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On the asymptotics of the negative eigenvalues for an axisymmetric ideal magnetohydrodynamic model

by

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ABSTRACT. — We study the negative spectrum of the linear self-adjoint operator A of the magnetohydrodynamic force in the case of toroidal geometry with axial symmetry. Applying variational methods, we obtain an asymptotic formula which describes the behaviour of the negative eigenvalues of the force operator $A^{(k)}$ with a fixed wavenumber k. This formula implies that the classic Mercier condition for magnetoactive plasma stability is a necessary condition which guarantees that the total multiplicity of the negative eigenvalues of $A^{(k)}$ is finite. Moreover, we establish a corresponding sufficient condition which is quite similar to the Mercier condition and from physical point of view coincides with it.

RÉSUMÉ. — Nous étudions le spectre négatif de l'opérateur linéaire auto-adjoint A de la force magnétohydrodynamique en cas de géométrie toroïdale avec une symétrie axiale. En appliquant des méthodes variationnelles nous obtenons une formule asymptotique qui décrit le comportement des valeurs propres négatives de l'opérateur de la force $A^{(k)}$ avec un nombre d'ondes toroïdal k fixé. Cette formule implique que la condition classique de Mercier pour la stabilité d'un plasma magnétoactif est une condition nécessaire pour garantir que la multiplicité totale des valeurs propres négatives de $A^{(k)}$ est finie. De plus, nous établissons une condition suffisante correspondante qui est très similaire à la condition de Mercier et de point de vue physique coïncide avec elle.

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0. INTRODUCTION

Ideal linear magnetohydrodynamics (MHD) is one of the basic models for description of the interaction between a magnetoactive plasma and an exterior magnetic field. Assume that the plasma is confined in a toroidal domain $\mathcal{O}$ surrounded by a perfectly conducting surface $\partial\mathcal{O}$. Then the modes associated to the negative spectrum of the linear MHD force operator drive various MHD instabilities (see Bateman [*1*]).

The static equilibrium plasma configuration is determined by the magnetic field $\vec{\mathcal{B}}$, pressure $\mathcal{P} \geqslant 0$ and density $\rho > 0$. The constant adiabaticity index is denoted by γ.

The quantities $\vec{\mathcal{B}}$ and $\mathcal{P}$ satisfy the equilibrium equations

$$\operatorname{div} \vec{\mathcal{B}} = 0\,, \tag{0.1}$$

$$\nabla\mathcal{P} = \vec{\mathcal{J}} \times \vec{\mathcal{B}} \tag{0.2}$$

where $\vec{\mathcal{J}} = \operatorname{rot} \vec{\mathcal{B}}$ is the equilibrium electric current density. The normal components of $\vec{\mathcal{B}}$ and $\vec{\mathcal{J}}$ vanish on $\partial\mathcal{O}$, i. e.

$$\vec{n} \cdot \vec{\mathcal{B}}_{\restriction \partial\mathcal{O}} = 0\,, \tag{0.3}$$

$$\vec{n} \cdot \vec{\mathcal{J}}_{\restriction \partial\mathcal{O}} = 0\,, \tag{0.4}$$

where $\vec{n}$ is the unit normal to $\partial\mathcal{O}$ vector.

The plasma displacement vector $\vec{\xi}$ (whose time derivative $\dot{\vec{\xi}}$ coincides with the linear perturbation of the macroscopic plasma velocity) satisfies the equation

$$\ddot{\vec{\xi}} + \mathrm{A}\vec{\xi} = 0 \tag{0.5}$$

where

$$-\,\rho\mathrm{A}\vec{\xi} = \nabla(\vec{\xi}\cdot\nabla\mathcal{P} + \gamma\mathcal{P}\operatorname{div}\vec{\xi}) + (\operatorname{rot}\operatorname{rot}(\vec{\xi}\times\vec{\mathcal{B}}))\times\vec{\mathcal{B}} + \\ + \vec{\mathcal{J}}\times\operatorname{rot}(\vec{\xi}\times\vec{\mathcal{B}}) \tag{0.6}$$

together with the initial conditions

$$\vec{\xi}(0) = \vec{\xi}_0,\ \dot{\vec{\xi}}(0) = \vec{\xi}_1\,, \tag{0.7}$$

and the boundary condition

$$\vec{n}\cdot\vec{\xi}_{\restriction \partial\mathcal{O}} = 0\,. \tag{0.8}$$

Assume, at first, that the force operator A is defined on a domain $\mathrm{D}_0(\mathrm{A})$ which consists of C^∞-functions $\vec{\xi} : \mathcal{O} \to \mathbb{C}^3$ satisfying (0.8). Then it follows from (0.1), (0.2) and (0.3) that A is symmetric and semibounded from below in the Hilbert space (HS) with an inner product generated by the quadratic form (QF)

$$b[\vec{\xi}\,] = \int_{\mathcal{O}} \rho\,|\,\vec{\xi}\,|^2\,d\mathcal{O}\,. \tag{0.9}$$

The operator A corresponds to the QF

$$a[\vec{\xi}] = (A\vec{\xi}, \vec{\xi}) = \int_{\mathcal{O}} \{ |\operatorname{rot}(\vec{\xi} \times \vec{\mathscr{B}})|^2 + \gamma\mathscr{P}|\operatorname{div}\vec{\xi}|^2 + \operatorname{Re}(\overline{\operatorname{div}\vec{\xi}}\,\vec{\xi}\cdot\nabla\mathscr{P} - \vec{\xi}\cdot\vec{\mathscr{J}} \times \operatorname{rot}(\vec{\xi} \times \vec{\mathscr{B}})) \} \, d\mathcal{O}, \quad \vec{\xi} \in D_0(A), \quad (0.10)$$

which determines the potential energy of the plasma. Then it is natural to replace A by its self-adjoint Friedrichs extension.

Consider the solution $\vec{\xi}$ of the initial boundary-value problem defined by (0.5), (0.7) and (0.8). It is clear that the QF $b[\vec{\xi}]$ (which determines the kinetic energy of the plasma) remains bounded with respect to the time iff the negative spectrum of the self-adjoint force operator A is empty. This represents the well-known fact that in the linear approximation the non-negative definiteness of the force operator is equivalent to plasma stability (cf. Mercier and Luc [2]).

There is a wide physical literature concerning the spectral properties of the force operator for various linear MHD models (see Freidberg [3] and the literature cited there). Amongst them, considerable attention is allocated to the axisymmetric model which is treated in the present paper. Axial symmetry means that the domain $\mathcal{O}$ containing the plasma can be represented in the form

$$\mathcal{O} = \{ (r, \varphi, z) : (r, z) \in \Omega, \varphi \in [0, 2\pi) \},$$

where (r, φ, z) are the cylindric coordinates and Ω is a bounded plane domain whose closure does not intersect with the symmetry axis $\{ r = 0 \}$. Moreover the equilibrium functions ρ, $\vec{\mathscr{B}}$ (and hence $\mathscr{P}$) are independent of the variable φ.

Consider the Fourier series $(2\pi)^{-1/2} \sum_{k\in\mathbb{Z}} \vec{\xi}^{(k)}(r, z)e^{ik\varphi}$ for the displacement vector $\vec{\xi}(r, \varphi, z)$. Axial symmetry allows us to establish unitary equivalence between the force operator A and an orthogonal sum of some operators $A^{(k)}$ which are labelled by the integer toroidal wavenumber k. These operators are self-adjoint in the HS with an inner product generated by the QF

$$\int_{\Omega} |\vec{\xi}|^2 \rho r dr dz, \qquad \vec{\xi}(r, z) = (\xi_1, \xi_2, \xi_3).$$

The operators $A^{(k)}$ can be obtained formally if we substitute in (0.6) the derivative with respect to φ for multiplication by ik. They can be treated as force operators with a fixed toroidal wavenumber k.

One of the earliest results concerning the spectrum of the force operator A for the axisymmetric MHD model is the classic necessary condition for plasma stability obtained by Mercier [4] (see below relation (2.1)).

Later, Mercier condition was extended for general toroidal configurations possessing no symmetry (cf. e. g. Mercier and Luc [*2*], § 2.4). Different derivations and detailed comments on the physical meaning of the Mercier condition can be found in Greene and Johnson [*5*] and Freidberg [*3*], section V.D.1.*b*.

However, the Mercier condition is not sufficient for plasma stability even in the case of axial symmetry (see Lortz [*6*] and Lortz and Nührenberg [*7*]). It became clear that the establishment of meaningful plasma stability criteria was closely related with the necessity of a deeper and more precise knowledge of the entire spectrum of the force operator. This was one of the main reasons for which Goedbloed [*8*] undertook a heuristic investigation of the essential spectrum of the force operator $A^{(k)}$ with an arbitrary fixed toroidal wavenumber $k \in \mathbb{Z}$. In particular, it was shown that the essential spectrum of $A^{(k)}$ is non-negative for each k. Besides, the lower bound of the essential spectrum of $A^{(k)}$ coincides with the origin of the spectral axis iff there exists a « rational magnetic surface » $S_{m,k}$ (see below subsection 2.1). Later, Descloux and Geymonat [*9*] obtained analogous results applying rigorous mathematical approach. These results were also confirmed by more recent but independent investigations of Lifshits [*10*] and Hamieri [*11*].

As the negative spectrum of $A^{(k)}$, $k \in \mathbb{Z}$, is discrete, it makes sense to study its behaviour near the origin of the spectral axis. Pao [*12*] established formally a necessary condition for the finiteness of the total multiplicity of the negative eigenvalues of the operator $A^{(k)}$ with an arbitrary fixed k. In other words, the violation of Pao condition (which is similar to Mercier condition but is not identical with it) implies that for some k the negative eigenvalues of $A^{(k)}$ accumulate at the origin. Pao's heuristic approach does not allow to obtain corresponding sufficient conditions which guarantee the finiteness of the total multiplicity of the negative eigenvalues of $A^{(k)}$ for each fixed k.

Goedbloed wrote in his paper [*8*] (which considerably stimulated the present work) that he also had planned to include some results on the behaviour of the negative spectrum of $A^{(k)}$ near the origin. He pointed, however, that the publication of these results had to be put off because of some contradictions with the existing literature which had not been properly overcome.

Note that the essential spectrum of the « total » force operator A may contain strictly negative points which are accumulation points of the « individual » operators $A^{(k)}$ with different k (see Hamieri [*11*] for rigorous results; see also the heuristic arguments in the earlier paper of Dewar and Glasser [*13*]). The points of the negative essential spectrum of A may form whole segments and the rightest of these segments may extend to the origin. Thus, the question about the accumulation at the origin of the negative eigenvalues of the total force operator is unreasonable. The more important and still open problem here is the precise localization of the entire essential spectrum, and especially the negative essential spectrum, of the force operator A.

The aim of the present paper is to study the behaviour of the negative eigenvalues of $A^{(k)}$, $k \in \mathbb{Z}$, near the origin of the spectral axis, applying mathematical methods. No rigorous results concerning this problem have ever been published except the author's short communication [*14*] which contains a weaker version of the results of the present paper.

Under some generic assumptions we obtain rigorous asymptotic formulae (see below theorems 2.1 and 2.2) which describe the behaviour of the negative eigenvalues of $A^{(k)}$ near the origin, k being fixed. It follows from these formulae that Mercier condition is a necessary condition for the finiteness of the total multiplicity of the negative eigenvalues of $A^{(k)}$ for each fixed k. Besides, we establish a corresponding sufficient condition which is quite similar to Mercier condition and from physical point of view coincides with it (see below corollary 2.4).

The results of the paper are obtained by means of a variational technique which has been traditionally used for the investigation of the discrete spectrum of various differential operators and, especially the operators of quantum mechanics (see Birman and Solomjak [*15*], and Reed and Simon [*17*], ch. XIII).

