem of the following type arises. There is a sequence of real functions defined with the help of a starting function and a sequence of operators $\mathbf{Q}_n$ *acting on the space of real functions by the recursive relation* $f_{n+1} = \mathbf{Q}_n f_n$, $n = 0, 1, \ldots$. *For large n the operator* $\mathbf{Q}_n$ *can be well approximated by an operator* $\mathbf{T}$, *independent of n. We want to prove that the sequence* $f_n(x)$ *has a limit. We say that such problems belong to the theory of asymptotic renormalization group theory, since it differs from problems of renormalization group theory by the dependence of the operators* $\mathbf{Q}_n$ *on n. Now we have to study a problem of the following type: find the limit of the sequence* $f_n(x)$ *defined by the recursive relations*

$$(*) \qquad f_{n+1} = \mathbf{T} f_n + \varepsilon_n ,$$

where ε_n *is a small error term. Let us remark that problems of this type play a most important role in several mathematical areas like dynamical systems, KAM theory, etc. There is a standard way of attacking such problems, but generally one has to overcome serious mathematical difficulties, connected with the special character of the given problem, when carrying out this program. First the solution of the fixed point equation* $f = \mathbf{T} f$ *has to be found, and then the stability of this fixed point with respect to the operator* $\mathbf{T}$ *has to be investigated, i. e. it has to be studied whether the sequence* $\mathbf{T}^n g$ *tends to the fixed point* f *for a general function g, and whether the convergence is fast enough. If it is so then it is natural to expect that the sequence* f_n *converges to the fixed point* f.

In Section 2 we find the solution of our fixed point equation $f = \mathbf{T} f$ *and show that it is sufficiently stable. This stability argument works only for* $1 < c < \sqrt{2}$, *and this is the reason why the results of Part I do not hold for* $\sqrt{2} < c < 2$ *any longer. Let us remark that we could express the right speed of convergence to the fixed point only in the space of the Fourier transforms, hence we met some technical problems when exploiting this property in the original space.*

We must admit that at this introductory level we have made a simplification in the formulation of the analytical problem. In order to guarantee the convergence of our sequence of functions we had to shift the function $f_n(x)$ *by an appropriate constant* $\mathbf{M}_n$, *which depends on n. The application of this shift is needed to guarantee that the functions* $f_n(x)$ *do not move away to plus or minus infinity. The stability of the fixed point with respect to the operator* $\mathbf{T}$

holds only after the application of this shift in the operator. Originally there was an unstable direction in the problem, and we eliminated it by the help of the sequence M_n. *In the precise proof it is not enough to know the convergence of the sequence* $f_n(x)$ *in itself, but we have to establish the convergence of the sequence of pairs* (f_n, M_n) *to a pair* (f, M) *with some* $\mathrm{M} > 0$. *Moreover, the operator* $\mathbf{T}$ *(defined in the main text) actually depends on* M_n. *On the other hand, studying the pairs* (f_n, M_n) *rather than the functions* f_n *does not cause any essential change in the proof, the convergence of the sequence* M_n *to* M *is sufficiently fast, and the dependence of the operator* $\mathbf{T}$ *on* M_n *is very weak. That is why we decided to disregard the dependence of* M *and* $\mathbf{T}$ *on n, at least at this heuristic level. Nevertheless, there are some interesting problems (the Thouless effect for instance), which are closely connected with the more intricate behaviour of the sequence* M_n. *(See Section 8 in Part II, where some conjectures and open problems are discussed.)*

A most important step in the proof of the result in Part I is to formulate a good inductive hypothesis about the behaviour of the functions $f_n(x)$, *which should reflect their most important properties, in particular their convergence to the fixed point. The formulation of a proper inductive hypothesis, which is closely related to the more intricate behaviour of the operator* $\mathbf{T}$, *is a highly non-trivial problem, and Section 3 is devoted to this question.*

We mention two peculiarities of the analytical problem we are dealing with in Part I, which may seem to be rather technical at first glance. Nevertheless, we think that these phenomena have deeper, non-technical causes. The first peculiarity is that we have to argue differently for small and large indices n. In problems of type (*) *appearing in the literature, the starting function generally contains a so-called small parameter, and the inductive procedure can be carried out for small n owing to this small parameter. In our problem the starting function also contains such a small parameter, and for small n the behaviour of our functions can be controlled by its help. But it does not enable us to carry out the induction procedure for small n, since during this procedure the operator* $\mathbf{Q}_n$ *should have been replaced by* $\mathbf{T}$, *and the resulting error is negligible only for large n. We prove the results necessary for us for small n in Section 4, and thereafter we can restrict our attention to large n. We believe that similar difficulties often appear in statistical physics when the large-scale limit is non-Gaussian.*