The essential spectrum of the linear MHD force operator can also be investigated by means of variational methods. For example, the precise localization of the essential spectrum of the operators $A^{(k)}$ in [*9*] is achieved by the use of a technique which has much in common with the methods applied in the present paper. We intend to develop our variational approach in order to study the essential spectrum of the total force operator A.

1. NOTATIONS AND PRELIMINARIES

1.1. Let $\mathscr{E} \subset \mathbb{R}^m$, $m \geqslant 1$, be a bounded domain; if $m > 1$, then its boundary $\partial\mathscr{E}$ is supposed to be piecewise smooth. We denote by $L_p^k(\mathscr{E})$ the usual Lebesgue spaces of $\mathbb{C}^k$-valued functions defined on $\mathscr{E}$ ($p \in [1, \infty]$, $k = 1, 2, \ldots$); if $k = 1$, we write $L_p(\mathscr{E})$ instead of $L_p^1(\mathscr{E})$. Let $\alpha \geqslant 0$ be a measurable function defined on $\mathscr{E}$; then $L_p^k(\mathscr{E}, \alpha)$ is the usual α-weighted Lebesgue space.

The standard Sobolev spaces of $\mathbb{C}$-valued functions defined on $\mathscr{E}$ are denoted by $H^l(\mathscr{E})$, $l = 1, 2, \ldots$; $\overset{\circ}{H}{}^l(\mathscr{E})$ is the closure of $C_0^\infty(\mathscr{E})$in the $H^l(\mathscr{E})$-norm. Also, the subspace of $H^l(0, 2\pi)$ which consists of periodic functions is denoted by $\dot{H}^l(0, 2\pi)$; similarly, if Π denotes the rectangle $\{ (\psi, \chi) : \psi \in I \subset \mathbb{R}, \chi \in (0, 2\pi) \}$, then $\dot{H}^l(\Pi)$ is the subspace of $H^l(\Pi)$ consisting of periodic with respect to χ functions.

Further, l_2 denotes the HS of square-summable complex-valued sequences. Similarly, if $\mathscr{H}$ is an arbitrary HS, then $l_2(\mathscr{H})$ denotes the HS of square-summable $\mathscr{H}$-valued sequences.

1.2. Let $Q = Q^*$ be a linear operator in a HS $\mathscr{H}$. Then $\sigma(Q)$ is the spectrum of Q and $E_\Delta(Q)$ is the spectral projection of Q corresponding to the set $\Delta \subset \mathbb{R}$. Put

$$N_\mu(Q) = \dim E_{(-\infty,\mu)}(Q)\mathscr{H}, \ \mu \in \mathbb{R} \ ; \tag{1.1}$$

if Q is compact, we also use the notation

$$n_\mu(Q) = \dim E_{(\mu,\infty)}(Q)\mathscr{H} \equiv N_{-\mu}(-Q), \quad \mu \geqslant 0 . \tag{1.2}$$

Let the inner product in $\mathscr{H}$ be generated by the QF q_1. Assume that q_2 is a closed semibounded QF defined in $\mathscr{H}$. Then q_2 generates by Lax-Milgram theorem a unique linear operator $Q = Q^*$ (see Reed and Simon [*16*], theorem VIII.15). We shall discuss the spectral properties of the QFs ratio q_2/q_1 meaning the corresponding properties of the operator Q and shall write q_2/q_1 instead of Q in the notations of the type of (1.1)-(1.2); if no special notation for the QF which generates the inner product in the HS $\mathscr{H}$ is introduced, we write $q_2/\mathscr{H}$ instead of q_2/q_1.

The domain of a linear operator $Q = Q^*$ is denoted by D(Q) and the domain of a closed semibounded QF q is denoted by $D[q]$. The value of the QF q for any $u \in D[q]$ is denoted by $q[u]$. If q depends on some additional parameters $\vec{p}$, we write $q[u; \vec{p}]$; when we need to indicate only the dependence of the QF q on the parameters $\vec{p}$, we write $q(\vec{p})$.

1.3. The spectral properties of the force operator A essentially depend on the topology of the equilibrium which is determined by the particular choice of the solution of the equations (0.1)-(0.2). Here we describe our assumptions about the equilibrium quantities.

We assume that $\vec{\mathscr{B}}(r, z)$ is a C^∞-function $\overline{\Omega} \rightarrow \mathbb{R}^3$ and $\rho(r, z)$ is a C^∞-function $\overline{\Omega} \rightarrow (0, \infty)$.

The axial symmetry allows to introduce the orthogonal « magnetic coordinate system » (MCS) (ψ, φ, χ) where $\psi = \psi(r, z)$, $\chi = \chi(r, z)$. Then Ω is parametrized by (ψ, χ) varying on the rectangle $\Pi = [0, \Psi) \times [0, 2\pi)$ and $\partial\Omega$ is defined by the equation $\psi = \Psi$. The pressure $\mathscr{P}$ and the quantity $T = r^2 \vec{\mathscr{B}} \cdot \nabla\varphi$ depend only on ψ and satisfy Grad-Shafranov equation

$$\mathscr{U}\psi + r^2 \frac{d\mathscr{P}}{d\psi} + T\frac{dT}{d\psi} = 0, \qquad \mathscr{U}\psi \equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2}.$$

Then we have

$$\vec{\mathscr{B}} = r^{-1}(\nabla\psi \times \vec{e}_\varphi + T\vec{e}_\varphi) \equiv \mathscr{B}_\varphi \vec{e}_\varphi + \mathscr{B}_\chi \vec{e}_\chi ,$$
$$\vec{\mathscr{J}} = r^{-1}(\nabla T \times \vec{e}_\varphi - \mathscr{U}\psi \vec{e}_\varphi) \equiv \mathscr{J}_\varphi \vec{e}_\varphi + \mathscr{J}_\chi \vec{e}_\chi ,$$

where $\vec{e}_\varphi$ and $\vec{e}_\chi$ are the unit vectors parallel respectively to $\nabla\varphi$ and $\nabla\chi$. Thus the equations (0.1)-(0.2) are satisfied. Since the vectors $\vec{\mathscr{B}}$ and $\vec{\mathscr{J}}$ are tangential to the « magnetic surfaces » $\{ \psi = \text{const.} > 0 \}$ and the outermost surface $\{ \psi = \Psi \}$ coincides with $\partial\mathcal{O}$, the boundary conditions (0.3)-(0.4) are satisfied too.

In what follows, for simplicity sake, we assume

$$\mathscr{P}, \mathrm{T} \in \mathrm{C}^\infty [0, \Psi], \tag{1.3}$$

$$\mathrm{T}(\psi) \neq 0, \quad \forall \psi \in [0, \Psi]. \tag{1.4}$$

We suppose that the MCS is non-degenerate everywhere on $\overline{\mathcal{O}}$ except the « magnetic axis », i. e. the circumference $\{ \psi = 0 \}$. It is known that the behaviour of the equilibrium quantities near the magnetic axis significantly influences the spectral properties of the force operator A (see Descloux and Geymonat [*9*] and Hamieri [*11*]). That is why here we describe in detail our assumptions in this respect. For simplicity sake we restrict our attention to an equilibrium configuration where the magnetic surfaces near the magnetic axis represent approximate circular tori. In other words, if the magnetic axis is defined by the equations $r = r_0 > 0$, $z = 0$, and $\mathrm{R}^2 = (r - r_0)^2 + z^2$, $\phi = \operatorname{arctg}(z(r - r_0)^{-1})$, then we assume ψ, $\chi - \phi \in \mathrm{C}^\infty(\overline{\Omega})$, and

$$\begin{aligned} \psi &= \mathrm{R}^2/2 + \psi^{(1)}(\mathrm{R}, \phi)\mathrm{R}^3, \qquad \psi^{(1)} \in \mathrm{C}^\infty, \\ \chi &= \phi + \chi^{(1)}(\mathrm{R}, \phi)\mathrm{R}, \qquad \chi^{(1)} \in \mathrm{C}^\infty, \end{aligned} \tag{1.5}$$

where $\psi^{(1)}$ and $\chi^{(1)}$ are 2π-periodic functions with respect to ϕ. Under these assumptions we have

$$|\nabla\psi|^2/\psi = 2 + \psi^{1/2}\psi^{(2)} + \psi\psi^{(3)}, \qquad \psi^{(2)}, \psi^{(3)} \in \mathrm{C}^\infty(\overline{\Pi}), \tag{1.6}$$

$$r(\psi, \chi) = r_0 + \psi^{1/2} r_1 + \psi r_2, \qquad r_1, r_2 \in \mathrm{C}^\infty(\overline{\Pi}). \tag{1.7}$$

Denote by J the Jacobian of the MCS. Then we have

$$\mathrm{J}(\psi, \chi) > 0, \qquad \forall (\psi, \chi) \in \overline{\Pi}, \tag{1.8}$$

$$\mathrm{J}(\psi, \chi) = r_0 + \psi^{1/2}\mathrm{J}_1 + \psi \mathrm{J}_2, \qquad \mathrm{J}_1, \mathrm{J}_2 \in \mathrm{C}^\infty(\overline{\Pi}). \tag{1.9}$$

It is well-known that there exist equilibrium configurations possessing all the assumed properties (cf. e. g. Mercier and Luc [*2*]; see also Landau and Lifshits [*18*], ch. VIII, § 68, for a simple explicit solution of the Grad-Shafranov equation for the case $d\mathscr{P}/d\psi = \text{const.}$ and $d\mathrm{T}/d\psi = \text{const.}$).

1.4. In the subsection we introduce some functions and differential operations which are connected with the equilibrium quantities and will be met frequently in what follows.

First of all introduce the short-hand notations for the partial derivatives with respect to ψ and χ:

$$\partial_\psi = \partial/\partial\psi, \qquad \partial_\chi = \partial/\partial\chi,$$
$$\partial_{\psi\psi} = \partial^2/\partial\psi^2, \qquad \partial_{\psi\chi} = \partial^2/\partial\psi\partial\chi, \qquad \partial_{\chi\chi} = \partial^2/\partial\chi^2,$$

Further, put

$$\alpha_0 = J\mathcal{B}^2, \quad \alpha_1 = r^{-1}J\mathcal{B}_\varphi, \quad \alpha_2 = \alpha_1 - \alpha_0 T^{-1},$$

and for each $k \in \mathbb{Z}$ introduce a pair of commuting differential operations

$$\mathcal{F}_j\eta = \mathcal{F}_j(k)\eta = \alpha_0^{-1}(-i\partial_\chi\eta + k\alpha_j\eta), \quad \eta : \overline{\Pi} \to \mathbb{C}, \quad j = 1, 2. \quad (1.10)$$

Next, set

$$\alpha_3 = 2\mathcal{B}^{-2}d\mathcal{P}/d\psi$$

and introduce the differential operation

$$\mathcal{D}\eta = \alpha_0^{-1}\partial_\psi(\alpha_0\eta) + \alpha_3\eta, \quad \eta : \overline{\Pi} \to \mathbb{C}.$$

At last, denote by

$$\mathcal{K}_\varphi = \mathcal{B}_\chi\partial_\psi r, \quad \mathcal{K}_\chi = rJ^{-1}\partial_\psi(J\mathcal{B}_\chi)$$

the principal curvatures of the magnetic surfaces and define the quantities

$$\alpha_4 = 2(\mathcal{B}^2 \mid \nabla\psi \mid)^{-1}\mathcal{B}_\varphi^2(\mathcal{K}_\varphi - \mathcal{K}_\chi),$$
$$\alpha_5 = 2(\mathcal{B} \mid \nabla\psi \mid)^{-2}(\mathcal{I}_\chi\mathcal{B}_\varphi\mathcal{K}_\varphi - \mathcal{I}_\varphi\mathcal{B}_\chi\mathcal{K}_\chi),$$
$$\alpha_6 = \mathcal{B}^{-1} \mid \nabla\psi \mid, \quad \alpha_7 = \mathcal{B}_\varphi\mathcal{B}_\chi^{-1}.$$

The asymptotics for $\psi \downarrow 0$ of all the functions $\alpha_j(\psi, \chi), j = 0\text{-}7$, and their derivatives can be found by the use of (1.3) and (1.5)-(1.9). In particular, the estimates

$$\alpha_6^2 = C_1\psi(1 + O(\psi^{1/2})), \quad \psi \downarrow 0, \quad (1.11)$$
$$\alpha_7^2 = C_2\psi^{-1}(1 + O(\psi^{1/2})), \quad \psi \downarrow 0, \quad (1.12)$$

hold with some positive constants $C_j, j = 1, 2$.

Note also that (1.6)-(1.7) imply that the equilibrium magnetic field $\vec{\mathcal{B}}$ has a non-vanishing geodesic curvature, i. e.

$$\partial_\chi\mathcal{B}^2 \not\equiv 0. \quad (1.13)$$

1.5. Define the QF a (see (0.10)) on a domain consisting of vectors $\vec{\xi}$ satisfying the boundary condition (0.8) with components $\xi_j \in H^1(\mathcal{O})$, $j = 1, 2, 3$, which vanish in some vicinity of the magnetic axis. Close then $a[\vec{\xi}]$ in $L_2^3(\mathcal{O}, \rho)$. It is clear that the operator generated by the QFs ratio a/b (see (0.9)) coincides with the self-adjoint force operator A.

Expand $\vec{\xi} \in D[a]$ into a Fourier series with respect to φ:

$$\vec{\xi}(\psi, \varphi, \chi) = (2\pi)^{-1/2} \sum_{k \in \mathbb{Z}} \vec{\xi}^{(k)}(\psi, \chi)^{eik\varphi}$$

and put

$$\eta_1 = \eta_1^{(k)} = \vec{\xi}^{(k)} \cdot \nabla\psi,$$
$$\eta_2 = \eta_2^{(k)} = iT \mid \nabla\psi \mid^{-2}\vec{\xi}^{(k)} \cdot (\nabla\psi \times \vec{\mathcal{B}}),$$
$$\eta_3 = \eta_3^{(k)} = i\vec{\xi}^{(k)} \cdot \vec{\mathcal{B}}.$$

Then we have

$$a[\vec{\xi}] = \sum_{k\in\mathbb{Z}} a_0[\vec{\eta}^{(k)}; k], \qquad b[\vec{\xi}] = \sum_{k\in\mathbb{Z}} b_0[\vec{\eta}^{(k)}],$$

where

$$a_0[\vec{\eta}; k] = \int_\Pi \{ \alpha_6^{-2} | \mathscr{F}_1\eta_1 |^2 + \alpha_7^{-2} | \mathscr{F}_1\eta_2 - \alpha_4\eta_1 |^2 + | \mathscr{D}\eta_1 + \mathscr{F}_2\eta_2 |^2$$
$$+ \gamma\mathscr{P}\mathscr{B}^{-2} | \mathscr{B}^2\mathscr{D}\mathscr{B}^{-2}\eta_1 - \alpha_3\eta_1 + \mathscr{B}^2\mathscr{F}_2\mathscr{B}^{-2}\eta_2 + \mathscr{B}^2\mathscr{F}_1\mathscr{B}^{-2}\eta_3 |^2$$
$$- \alpha_5 | \eta_1 |^2 \} \alpha_0 d\Pi, \; d\Pi \equiv d\psi d\chi, \tag{1.14}$$

$$b_0[\vec{\eta}] = \int_\Pi (\alpha_6^{-2} | \eta_1 |^2 + \alpha_7^{-2} | \eta_2 |^2 + | \eta_3 |^2)\rho\mathscr{B}^{-4}\alpha_0 d\Pi. \tag{1.15}$$

Put

$$D_0[a_0] = \{ \vec{\eta} = (\eta_1, \eta_2, \eta_3) : \eta_j \in \dot{H}^1(\Pi),$$
$$\{ \psi = 0 \} \sqsubset \operatorname{supp} \eta_j, \; j = 1, 2, 3, \; \eta_1 \restriction_{\{\psi = \Psi\}} = 0 \}$$

and define $D[b_0]$ as the closure of $D_0[a_0]$ in the norm generated by the QF b_0 (or, more briefly, in the b_0-norm). Since $\tilde{\alpha}_0 \equiv \rho\mathscr{B}^{-4}\alpha_0$ and $\tilde{\alpha}_0^{-1}$ are in $L_\infty(\Pi)$, it follows from (1.11)-(1.12) that the HS $D[b_0]$ coincides algebraically with the set

$$\{ \vec{\eta} = (\eta_1, \eta_2, \eta_3) : \eta_1 \in L_2(\Pi, \psi^{-1}), \; \eta_2 \in L_2(\Pi, \psi), \; \eta_3 \in L_2(\Pi) \}.$$

Define now the QF $a_0(k)$ on $D_0[a_0]$, close it in $D[b_0]$ and denote by $\mathscr{A}^{(k)}$ the operator generated by the QFs ratio $a_0(k)/b_0$. The force operators $A^{(k)}$ with a fixed toroidal wavenumber k which were discussed in the introduction are unitarily equivalent to the operators $\mathscr{A}^{(k)}$, so that the total force operator A is unitarily equivalent to the orthogonal operator sum $\sum_{k\in\mathbb{Z}} \oplus \mathscr{A}^{(k)}$ defined in the HS $l_2(D[b_0])$.

Put $\qquad \mathscr{N}_\lambda^{(k)} = N_{-\lambda}(\mathscr{A}^{(k)}) \equiv N_{-\lambda}(A^{(k)}), \quad k \in \mathbb{Z}, \quad \lambda > 0,$

i. e. $\mathscr{N}_\lambda^{(k)}$ is the total multiplicity of the eigenvalues of $\mathscr{A}^{(k)}$ (or of $A^{(k)}$) which are smaller than $-\lambda < 0$. The aim of this paper is to study the behaviour of $\mathscr{N}_\lambda^{(k)}$ when $\lambda \downarrow 0$ and $k \in \mathbb{Z}$ is fixed.

2. STATEMENT OF THE MAIN RESULTS

2.1. Set

$$\beta_j(\psi) = (2\pi)^{-1} \int_0^{2\pi} \alpha_j(\psi, \chi) d\chi, \qquad j = 0, 1, 2.$$

The quantity $\beta_1(\psi)$ (whose derivative β_1' is known as the « shear » of the equilibrium magnetic field $\vec{\mathscr{B}}$) has an obvious geometric meaning. If we move along a fixed magnetic field force line lying on the magnetic surface $S(\psi_0) = \{ \psi = \psi_0 \}$, then a shift of the poloidal angle χ by 2π corresponds to a shift of the toroidal angle φ by $-2\pi\beta_1(\psi_0)$. Let $\beta_1(\psi) = -m/n$, $(m, n) \in \mathbb{Z}^2$, $n \neq 0$, for some fixed $\psi \in (0, \Psi]$; then we call $S(\psi)$ a rational magnetic surface and denote it by $S_{m,n}$. If m/n is an irreductible fraction, the magnetic field force lines lying on $S_{m,n}$ close on themselves after m turns in the toroidal direction and n turns in the poloidal direction.

For simplicity sake we assume that the functions β_j, $j = 1, 2$, can take any fixed rational value only on a finite subset of $[0, \Psi]$.

For each $\psi \in (0, \Psi]$ define the quantity

$$\Xi(\psi) = -\left(\int_{S(\psi)} \alpha_5 \mathscr{B}^2 \,|\, \nabla\psi \,|^{-1} dS\right) \int_{S(\psi)} \mathscr{B}^2 \,|\, \nabla\psi \,|^{-3} dS + \frac{1}{4}\left(2\int_{S(\psi)} \vec{\mathscr{J}} \cdot \vec{\mathscr{B}} \,|\, \nabla\psi \,|^{-3} dS - 4\pi^2 \beta_1'(\psi)\right)^2 .$$

Mercier necessary condition for plasma stability means that

$$\Xi(\psi) \geqq 0, \qquad \forall \psi \in (0, \Psi]. \tag{2.1}$$

Let $S_{m,n} = S(\psi)$ be a fixed rational magnetic surface such that $\beta_1'(\psi) \neq 0$. If $S_{m,n} \subset \mathcal{O}$, put

$$\mathscr{C}_{m,n} = (-\,\Xi(\psi))_+^{1/2} / 4\pi^3 \,|\, \beta_1'(\psi) \,| \,;$$

if $S_{m,n} = \partial\mathcal{O}$, the coefficient $\mathscr{C}_{m,n}$ is defined in the same way but it twice smaller.

2.2. THEOREM 2.1. — *i*) Let $0 \neq k \in \mathbb{Z}$, $k\beta_1(0) \bar{\in} \mathbb{Z}$ and $\beta_1' \neq 0$ for each rational magnetic surface $S_{m,k}$. Then we have

$$\lim_{\lambda \downarrow 0} |\log \lambda\,|^{-1} \mathcal{N}_\lambda^{(k)} = \mathscr{C}_k\,, \tag{2.2}$$

where

$$\mathscr{C}_k = \sum_{m \in \mathbb{Z} : \exists S_{m,k} \subset \bar{\mathcal{O}}} \mathscr{C}_{m,k}\,. \tag{2.3}$$

ii) Besides, if $\Xi > 0$ on each rational magnetic surface $S_{m.k}$, we have

$$\mathcal{N}_\lambda^{(k)} = 0(1), \qquad \lambda \downarrow 0\,, \tag{2.4}$$

i. e. the negative eigenvalues of $\mathscr{A}^{(k)}$ do not accumulate at the origin.

REMARK 2.2. — The sum at the right hand side of (2.3) is taken over the integer values of the function $k\beta_1(\psi)$, $\psi \in [0, \Psi]$. Since $0 \neq k \in \mathbb{Z}$ is fixed and β_1 is a bounded function which by assumption can take any fixed rational value only on a finite subset of $[0, \Psi]$, the sum in (2, 3) contains a finite number of terms.

THEOREM 2.3. — The negative eigenvalues of $\mathscr{A}^{(0)}$ do not accumulate at the origin, i. e.

$$\mathcal{N}_\lambda^{(0)} = 0(1), \quad \lambda \downarrow 0. \tag{2.5}$$

COROLLARY 2.4. — Suppose that $\beta_1'(\psi) \neq 0$ for each $\psi \in (0, \Psi]$ and $\beta_1(0)$ is an irrational number.

i) Assume that the total multiplicity of the negative eigenvalues of the operator $A^{(k)}$ is finite for each $k \in \mathbb{Z}$. Then we have $\Xi(\psi) \geqq 0$, $\forall \psi \in (0, \Psi]$.

ii) Let $\Xi(\psi) > 0$, $\forall \psi \in (0, \Psi]$. Then for each $k \in \mathbb{Z}$ the total multiplicity of the negative eigenvalues of $A^{(k)}$ is finite.

2.3. The proofs of theorems 2.1 and 2.3 are based on some auxilliary results established in sections 3-4. In sections 5-7 we estimate $\mathcal{N}_\lambda^{(k)}$ from above and find that the inequality

$$\limsup_{\lambda \downarrow 0} |\log \lambda|^{-1} \mathcal{N}_\lambda^{(k)} \leqslant \mathscr{C}_k \tag{2.6}$$

holds under the hypothesis *i*) of theorem 2.1. In these sections we also show that under the hypothesis *ii*) of theorem 2.1 the estimate (2.4) is valid, and under the hypothesis of theorem 2.3 the estimate (2.5) holds.

In sections 8-9 we estimate $\mathcal{N}_\lambda^{(k)}$ from above in order to demonstrate that the inequality

$$\liminf_{\lambda \downarrow 0} |\log \lambda|^{-1} \mathcal{N}_\lambda^{(k)} \geqslant \mathscr{C}_k \tag{2.7}$$

is valid under the hypothesis *i*) of theorem 2.1. Then (2.6) and (2.7) entail (2.2).