In order to prove the convergence of the sequence $f_n(x)$ *to the fixed point, we also have to show that the error term* ε_n *in* (*) *is really small, and this demands unexpectedly serious efforts. This is the second peculiarity of the problem, and it is closely related to the unusual normalization. We discuss this question in more detail. In the integral expression* $\mathbf{Q}_n f_n$, *the function* f_n *has a rather complicated argument*

$$t_n(x, u, v) = c^{-n}\left(\sqrt{\left(\mathrm{M} \pm \frac{x}{c^{n+1}} + \frac{u}{c^n}\right)^2 + \frac{v^2}{c^n}} - \mathrm{M}\right).$$

When we turn to the operator $\mathbf{T}$ *instead of* $\mathbf{Q}_n$ *then we replace this argument by the simpler expression* $\frac{x}{c} \pm u + \frac{v^2}{2M}$, *i. e. by the main term in its Taylor expansion. The smallness of the error term* ε_n *depends on whether the error caused by the replacement of the argument of* f_n *in the expression* $\mathbf{Q}_n f_n$ *is small or not. A relatively simple calculation shows that, roughly speaking, it has the order of* const. $c^{-n} f_n'(x)$. *Here the term* c^{-n} *should guarantee the smallness of this expression, but we also have to know that* $f_n'(x)$ *is not too large. The final result of Part I shows that this statement really holds, because the functions* f_n *converge together with their derivatives. But this information cannot be applied at the start. Hence a separate argument is needed in order to bound the functions* $f_n(x)$ *together with their derivatives. The bound we have given also implies that our sequence of functions is not degenerate, i. e. we have applied the right scaling. The proof of such a bound is one of the most essential steps in the proof, and most technical difficulties arise at this point. We proved this bound in Sections 5 and 6. We had to exploit the stability of the operator* $\mathbf{T}$ *and the heuristic argument leading to the convergence of our sequence to the fixed point in an implicit way. The behaviour of the functions* $f_n(x)$ *and their Fourier transforms had to be controlled simultaneously.*

In Section 7 we prove some properties of the solution of the fixed point equation. With the help of this information we can turn the heuristic argument about the convergence of the sequence f_n *to the fixed point* f *to a rigorous proof. This is done in Section 8. We also need some estimates on the decrease of the functions* f_n *at plus and minus infinity. They are obtained in Sections 9 and 10.*

In Part II, we first construct the limit Gibbs state we are investigating, and then determine its large-scale limit. We make the following construction. We take a small external magnetic field and construct the Gibbs state in a finite volume in the presence of this external field. Then we get the limit Gibbs state by letting the volume tend to infinity and the external field to zero. This construction is explained in the main text in more detail.

Both in the construction of the limit Gibbs field and in the investigation of the large-scale limit, the solution of the following problem plays an important role: take an external magnetic field with some h, construct the Gibbs state in a large volume in the presence of this field, and restrict it to a smaller volume. Consider the Radon-Nikodym derivative of this restricted measure with respect to the Gibbs state without boundary condition in the smaller volume, and give a good asymptotic formula for it. This problem leads to a purely analytical question where the asymptotic behaviour of a sequence of real functions defined recursively by means of an integral operator has to be studied. This problem was solved in the case $\sqrt{2} < c < 2$ *in our paper* [5], *and since both the result and the method of proof is very similar in the present case, we discuss this problem in this introductory part more briefly.*

The most striking feature of this problem is that the Radon-Nikodym derivative has the same asymptotic behaviour in the cases $1 < c < \sqrt{2}$ *and* $\sqrt{2} < c < 2$. *More precisely, there is a typical domain, where we give a good asymptotic formula on the Radon-Nikodym derivative, and the main term in this asymptotic formula is the same in the two cases. In the non-typical domain it is enough to give some upper bounds. On the other hand, the kernel of the integral operator through which the Radon-Nikodym derivative is computed contains the density function of the average spin of the Gibbs state without boundary condition investigated in Part I, and this function is essentially different in the two cases. This means that the integral operators we are working with are different in the two cases, their action nevertheless is the same, at least asymptotically. The reason for this surprising fact is explained in our paper [6]. The right formulation and proof of the upper bound in the non-typical domain is also an important part of the proof, but since it has a rather technical character we do not discuss it here. We only mention that this upper bound is very similar in the cases* $1 < c < \sqrt{2}$ *and* $\sqrt{2} < c < 2$. *The main part, the first five sections of Part II, deals with the investigation of the Radon-Nikodym derivative. Then, with the help of this result and the result of Part I, we can prove the existence of the limit field in Section 6 and describe its large-scale limit in Section 7. Finally, in Section 8 we formulate some conjectures and open problems.*

The results proved in the Appendix can be found in different papers. The only result with some novelty value is the statement that the measure we have constructed through a limit procedure in Part II is really a Gibbs state. There are several similar results in the literature, but we have found none which could have been directly applied in our case. To prove such a statement one has to justify a formal limit procedure, and the main technical difficulty in our case is caused by the facts that the potential of the model has an infinite range interaction, and the spins take values in a non-compact space.

In the Preface we wanted to explain the main results of this work and to discuss the most important and interesting analytical problems needed in the proofs. We have tried to explain our approach to the problems we are investigating without being too technical.