REMARK 2.5. — For the sake of the clarity of exposition, in sections 5-9 we assume that for each $0 \neq k \in \mathbb{Z}$ the functions $k\beta_1(\psi)$ and $k\beta_2(\psi)$ do not take integer values for one and the same $\psi \in (0, \Psi]$. The complementary arguments needed in the general case when this assumption may not be satisfied, are described briefly in section 10.

3. ABSTRACT AUXILLIARY RESULTS

3.1. We begin with a variational lemma (cf. e. g. Birman and Solomjak [*15*]) known as Glasman lemma.

LEMMA 3.1. — Assume that q_1 is a semibounded from below QF defined in a HS with an inner product generated by the QF q_2. Then for each $\mu \in \mathbb{R}$ the quantity $N_\mu(q_1/q_2)$ coincides with the maximum dimension of the linear subsets of $D[q_1]$ whose non-zero elements u satisfy the inequality

$$q_1[u] < \mu q_2[u];$$

in particular, we have

$$N_\mu(q_1/q_2) = N_0(q_1 - \mu q_2/q_2). \tag{3.1}$$

The following corollaries can be easily deduced from lemma 3.1.

COROLLARY 3.2. — Let q_i, $i = 1, 2$, be closed semibounded from below QFs in the HSs $\mathscr{H}_1$ and $\mathscr{H}_2$. Define $\tilde{\mathscr{H}}_i$, $i = 1, 2$, as the HSs with inner products generated by the QFs $q_i[u_i]+t_i \| u_i \|^2_{\mathscr{H}_i}$, $u_i \in D[q_i]$, with sufficiently great t_i. Suppose that there exists a linear bounded operator $\mathscr{W}: \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}_2$ such that Ker $\mathscr{W} = \{ 0 \}$ and the inequality

$$tq_2[\mathscr{W} u_1] \leqq q_1[u_1], \qquad \forall u_1 \in D[q_1],$$

holds with some $t > 0$. Then we have

$$N_0(q_1/\mathscr{H}_1) \leqq N_0(q_2/\mathscr{H}_2).$$

COROLLARY 3.3. — Assume that q is a closed positively definite QF in some HS $\mathscr{H}$ and $\tilde{q}$ is a real-valued QF which is compact in $D[q]$. Then for each $\varepsilon > 0$ we have

$$N_0(\varepsilon q - \tilde{q}/\mathscr{H}) = n_\varepsilon(\tilde{q}/q);$$

hence the quantity $N_0(\varepsilon q - \tilde{q}/\mathscr{H})$ is independent of the particular HS $\mathscr{H}$.

COROLLARY 3.4. — Assume that $Q = Q^*$ is a semi-bounded from below linear operator in some HS $\mathscr{H}$ and $\mathscr{H}_1$ is a subspace of $\mathscr{H}$ such that $\mathscr{H}_2 = \mathscr{H} \ominus \mathscr{H}_1 \subset D(Q)$, dim $\mathscr{H}_2 = d < \infty$. Let P be the orthogonal projection $\mathscr{H} \to \mathscr{H}_1$ and $Q_1 = PQ$ be the self-adjoint in $\mathscr{H}_1$ operator with domain $D(Q_1) = D(Q) \cap \mathscr{H}$. Then we have

$$0 \leqq N_\mu(Q) - N_\mu(Q_1) \leqq d, \qquad \forall \mu \in \mathbb{R}.$$

COROLLARY 3.5. — Assume that $Q_i = Q_i^*$, $i = 1, 2$, are linear operators in some HS and $Q = Q_2 - Q_1$ is a compact operator. Then we have

$$N_\mu(Q_1) \leqq N_{\mu+\varepsilon}(Q_2) + n_\varepsilon(Q), \qquad \forall \mu \in \mathbb{R}, \qquad \forall \varepsilon > 0.$$

3.2. The following lemma is equivalent to the well-known Weyl inequalities for the singular values of compact operators (cf. e. g. Birman and Solomjak [*15*]).

LEMMA 3.6. — Assume that $Q_i = Q_i^*$, $i = 1,2$, are compact linear operators in some HS and $Q = Q_1 + Q_2$. Then we have

$$n_\varepsilon(Q) \leqq n_{\varepsilon(1-\mu)}(Q_1) + n_{\varepsilon\mu}(Q_2), \qquad \forall \varepsilon > 0, \qquad \forall \mu \in (0, 1).$$

3.3. Until the end of this section we study the asymptotics of the negative spectrum of some simple model operators which depend on a small parameter.

For $\tau \in (0, \infty)$ and $\Lambda \in (0, \tau)$ introduce the QFs

$$\gamma_1[f: \Lambda, \tau] = \int_\Lambda^\tau x^2 |f'|^2 dx, \qquad f' \equiv df/dx, \qquad f \in \mathring{H}^1(\Lambda, \tau), \quad (3.2)$$

$$\gamma_2[f; \Lambda, \tau] = \int_\Lambda^\tau |f|^2 dx. \quad (3.3)$$

LEMMA 3.7. — Let $y \in \mathbb{R}$ and

$$\Phi(y) = (2\pi)^{-1}(4y-1)_+^{-1/2}. \quad (3.4)$$

Then for each $\tau \in (0, \infty)$ we have

$$N_0(\gamma_1(\Lambda, \tau) - y\gamma_2(\Lambda, \tau)/\gamma_1(\Lambda, \tau)) = \Phi(y)|\log \Lambda| + 0(1), \qquad \Lambda \downarrow 0. \quad (3.5)$$

Proof. — By the use of corollary 3.3 we obtain

$$N_0(\gamma_1(\Lambda, \tau) - y\gamma_2(\Lambda, \tau)/\gamma_1(\Lambda, \tau)) = n_1(y\gamma_2(\Lambda, \tau)/(\gamma_1(\Lambda, \tau)). \quad (3.6)$$

Change the variables $x \mapsto \tau^{-1}x$ in the QFs $\gamma_j(\Lambda, \tau)$, $j = 1, 2$, and verify the identity

$$n_1(y\gamma_2(\Lambda, \tau)/\gamma_1(\Lambda, \tau)) = n_1(y\gamma_2(\Lambda\tau^{-1}, 1)/\gamma_1(\Lambda\tau^{-1}, 1)). \quad (3.7)$$

The eigenvalues λ_k and the eigenfunctions f_k of the QFs ratio $y\gamma_2(\Lambda\tau^{-1},1)/\gamma_1(\Lambda\tau^{-1})$ satisfy the boundary-value problem

$$\begin{aligned} -\lambda_k(x^2 f_k')' &= y f_k, \\ f_k(\Lambda\tau^{-1}) = f_k(1) &= 0. \end{aligned} \quad (3.8)$$

The equation in (3.8) is an Euler equation; hence it is easy to compute λ_k explicitly and obtain the equality

$$n_1(y\gamma_2(\Lambda\tau^{-1}, 1)/\gamma_1(\Lambda\tau^{-1}, 1)) = \text{ent}\,\{\Phi(y)|\log(\Lambda\tau^{-1})|\}, \quad (3.9)$$

where ent denotes the integer part. Now (3.3) follows directly from (3.6), (3.7) and (3.9).

3.4. Let $\tau > 0$, $c > 0$, $\lambda > 0$. Define the QFs

$$\mathscr{S}_1^{(1)}[f; \lambda, \tau, c] = \int_0^\tau (x^2 + \lambda c)|f'|^2 dx, \qquad f \in \mathring{H}^1(0, \tau), \quad (3.10)$$

$$\mathscr{S}_2^{(1)}[f; \tau] = \int_0^\tau |f|^2 dx; \quad (3.11)$$

the QFs $\mathscr{S}_1^{(2)}(\lambda, \tau, c)$ and $\mathscr{S}_2^{(2)}(\tau)$ are defined analogously except that the interval of integration $(0, \tau)$ is replaced by $(-\tau, \tau)$.

LEMMA 3.8. — Let $y \in \mathbb{R}$. Then we have

$$\begin{aligned} &N_0(\mathscr{S}_1^{(j)}(\lambda, \tau, c) - y\mathscr{S}_2^{(j)}(\tau)/\mathscr{S}_1^{(j)}(\lambda, \tau, c)) \\ &\leqq 2^{-1}j\Phi(y)|\log \lambda| + 0(1), \qquad \lambda \downarrow 0, \quad \forall \tau > 0, \quad \forall c > 0, \quad j = 1, 2. \end{aligned} \quad (3.12)$$

Proof. — First of all, represent the QFs $\mathcal{S}_l^{(2)}$, $l = 1, 2$, as sums of integrals over $(-\tau, 0)$ and $(0, \tau)$, then change the variable $x \mapsto -x$ in the integrals over $(-\tau, 0)$ and apply corollary 3.4. Thus we get

$$\mathrm{N}_0(\mathcal{S}_1^{(2)}(\lambda, \tau, c) - y\mathcal{S}_2^{(2)}(\tau)/\mathcal{S}_2^{(2)}(\lambda, \tau, c)) \\ \leqq 2\mathrm{N}_0(\mathcal{S}_1^{(1)}(\lambda, \tau, c) - y\mathcal{S}_2^{(1)}(\tau)/\mathcal{S}_1^{(1)}(\lambda, \tau, c)) + 1 .$$

Hence it suffices to demonstrate (3.12) for $j = 1$.

Assume $\lambda < \tau^2$. Applying lemma 3.1 and corollaries 3.3 and 3.4 we obtain the estimate

$$\mathrm{N}_0(\mathcal{S}_1^{(1)}(\lambda, \tau, c) - y\mathcal{S}_2^{(1)}(\tau)/\mathcal{S}_1^{(1)}(\lambda, \tau, c)) \leqq \mathrm{N}_0(\gamma_1(\lambda^{1/2}, \tau) \\ - y\gamma_2(\lambda^{1/2}, \tau)/\gamma_1(\lambda^{1/2}, \tau)) + n_1(y\mathcal{S}_2^{(1)}(\lambda^{1/2})/\mathcal{S}_1^{(1)}(\lambda, \lambda^{1/2}, c)) + 1 . \quad (3.13)$$

Change the variable $x \mapsto \lambda^{1/2}x$ in the QFs $\mathcal{S}_1^{(1)}(\lambda, \lambda^{1/2}, c)$ and $\mathcal{S}_2^{(1)}(\lambda^{1/2})$ and verify the identity

$$n_1(y\mathcal{S}_2^{(1)}(\lambda^{1/2})/\mathcal{S}_1^{(1)}(\lambda, \lambda^{1/2}, c)) = n_1(y\mathcal{S}_2^{(1)}(1)/\mathcal{S}_1^{(1)}(1, 1, c)) \quad (3.14)$$

The quantity at the right hand side of (3.14) is independent of λ and finite since the QFs ratio $\mathcal{S}_2^{(1)}(1)/\mathcal{S}_1^{(1)}(1, 1, c)$ generates a compact operator. Now (3.12) for $j = 1$ follows directly from (3.13), (3.14) and lemma 3.7.

4. THE SPECTRAL PROPERTIES OF THE OPERATORS $\mathcal{F}_j$

4.1. In this section we discuss the spectral properties of the first-order differential operators $\mathcal{F}_j(k)$, $k \in \mathbb{Z}$, $j = 1, 2$, (see (0.10)) as the spectral analysis of the operators $\mathcal{A}^{(k)}$ is closely connected with them. Throughout the section, except subsection 4.3, the integer parameter $k \neq 0$ is supposed to be fixed.

At first fix some $\psi \in (0, \Psi]$ and define the first-order ordinary differential operators

$$\mathrm{F}_j(\psi) = \mathrm{F}_j(\psi, k) = \alpha_0^{-1}(\psi)\left(-i\frac{d}{d\chi} + k\alpha_j(\psi)\right), \quad \mathrm{D}(\mathrm{F}_j) = \dot{\mathrm{H}}^1(0, 2\pi), \quad j = 1, 2 .$$

Obviously, $\mathrm{F}_j(\psi)$, $j = 1, 2$, are commuting self-adjoint operators in the HS $\mathcal{L}(\psi) = \mathrm{L}_2((0, 2\pi), \alpha_0(\psi))$. Their eigenvalues $\Theta_m^{(j)}(\psi)$ and normalized eigenfunctions $\mathrm{U}_m(\chi\,; \psi)$ can be written explicitly:

$$\Theta_m^{(j)}(\psi) = \beta_0^{-1}(\psi)(m + k\beta_j(\psi)), \quad (4.1)$$

$$\mathrm{U}_m(\chi\,; \psi) = (2\pi\beta_0(\psi))^{-1/2} \exp i\left\{\int_0^\chi (\Theta_m^{(1)}(\psi)\alpha_0(\psi, \sigma) - k\alpha_1(\psi, \sigma))d\sigma\right\}, \quad (4.2)$$

$$(\mathrm{F}_j(\psi)\mathrm{U}_m)(\chi) = \Theta_m^{(j)}(\psi)\mathrm{U}_m(\chi\,; \psi), \quad j = 1, 2, \quad m \in \mathbb{Z} . \quad (4.3)$$

Hence the zero is an eigenvalue of $F_j(\psi)$, $j = 1, 2$, iff the equality

$$\beta_j(\psi) = - m/k \qquad (4.4)_j$$

holds for some $m \in \mathbb{Z}$. We shall call the values ψ for which $(4.4)_j$ holds with some $m \in \mathbb{Z}$, j-degenerate values of the variable ψ; thus the 1-degenerate values ψ correspond to the rational magnetic surfaces $S_{m,k}$. If ψ is not a j-degenerate value, we shall call it a j-regular value. Consequently $F_j^2(\psi)$ is positively definite for j-regular ψ and the resolvent of $F_j(\psi)$ can be written explicitly:

$$(F_j^{-1}(\psi)\eta)(\chi) = \frac{1}{2}\int_0^{2\pi}\left\{\exp i\left(-k\int_\sigma^\chi \alpha_j(\psi, t)dt\right)\right\}$$
$$\times \{ i \operatorname{sign}(\chi-\sigma) + \operatorname{cotg}(\pi k \beta_j(\psi)) \} \eta(\sigma)\alpha_0(\psi, \sigma)d\sigma, \qquad j = 1, 2. \quad (4.5)$$

For 2-regular ψ we define the bounded self-adjoint operator

$$F(\psi) = F_1(\psi)F_2^{-1}(\psi)$$

and denote by $\Theta_m(\psi) = \Theta_m^{(1)}(\psi)/\Theta_m^{(2)}(\psi)$ its eigenvalues; obviously, the corresponding eigenfunctions are $U_m(\psi)$ (see (4.2)). It is clear that if ψ is 1-regular, then the operator $F^2(\psi)$ is positively definite.

Assume that ψ_1 is a 1-degenerate and 2-regular value. Put $k\beta_1(\psi_1) = -m_1$. Denote by $\langle \cdot, \cdot \rangle_1$ and $\| \cdot \|_1$ respectively the inner product and the norm in the HS $\mathscr{L}(\psi_1)$. It is easy to verify that the estimates

$$\| F_1(\psi)\eta - F_1(\psi_1)\eta \|_1^2 \leqq c(\psi - \psi_1)^2(\| F_1(\psi_1)\eta \|_1^2$$
$$+ | \langle \eta, U_{m_1}(\psi_1) \rangle_1 |^2), \qquad \eta \in \dot{H}^1(0, 2\pi), \quad (4.6)$$

$$\| F(\psi)\eta - F(\psi_1)\eta \|_1 \leqq c | \psi - \psi_1 | \, \| \eta \|_1, \qquad \eta \in L_2(0, 2\pi), \quad (4.7)$$

$$\| F_2^{-1}(\psi)\eta - F_2^{-1}(\psi_1)\eta \|_1 \leqq c | \psi - \psi_1 | \, \| F_2^{-1}(\psi_1)\eta \|_1, \qquad \eta \in L_2(0, 2\pi), \quad (4.8)$$

hold with some c which is independent of η and ψ, if $| \psi - \psi_1 |$ is sufficiently small. Checking (4.6)-(4.8), take into account that α_0^{-1}, α_j, $j = 0 - 2$, and their derivatives are continuous with respect of ψ. Checking (4.7)-(4.8), apply also the standard resolvent identity.

Generally, a 1-degenerate value ψ may be also 2-degenerate. Such values ψ will be referred to as completely degenerate values.

REMARK 4.1. — The assumption described in remark 2.5 means that there is no completely degenerate values $\psi \in [0, \Psi]$. As stated in remark 2.5, the general case when completely degenerate values ψ may be present on $(0, \Psi]$ is treated in section 10. It is essential that for completely degenerate values ψ we have

$$- k\beta_1(\psi) = m_1 \neq m_2 = - k\beta_2(\psi), \qquad (4.9)$$

i.e. different eigenfunctions $U_{m_1}(\psi)$ and $U_{m_2}(\psi)$ correspond to the zero eigenvalues of $F_1(\psi)$ and $F_2(\psi)$.

4.2. Put $\tilde{\Pi} = (\psi', \psi'') \times (0, 2\pi)$ where $0 \leqslant \psi' < \psi'' \leqslant \Psi$. Then the HSs $L_2(\tilde{\Pi}, \alpha_0 \psi^{j-1})$, $j = 1, 2$, can be represented as direct integrals

$$L_2(\tilde{\Pi}, \alpha_0 \psi^{j-1}) = \int_{\psi'}^{\psi''} \oplus \mathscr{L}(\psi) \psi^{j-1} d\psi .$$

Define the self-adjoint in $L_2(\tilde{\Pi}, \alpha_0 \psi^{j-1})$ operators

$$\mathscr{F}_j = \mathscr{F}_j(\psi', \psi''; k) = \int_{\psi'}^{\psi''} \oplus F_j(\psi, k) \psi^{j-1} d\psi, \qquad j = 1, 2, \quad (4.10)$$

where $D(F_j(\psi, k)) = \dot{H}^1(0, 2\pi)$ for almost every $\psi \in (\psi', \psi'')$. Then (4.3) and (4.10) entail

$$\sigma(\mathscr{F}_j(\psi', \psi''; k)) = \overline{\bigcup_{m \in \mathbb{Z}} \bigcup_{\psi \in [\psi', \psi'']} \Theta_m^{(j)}(\psi; k)}$$

(see Reed and Simon [*17*], theorem XIII.85). Hence, if all $\psi \in [\psi', \psi'']$ are j-regular $\mathscr{F}_j^2$ is positively definite and $\mathscr{F}_j^{-1}$ is bounded. If there are j-degenerate values $\psi \in [\psi', \psi'']$, then $0 \in \sigma(\mathscr{F}_j(\psi', \psi''))$. However, the set of all j-degenerate values $\psi \in [0, \Psi]$ is finite, i.e. it is a set of vanishing measure, so that the zero is not an eigenvalue of $\mathscr{F}_j(\psi', \psi'')$; hence the operators $\mathscr{F}_j^{-1}$, $j = 1, 2$, are well defined although not bounded. The self-adjoint in $L_2(\tilde{\Pi}, \alpha_0 \psi)$ operator $\mathscr{F} = \overline{\mathscr{F}_1 \mathscr{F}_2^{-1}}$ is well defined on $D(\mathscr{F}_2^{-1})$; if there are no 1-degenerate values $\psi \in [\psi', \psi'']$, the operator $\mathscr{F}^2(\psi', \psi'')$ is positively definite (the essential domain of $\mathscr{F}$ is $D(\mathscr{F}_2^{-1}) \cap L_2(\tilde{\Pi})$).

4.3. In the case $k = 0$ the self-adjoint in $L_2(\Pi, \alpha_0 \psi^{j-1})$ operators $\mathscr{F}_j(0) = -i\alpha_0^{-1}\partial_\chi$, $j = 1, 2$, are defined as in (4.10) with $\psi' = 0$, $\psi'' = \Psi$, $F_j(\psi, 0) = -i\alpha_0^{-1}(\psi) d/d\chi$, and $D(F_j(\psi, 0)) = \dot{H}^1(0, 2\pi)$ for almost every $\psi \in (0, \Psi)$. Set

$$\tilde{\mathscr{L}} = \int_0^{\psi} \oplus (\mathscr{L}(\psi) \ominus \{ 1 \}) \psi d\psi .$$

Note that $\mathscr{F}_2(0)$ is invertible in $\tilde{\mathscr{L}}$ and the closure of the operator $\mathscr{F}(0) = \mathscr{F}_1(0)\mathscr{F}_2^{-1}(0)$ which is defined at first on $\tilde{\mathscr{L}} \cap L_2(\Pi)$, coincides with the identity operator in $\tilde{\mathscr{L}}$.

5. ESTIMATION OF $\mathscr{N}_\lambda^{(k)}$ FROM ABOVE

i) Reduction to a scalar operator.

5.1. The aim of this section is to introduce a QFs ratio $a_1^{(+)}(\lambda)/b_1^{(+)}$ (see subsection 5.2), $\zeta \in D[a_1^{(+)}]$ being a scalar function, and establish the estimate $\mathscr{N}_\lambda^{(k)} \leqslant N_0(a_1^{(+)}(\lambda)/b_1^{(+)}) + 0(1)$, $\lambda \downarrow 0$.

In this subsection we define an auxilliary second order quasielliptic operator $\mathcal{G}$.

Introduce the variables $x_1 = (2\psi)^{1/2} \cos \chi$, $x_2 = (2\psi)^{1/2} \sin \chi$. The change of the variables $(\psi, \chi) \mapsto \vec{x}=(x_1, x_2)$ maps the rectangle $\Pi=(0, \Psi) \times (0, 2\pi)$ onto the circle B with a radius $\mathcal{R} = (2\Psi)^{1/2}$, and the operator

$$(\mathcal{T} w)(\psi, \chi) = w(\vec{x}(\psi, \chi))$$

is an isometric mapping from $L_2(B, \alpha_0)$ onto $L_2(\Pi, \alpha_0)$,

Define the differential operators

$$\mathcal{D}^* = -\partial_\psi + \alpha_3, \qquad \mathcal{G} = \mathcal{D}\alpha_6^2\mathcal{D}^* + \mathcal{F}_2\alpha_7^2\mathcal{F}_2 .$$

Set $\tilde{\mathcal{G}} = \mathcal{T}^*\mathcal{G}\mathcal{T}$, $D(\tilde{\mathcal{G}}) = \{ w \in H^2(B) : (\vec{v}\cdot\nabla w - \mathcal{R}\alpha_3 w) \} \restriction_{\partial B} = 0 \}$ where $\vec{v} = |\vec{x}|^{-1}\vec{x}$, and define the domain of the operator $\mathcal{G}$ as $D(\mathcal{G}) = \mathcal{T} D(\tilde{\mathcal{G}})$.

LEMMA 5.1. — *i)* The operator $\mathcal{G} = \mathcal{G}(k)$, $k \in \mathbb{Z}$, with a domain $D(\mathcal{G})$ is self-adjoint in $L_2(\Pi, \alpha_0)$;

ii) The resolvent of $\mathcal{G}$ is compact;

iii) $\mathcal{G}$ is a positively definite operator;

iv) For each $v \in D(\mathcal{G})$ we have

$$c_0 \int_\Pi (\psi \,|\, \partial_{\psi\chi} v \,|^2 + \psi^{-1} \,|\, \partial_{\chi\chi} v \,|^2) d\Pi \leqq \| \mathcal{G} v \|^2_{L_2(\Pi,\alpha_0)}, \quad c_0 > 0, \tag{5.1}$$

$$c_1 \int_\Pi (\psi \,|\, \partial_\psi v \,|^2 + \psi^{-1} \,|\, \partial_\chi v \,|^2 + \psi^{-1/2} \,|\, v \,|^2) d\Pi \leqq \| \mathcal{G}^{1/2} v \|^2_{L_2(\Pi,\alpha_0)}, \quad c_1 > 0. \tag{5.2}$$

Proof. — *i)* Straightforward calculations show that the operator $\alpha_0\tilde{\mathcal{G}}$ can be written in the form $\alpha_0\tilde{\mathcal{G}} = \mathcal{G}_1 + \mathcal{G}_2$, where

$$\mathcal{G}_1 w = -\operatorname{div} \Gamma\nabla w + G_0 w, \qquad \mathcal{G}_2 w = i(\vec{G}\cdot\nabla w + \operatorname{div}(\vec{G} w)),$$

$\Gamma(\vec{x}) = \{ \Gamma_{q,s} \}_{q,s=1,2}$ is a matrix-valued function with coefficients

$$\Gamma_{11} = g_1 x_1^2 + g_2 x_2^2, \qquad \Gamma_{12} = \Gamma_{21} = (g_1 - g_2) x_1 x_2,$$
$$\Gamma_{22} = g_1 x_2^2 + g_2 x_1^2, \qquad g_1 = \alpha_0 \,|\, \vec{x} \,|^{-4} \alpha_6^2, \qquad g_2 = \alpha_0^{-1} \alpha_7^2,$$

and

$$G_0 = g_1 \alpha_3^2 \,|\, \vec{x} \,|^4 + k^2 g_2 \alpha_2^2 + \operatorname{div} \vec{\gamma}, \qquad \vec{\gamma} = (\gamma_1, \gamma_2) = g_1 \alpha_3 \,|\, \vec{x} \,|^2 \vec{x},$$
$$\vec{G} = (G_1, G_2) = (k\alpha_2 g_2 x_2, -k\alpha_2 g_2 x_1).$$

Besides the boundary condition

$$(\vec{v}\cdot\nabla w - \mathcal{R}\alpha_3 w)\restriction_{\partial B} = 0$$

satisfied by $w \in D(\mathcal{G})$ is equivalent to the condition

$$(\vec{v}\cdot\Gamma\nabla w - \gamma_0 w)\restriction_{\partial B} = 0, \qquad \gamma_0 = \vec{\gamma}\cdot\vec{v}. \tag{5.3}$$

The assumptions about the analytic properties of the equilibrium functions described in subsection 1.3 imply that the matrix $\Gamma(\vec{x})$ is positively definite for every $\vec{x} \in \overline{B}$. Moreover, the coefficients $\Gamma_{q,s}$, $q, s = 1, 2$, and their first derivatives are in $L_\infty(B)$. Similarly we have $G_0 \in L_\infty(B)$ and $\gamma_0 \in C^\infty(\partial B)$. Then it follows from the general spectral theory of second-order elliptic operators that the operator $\mathscr{G}_1$ with a domain consisting of $H^2(B)$-functions which satisfy the boundary condition (5.3), is self-adjoint in $L_2(B)$ (see Berezanskii [*19*], ch. VI, theorem 1.5, where the self-adjointness of $\mathscr{G}_1$ is proved under slightly more restrictive assumptions about the smoothness of the coefficients than necessary for our purposes; the cited theorem, however, can be extended under considerably less restrictive assumptions in this respect-cf. e. g. Ladyženskaya and Uraltseva [*20*], § 17).

Since $\vec{G} \cdot \vec{v} = 0$, it follows from Greene formula that the operator $\mathscr{G}_2$ is formally self-adjoint in $L_2(B)$. As far as G_s and $\partial G_s/\partial x_j$, $s=1, 2$, $j=1, 2$, are in $L_\infty(B)$, $\mathscr{G}_2$ is a continuous mapping from $\overset{\circ}{H}{}^1(B)$ to $L_2(B)$. Therefore $\alpha_0 \mathscr{G} = \mathscr{G}_1 + \mathscr{G}_2$ is self-adjoint in $L_2(B)$ and $\mathscr{G}$ is selfadjoint in $L_2(B, \alpha_0)$. Since $\mathscr{G}$ is unitarily equivalent to $\tilde{\mathscr{G}}$, it is self-adjoint in $L_2(\Pi, \alpha_0)$.

ii) Next we consider the QF of the operator $\mathscr{G}$. Fix $w \in D(\mathscr{G})$ and set $v = \mathscr{T} w$. Then we have

$$(\tilde{\mathscr{G}} w, w)_{L^2(B,\alpha_0)} = (\mathscr{G} v, v)_{L^2(\Pi,\alpha_0)} = \int_\Pi \alpha_0(\alpha_6^2 \mid \mathscr{D}^* v \mid^2 + \alpha_7^2 \mid \mathscr{F}_2 v \mid^2) d\Pi \; ; \quad (5.4)$$

hence $\mathscr{G}$ is non-negatively definite. Apply the estimates (1.11)-(1.12) and take into account that α_0, $\alpha_7^2\alpha_2^2$ and $\alpha_6^2\alpha_3^2$ are in $L_\infty(\Pi)$. Then we obtain the following Gårding type estimate

$$(\mathscr{G} v, v)_{L^2(\Pi,\alpha_0)} \geqq c' \int_\Pi (2\psi \mid \partial_\psi v \mid^2 + (2\psi)^{-1} \mid \partial_\chi v \mid^2 + \mid v \mid^2) d\Pi - c'' \int_\Pi \mid v \mid^2 d\Pi$$
$$= c' \parallel w \parallel^2_{H^1(B)} - c'' \parallel w \parallel^2_{L^2(B)}, \qquad c' > 0, \qquad c'' > 0 . \qquad (5.5)$$

Since $H^1(B)$ is compactly embedded in $L_2(B)$, it follows from (5.5) that the resolvent of $\mathscr{G}$ is compact.

iii) Since the spectrum of $\mathscr{G}$ is purely discrete and non-negative, it suffices to show that the equality

$$(\mathscr{G} v, v)_{L^2(\Pi,\alpha_0)} = 0 \qquad (5.6)$$

implies $v = 0$ in order to demonstrate the positive definiteness of $\mathscr{G}$. It follows from (5.4) that (5.6) entails

$$\mathscr{D}^* v = 0, \qquad \mathscr{F}_2 v = 0 . \qquad (5.7)$$

If $k \neq 0$, the second equality in (5.7) implies that $v = 0$, since the zero is not an eigenvalue of the operator $\mathscr{F}_2$. If $k = 0$, the equalities (5.7) imply that $\partial_\chi v = 0$ and $v = v_0 \mathscr{B}^{-2} \exp(2\mathscr{P} + \mathscr{B}^2)$ where $v_0 = v_0(\chi)$ is independent of ψ. Since the quantities $\mathscr{P}(\psi)$ and $\lim_{\psi \downarrow 0} \mathscr{B}^2(\psi, \chi)$ are independent of χ, v_0 should be independent of χ as well. At last, (1.13) entails $v_0 = 0$.

iv) Let $w \in D(\tilde{\mathscr{G}})$. Since $\tilde{\mathscr{G}}$ is positively definite, the following *a priori* estimate

$$c_0' \| w \|^2_{H^2(B)} \leqq \| \tilde{\mathscr{G}} w \|^2_{L^2(B,\alpha_0)} \tag{5.8}$$

holds with some positive constant c_0' (see Ladyženskaya and Uraltseva [20], § 8). Set $v = \mathscr{T} w$. Then we have

$$\begin{gathered} c_0'' \int_\Pi (\psi \mid \partial_{\psi\chi} v \mid^2 + \psi^{-1} \mid \partial_{\chi\chi} v \mid^2) d\Pi \leqq \| w \|^2_{H^2(B)} \quad c_0'' > 0 \,, \\ \| \tilde{\mathscr{G}} v \|^2_{L^2(B,\alpha_0)} = \| \mathscr{G} w \|^2_{L^2(\Pi,\alpha_0)} \,. \end{gathered} \tag{5.9}$$

Now (5.8)-(5.9) entail (5.1) with $c_0 = c_0' c_0''$.

In order to check (5.2), take into account the estimate (5.5) and verify that Sobolev embedding theorem and Hölder inequality imply that the QF.

$$\int_\Pi (2\psi)^{-1/2} \mid v \mid^2 d\Pi = \int_B \mid \vec{x} \mid^{-1} \mid w \mid^2 d\vec{x}$$

is bounded in $H^1(B)$.

5.2. Introduce the QF

$$\tilde{a}_1^{(+)}[\zeta] = \int_\Pi \{ \psi^{-1} (\mid \partial_\chi \zeta \mid^2 + \mid \zeta \mid^2) + \psi (\mid \partial_\psi \zeta \mid^2 + \mid \mathscr{F}_2^{-1} \mathscr{D} \zeta \mid^2) \} \, d\Pi$$

and define $D[\tilde{a}_1^{(+)}]$ as the closure of the set

$D_0[\tilde{a}_1^{(+)}] = \{ \zeta \in \dot{H}^1(\Pi), \zeta_{\restriction \{\psi = \Psi\}} = 0, \{ \psi = 0 \} \in \operatorname{supp} \zeta, \mathscr{D}\zeta \in D(\mathscr{F}_2^{-1}) \}$.

in the $\tilde{a}_1^{(+)}$-norm. It is easy to see that $D[\tilde{a}_1^{(+)}]$ can be described as the set of distributions ζ such that:

i $\tilde{a}_1^{(+)}[\zeta] < \infty$;
ii) $\zeta(\psi, 0) = \zeta(\psi, 2\pi)$ for almost every $\psi \in (0, \Psi)$;
iii) $\zeta(\Psi, \chi) = 0$ for almost every $\chi \in (0, 2\pi)$.

Note that if ζ satisfies *i*), then *ii*) and *iii*) make sense and, besides, we have $\mathscr{D}\zeta \in D(\mathscr{F}_2^{-1})$.

Define the QF

$$b_1^{(+)}[\zeta] = b_1^{(+)}[\zeta; k] = \int_\Pi (\alpha_6^{-2} \mid \zeta \mid^2 + \alpha_7^{-2} \mid \mathscr{F}_2^{-1} \mathscr{D} \zeta \mid^2) \alpha_0 d\Pi \tag{5.10}$$

whose domain is the closure of $D[\tilde{a}_1^{(+)}]$ in the $b_1^{(+)}$-norm. Set

$$\begin{aligned} a_1^{(+)}[\zeta] = a_1^{(+)}[\zeta; \lambda, \varepsilon, k] = \int_\Pi \{ & (1-\varepsilon)(\alpha_6^{-2} \mid \mathscr{F}_1 \zeta \mid^2 + \alpha_7^{-2} \mid \mathscr{F} \mathscr{D} \zeta \mid^2) \\ & + 2\alpha_8 \operatorname{Re} \mathscr{F} \mathscr{D} \zeta \bar{\zeta} - (\alpha_5 - \alpha_4 \alpha_8 + \varepsilon \psi^{-1/2}) \mid \zeta \mid^2 \} \alpha_0 d\Pi + \lambda c_1^{(+)} b_1^{(+)}[\zeta] \end{aligned} \tag{5.11}$$

where $D[a_1^{(+)}] = D[\tilde{a}_1^{(+)}]$, $\varepsilon < 1$, $\lambda > 0$, $c_1^{(+)} = \min_{(\psi,\chi) \in \bar{\Pi}} \rho \mathscr{B}^{-4}$ and

$$\alpha_8 = \alpha_7^{-2} \alpha_4 \,. \tag{5.12}$$

The QF $a_1 + tb_1$ generates an equivalent norm in $D[\tilde{a}_1^{(+)}]$, provided that $t > 0$ is sufficiently great. Hence the QF $a_1^{(+)}$ is closed and semi-bounded from below in $D[b_1^{(+)}]$.

5.3. Introduce the operator

$$\mathscr{W}_1 = \mathscr{W}_1(k) = \begin{pmatrix} \alpha_6^2\mathscr{D}^* & 1 & 0 \\ \alpha_7^2\mathscr{F}_2 & -\mathscr{F}_2^{-1}\mathscr{D} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad k \in \mathbb{Z}, \tag{5.13}$$

with a domain

$$D(\mathscr{W}_1) = \{ \vec{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}) : \zeta^{(1)} \in D(\mathscr{G}),\ \zeta^{(2)} \in D[a_1^{(+)}],\ \zeta^{(3)} \in D(\mathscr{F}_1) \}.$$

We consider $D(\mathscr{W}_1)$ as a HS with an inner product generated by the QF

$$W_1(\vec{\zeta}\,) = \int_\Pi |\mathscr{G}\zeta^{(1)}|^2 \alpha_0 d\Pi + \tilde{a}^{(+)}[\zeta^{(2)}] + \int_\Pi (|\partial_\chi \zeta^{(3)}|^2 + |\zeta^{(3)}|^2) d\Pi. \tag{5.14}$$

LEMMA 5.2. — For each $k \in \mathbb{Z}$ the operator $\mathscr{W}_1(k)$ is an isomorphism from $D(\mathscr{W}_1)$ to $D[a_0^{(+)}]$.

Proof. — *i*) At first we describe a functional set which is wider than $D_0[a_0]$ but still is included in $D[a_0]$. Fix some $t > 0$ such that the QF $a_0 + tb_0$ (see (1.14)-(1.15)) generates an inner product in $D[a_0]$; note that this QF is equivalent to the QF

$$\tilde{a}_0[\vec{\eta}\,] = \int_\Pi \{ |\mathscr{D}\eta_1 + \mathscr{F}_2\eta_2|^2 + \psi|\partial_\psi \eta_1|^2 + \psi^{-1}(|\partial_\chi \eta_1|^2 + |\eta_1|^2)$$
$$+ \psi(|\partial_\chi \eta_2|^2 + |\eta_2|^2) + |\partial_\chi \eta_3|^2 + |\eta_3|^2 \} d\Pi.$$

We define $D_0[\tilde{\alpha}_0]$ as the set of vector distributions $\vec{\eta}$ such that:

i) $$\tilde{a}_0[\vec{\eta}\,] < \infty; \tag{5.15}$$

ii) $$\operatorname{supp} \eta_i \not\supset \{\psi = 0\}, \qquad i = 1, 2; \tag{5.16}$$

iii) $$\eta_i(\psi, 0) = \eta_i(\psi, 2\pi), \qquad i = 1, 2, 3, \tag{5.17}$$

for almost every $\psi \in (0, \Psi)$;

iv) $$\eta_1(\psi, \chi) = 0 \qquad \text{for almost every } \chi \in (0, 2\pi). \tag{5.18}$$

It is clear that $D_0[\tilde{a}_0]$ is dense in $D[a_0]$.

Fix some $\vec{\zeta} \in D(\mathscr{W}_1)$. The estimates (5.1)-(5.2) imply that the inequalities

$$\pm\, \tilde{a}_0[\mathscr{W}_1\vec{\zeta}\,] \leqq \pm\, c_\pm W_1[\zeta] \tag{5.19$_\pm$}$$

hold with some positive constants $c_\pm$. Set $\vec{\eta} = \mathscr{W}_1\vec{\zeta}$. Then $(5.19)_+$ entails (5.15). It is clear that $\vec{\eta}$ satisfies (5.17)-(5.18) but, generally, does not satisfy (5.16). Nevertheless it is easy to see that $\vec{\eta}$ can be approximated in respect to the $\tilde{a}_0$-norm by functions which are in $D_0[a_0]$. Hence $(5.19)_+$ implies that $\mathscr{W}_1 : D(\mathscr{W}_1) \to D[a_0]$ is a bounded operator.

ii) Now fix some $\vec{\eta} \in D[a_0]$ and define $\zeta^{(1)}$ as the unique solution of the operator equation

$$\mathscr{G}\zeta^{(1)} = \mathscr{D}\eta_1 + \mathscr{F}_2\eta_2 .$$

Next, set $\zeta^{(2)} = \eta_1 - \alpha_6^2\mathscr{D}^*\zeta^{(1)}$. Then we have $\mathscr{F}_2^{-1}\mathscr{D}\zeta^{(2)} = \alpha_7^2\mathscr{F}_2\zeta^{(1)} - \eta_2$. Hence it is clear that $\zeta^{(2)} \in D[a_1^{(+)}]$. At last, put $\zeta^{(3)} = \eta_3$ and $\vec{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)})$. Obviously we have $\vec{\zeta} \in D(\mathscr{W}_1)$ and $\mathscr{W}_1\vec{\zeta} = \vec{\eta}$; besides, $\vec{\eta} = 0$ implies $\vec{\zeta} = 0$. Consequently the inverse operator $\mathscr{T}_1^{-1} : D[a_0] \to D(\mathscr{T}_1)$ exists. Moreover, the estimate (5.19)$_-$ implies that $\mathscr{W}_1^{-1}$ is a bounded operator.

5.4. Lemma 5.3. — The estimate

$$\mathscr{N}_\lambda^{(k)} \leqq N_0(a_1^{(+)}(\lambda, \varepsilon, k)/b_1^{(+)}(k)) + 0(1), \qquad \lambda \downarrow 0, \tag{5.20}$$

holds for each $k \in \mathbb{Z}$ and $\varepsilon \in (0, 1)$.

Proof. — First of all, note that the QF

$$\tilde{b}_0[\vec{\eta}] = \int_\Pi (\alpha_6^{-2}|\eta_1|^2 + \alpha_7^{-2}|\eta_2|^2 + |\eta_3|^2)\alpha_0 d\Pi$$

generates an equivalent norm in $D[b_0]$. Hence we have

$$\mathscr{N}_\lambda^{(k)} = N_0(a_0 + \lambda b_0/\tilde{b}_0), \qquad \forall\lambda \geqq 0 .$$

Further, fix some $\vec{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}) \in D(\mathscr{W}_1)$ such that $\zeta^{(2)} \in D_0[\tilde{a}_1^{(+)}]$. Integrating by parts and using (5.1)-(5.2) we find that the estimate

$$a_0[\mathscr{W}_1\vec{\zeta}] + \lambda b_0[\mathscr{W}_1\vec{\zeta}] \geqslant a_1^{(+)}[\zeta^{(2)}; \lambda, \varepsilon, k] \\ + c' \int_\Pi \alpha_0(|\mathscr{G}\zeta^{(1)}|^2 - c|\mathscr{G}^{1/2}\zeta^{(1)}|^2)d\Pi, \quad \forall\lambda > 0, \quad \forall\varepsilon \in (0, 1), \tag{5.21}$$

holds with some independent of λ positive constants c' and $c = c(\varepsilon)$. Checking (5.21), take into account that the only term in the QF $a_0 + \lambda b_0$ containing η_3, i. e. the term

$$\int_\Pi \{ \gamma\mathscr{P}\mathscr{B}^{-2}|\mathscr{B}^2\mathscr{D}\mathscr{B}^{-2}\eta_1 - \alpha_3\eta_1 + \mathscr{B}^2\mathscr{F}_2\mathscr{B}^{-2}\eta_2 + \mathscr{B}^2\mathscr{F}_1\mathscr{B}^{-2}\eta_3|^2 \\ + \lambda\rho\mathscr{B}^{-4}|\eta_3|^2 \} \alpha_0 d\Pi, \tag{5.22}$$

is non-negative. Since $D_0[\tilde{a}_1^{(+)}]$ is dense in $D[a_1^{(+)}]$, the estimate (5.21) can be extended to all $\vec{\zeta} \in D(\mathscr{W}_1)$.

Besides we have

$$\tilde{b}_0[\mathscr{W}_1\vec{\zeta}] = \int_\Pi (|\mathscr{G}^{1/2}\zeta^{(1)}|^2 + |\zeta^{(3)}|^2)\alpha_0 d\Pi + b_1^{(+)}[\zeta^{(2)}] .$$

Thus we obtain

$$N_0(a_0 + \lambda b_0/\tilde{b}_0)) \leqq N_0(a_1^{(+)}(\lambda, \varepsilon, k)/b_1^{(+)}(k)) + n_{1/c}(\mathscr{G}^{-1/2}) . \tag{5.23}$$

The quantity $n_{1/c}(\mathscr{G}^{-1/2})$ is independent of λ and is finite as far as $\mathscr{G}^{-1/2}$ is a compact operator. Therefore, (5.23) entails (5.20).

5.5. *Proof of Theorem 2.3.* — Integrating by parts, we obtain

$$a_1^{(+)}[\zeta;\lambda,\varepsilon,0] \geqslant a_1^{(+)}[\zeta;0,\varepsilon,0]$$
$$= \int_\Pi \{ (1-\varepsilon)(\alpha_7^{-2} \mid \mathscr{D}\zeta \mid^2 + \alpha_0^{-2}\alpha_6^{-2} \mid \partial_\chi \zeta \mid^2) - \tilde{\alpha}_5(\varepsilon) \mid \zeta \mid^2 \} \alpha_0 d\Pi,\ \zeta \in \mathrm{D}_0[\tilde{a}_1^{(+)}].$$

where $\tilde{\alpha}_5 = \alpha_5 - \alpha_4\alpha_8 - 2\alpha_3\alpha_8 + \partial_\psi(\alpha_8/\alpha_0) + \varepsilon\psi^{-1/2}$. Note that $\psi^{1/2}\alpha_5 \in \mathrm{L}_\infty(\Pi)$ (see subsections 1.3-1.4 and (5.12)); then we have

$$a_1^{(+)}[\zeta;0,\varepsilon,0] \geqq c' \int_\Pi \{ \psi^{-1} \mid \partial_\chi\zeta \mid^2 + \psi \mid \partial_\psi \zeta \mid^2 + \mid \zeta \mid^2 \} d\Pi$$
$$- c'' \int_\Pi \psi^{-1/2} \mid \zeta \mid^2 d\Pi,\ c'(\varepsilon) > 0,\ c''(\varepsilon) > 0,\ \forall \varepsilon \in (0,1),\ \zeta \in \mathrm{D}_0[a_1^{(+)}]. \quad (5.24)$$

It is easy to check that the independent of λ operator $\mathscr{Q}$ which is generated by the QF*s* ratio

$$\int_\Pi \psi^{-1/2} \mid \zeta \mid^2 d\Pi \Big/ \int_\Pi \{ \psi^{-1} \mid \partial_\chi \zeta \mid^2 + \psi \mid \partial_\psi \zeta \mid^2 + \mid \zeta \mid^2 \} d\Pi, \qquad \zeta \in \mathrm{D}[a_1^{(+)}],$$

is compact. Then the crucial estimate (5.24) together with lemma 3.1 and and corollary 3.3 imply that the inequality

$$\mathrm{N}_0(a_1^{(+)}(0,\varepsilon,0)/b_1^{(+)})) \leqq n_\mu(\mathscr{Q}), \qquad \mu = c''/c', \quad (5.25)$$

holds for each $\lambda \geqq 0$. Thus (5.20) for $k = 0$ and (5.25) entail (2.5).

In what follows we fix $0 \neq k \in \mathbb{Z}$ and, as a rule, omit it in the notations.

6. ESTIMATION OF $\mathscr{N}_\lambda^{(k)}$ FROM ABOVE

ii) Localization with respect of ψ.

6.1. Let the open intervals $\{ \mathrm{I}_l \}$, $l \geqq 0$, form a finite partition of $(0, \Psi)$:

$$\mathrm{I}_l \cap \mathrm{I}_j = \varnothing, \qquad l \neq j;$$
$$\bigcup \bar{\mathrm{I}}_l = [0, \Psi]. \quad (6.1)$$

We assume that $0 \in \bar{\mathrm{I}}_0$ and $\bar{\mathrm{I}}_0 \backslash \{ 0 \}$ does not contain any *j*-degenerate values ψ, $j = 1, 2$. Note that under hypothesis *i*) of theorem 2.1 the operator $\mathrm{F}_1^2(\psi)$ is positively definite for every $\psi \in \bar{\mathrm{I}}_0$. Besides, we assume that if some $\bar{\mathrm{I}}_l$, $l \geqq 1$, contains a 1-degenerate value ψ, then $\psi \in \mathrm{I}_l$ (unless $\psi = \Psi$) and $\bar{\mathrm{I}}_l$ contains no other 1-degenerate values and no 2-degenerate values. Such partition of $(0, \Psi)$ is possible since in remark 2.5 we assumed that $[0, \Psi]$ does not contain any completely degenerate values (see also remark 4.1).

Put $\Pi_l = I_l \times (0, 2\pi)$, $l \geqq 0$. Introduce the QF $a_2^{(+)}(\lambda, \varepsilon, I_l)$ and $b_2^{(+)}(I_l)$ which are completely analogous respectively to the QFs $a_1^{(+)}(\lambda, \varepsilon)$ and $b_1^{(+)}$ except that the domain of integration Π is replaced by Π_l.

If $l \geqslant 1$, we set

$$D[a_2^{(+)}(I_l)] = \{ \zeta \in \dot{H}^1(\Pi_l) : \mathscr{D}\zeta \in D(\mathscr{F}_2^{-1}) \} ;$$

if $l = 0$, we define $D[a_2^{(+)}(I_0)]$ as the closure of the set

$$\{ \zeta \in \dot{H}^1(\Pi_0) : \{ \psi = 0 \} \subset \operatorname{supp} \zeta \}$$

with respect to the norm generated by the QF

$$\int_{\Pi} \{ \psi(|\partial_\psi \zeta|^2 + |\mathscr{F}_2^{-1}\mathscr{D}_\zeta|^2) + \psi^{-1}(|\partial_\chi \zeta|^2 + |\zeta|^2) \} \, d\Pi .$$

For all $l \geqslant 0$ we define $D[b_2^{(+)}(I_l)]$ as the closure of $D[a_2^{(+)}(I_l)]$ in the $b_2^{(+)}$-norm.

Obviously we have

$$N_0(a_1^{(+)}(\lambda, \varepsilon)/b_1^{(+)}) \leqq \sum_{l \geqq 0} N_0(a_2^{(+)}(\lambda, \varepsilon, I_l)/b_2^{(+)}(I_l)), \qquad \forall \lambda \geqq 0, \qquad \forall \varepsilon \in (0, 1). \quad (6.2)$$

6.2. We group the intervals $I_l, \mathscr{I} \geqq 0$, into two disjoint sets $\mathscr{I}_j, j = 1, 2$, putting $I_l \in \mathscr{I}_1$, if I_l contains a 1-degenerate value, and $I_l \in \mathscr{I}_2$, if all $\psi \in \bar{I}_l$ are 1-regular.

LEMMA 6.1. — Let $I_l \in \mathscr{I}_2$. Then we have

$$N_0(a_2^{(+)}(\lambda, \varepsilon, I_l)/b_2^{(+)}(I_l)) = 0(1), \qquad \lambda \downarrow 0, \qquad \forall \varepsilon \in (0, 1). \quad (6.3)$$

Proof. — Assume at first $l \geqslant 1$. Since the operator $\mathscr{F}^2$ is positively defined in $L_2(\Pi_l, \alpha_0)$, we get the estimate

$$a_2^{(+)}[\zeta; \lambda, \varepsilon, I_l] \geqslant a_2^{(+)}[\zeta; 0, \varepsilon, I_l] \geqslant \geqq c' \| \zeta \|^2_{H^1(\Pi_l)} - c'' \| \zeta \|^2_{L^2(\Pi_l)}, \qquad c' > 0, \qquad c'' > 0. \quad (6.4)$$

As far as $H^1(\Pi_l)$ is compactly embedded in $L_2(\Pi_l)$, the estimate (6.3) with $l > 0$ follows from (6.4) and corollaries 3.2 and 3.3.

The proof for $l = 0$ is quite similar but a trifle more complicated due to the degeneracy of the MCS on the magnetic axis. Moreover we have $F_2(0)U_0(0) = 0$ (see (4.2)-(4.3)), so that $\psi = 0$ is always a 2-degenerate value.

Set

$$\zeta_0(\psi) = \langle \zeta, U_0 \rangle_{\mathscr{L}(\psi)}, \qquad \zeta \in D[a_2^{(+)}(I_0)],$$

$$a_{2,1}[\zeta] = \int_{\Pi_0} (\psi \mid \partial_\psi \zeta \mid^2 + \psi^{-1} \mid \partial_\chi \zeta \mid^2 + \mid \zeta \mid^2) d\Pi + \int_{I_0} \psi^{-1} \mid d\zeta_0/d\psi \mid^2 d\psi ,$$

$$a_{2,2}[\zeta] = \int_{\Pi_0} \psi^{-1/2} \mid \zeta \mid^2 d\Pi + \int_{I_0} \psi^{-2} \mid \zeta_0 \mid^2 d\psi + \int_0^{2\pi} \mid \zeta(\tilde{\psi}, \chi) \mid^2 d\chi ,$$

where $\tilde{\psi}$ denotes the right end of the interval I_0. Then we have an analogous to (6.4) but more sophisticated estimate

$$a_2^{(+)}[\zeta ; \lambda, \varepsilon, I_0] \geqq a_2^{(+)}[\zeta ; 0, \varepsilon, I_0] \geqq c' a_{2,1}[\zeta] - c'' a_{2,2}[\zeta], \quad c' > 0, \quad c'' > 0 . \tag{6.5}$$

It is not difficult to verify that the QFs ratio $a_{2,2}/a_{2,1}$ generates a compact operator; hence (6.5) entails (6.3) for $l = 0$.

Combining (6.2) and (6.3), we get the estimate

$$N_0(a_1^{(+)}(\lambda, \varepsilon)/b_1^{(+)}) \leqq \sum_{l : I_l \in \mathscr{I}_1} N_0(a_2^{(+)}(\lambda, \varepsilon, I_l)/b_2^{(+)}(I_l)) + 0(1), \quad \lambda \downarrow 0, \quad \forall \varepsilon \in (0, 1) . \tag{6.6}$$

6.3. Further we study the asymptotic behaviour of the quantities $N_0(a_2^{(+)}(\lambda, I_l)/b_2^{(+)}(I_l))$, $I_l \in \mathscr{I}_1$. Their analysis is uniform with respect to l; therefore, without any loss of generality, we may assume that $\mathscr{I}_1$ consists of a single interval I_1. For definiteness sake we suppose that $\bar{I}_1 \subset (0, \Psi)$; the insignificant differences in the argument when Ψ, being a 1-degenerate value, is the right end of I_1 are discussed in remark 7.1 (see below). Denote by ψ_1 the 1-degenerate value contained in I_1 and set $I_1 = (\psi_1 - \tau, \psi_1 + \tau)$, where $\tau > 0$ is sufficiently small.

Define the QF $a_3^{(+)}(\lambda, \varepsilon, \tau)$ as the QF $a_2^{(+)}(\lambda, \varepsilon, I_1)$ with a domain

$$D[a_3^{(+)}] = \{ \zeta \in D[a_2^{(+)}(I_1)] : \zeta_{\restriction\{\psi = \psi_1 \pm \tau\}} = 0 \} .$$

Put

$$b_3^{(+)}(\tau) = b_2^{(+)}(I_1) .$$

Although the inclusion $D[a_3^{(+)}] \subset D[a_2^{(+)}]$ is in the « wrong » direction, the estimate

$$N_0(a_2^{(+)}(\lambda, \varepsilon', I_1)/b_2^{(+)}(I_1)) \leqq N_0(a_3^{(+)}(\lambda, \varepsilon, \tau)/b_3^{(+)}(\tau)) + 0(1), \quad \lambda \downarrow 0 . \tag{6.7}$$

holds for each $1 > \varepsilon > \varepsilon' > 0$. The proof of (6.7) is quite simple and purely technical, so that we omit it.

7. ESTIMATION OF $\mathcal{N}_\lambda^{(k)}$ FROM ABOVE

iii) Localization with respect to the Fourier component.

7.1. Set $\alpha_9 = \alpha_3 + \partial_\psi(\log \alpha_0)$. In this section we denote by $\alpha_{j,1}(\chi)$ and $\beta_{j,1}$ respectively the values of $\alpha_j(\psi, \chi)$, $j = 0\text{-}9$ and $\beta_j(\psi)$, $j = 0\text{-}2$, at $\psi = \psi_1$. Besides, $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ (or $\| \cdot \|_1$ and $\| \cdot \|_2$) denote the inner product (or the norm) respectively in the HSs $L_2((0, 2\pi), \alpha_{0,1})$ and $L_2((0, 2\pi), \alpha_{0,1}\alpha_{7,1}^{-2})$.

Put $k\beta_{1,1} = -m_1$ (hence $m_1 \in \mathbb{Z}$) and introduce the notations

$$\theta_m = \Theta_m(\psi_1), \qquad m \neq m_1, \qquad \theta_{m_1} = \frac{d\Theta_{m_1}}{d\psi}(\psi_1);$$

$$u_m = u_m(\chi) = U_m(\chi; \psi_1), \qquad m \in \mathbb{Z};$$

$$v = v(\chi) = F(\psi_1)\partial_\psi U_{m_1}(\chi; \psi_1);$$

$$w_m = w_m(\chi) = F(\psi_1)\alpha_{9,1}u_m + \alpha_{4,1}u_m, \qquad m \in \mathbb{Z};$$

$$\zeta_m = \zeta_m(\psi) = \langle \zeta, u_m \rangle_1, \qquad m \in \mathbb{Z}, \qquad \zeta \in D[a_3^{(+)}].$$

Change the variable $x = \psi - \psi_1$ and denote by « prime » the derivative with respect to x. Then we have

$$\mathcal{F}\mathcal{D}\zeta + \alpha_4\zeta = (x(\theta_{m_1}u_{m_1} + v) + \tilde{u}_{m_1})\zeta'_{m_1} + \sum_{m \neq m_1} (\theta_m u_m + \tilde{u}_m)\zeta'_m + \sum_{m \in \mathbb{Z}} (w_m + \tilde{w}_m)\zeta_m,$$

where $\tilde{u}_m$ and $\tilde{w}_m$ satisfy the estimates

$$\| \tilde{u}_{m_1}(x) \|_1 = 0(x^2), \quad \| \tilde{u}_m(x) \|_1 = 0(x), \quad x \to 0, \quad m_1 \neq m \in \mathbb{Z}, \tag{7.1}$$

$$\| \tilde{w}_m(x) \|_1 = 0(x), \quad x \to 0, \quad m \in \mathbb{Z}. \tag{7.2}$$

In order to verify (7.1)-(7.2), apply the estimate (4.7) and take into account that the equilibrium quantities are smooth with respect to $\psi \in \bar{I}_1$.

7.2. Introduce the QFs

$$a_{4,1}^{(+)}[\zeta; \lambda, \varepsilon, \tau] = (1 - \varepsilon)\int_{-\tau}^{\tau} \left\{ \left\| x(\theta_{m_1}u_{m_1} + v)\zeta'_{m_1} + \sum_{m \neq m_1} \theta_m u_m \zeta'_m \right\|_2^2 + \left\| \alpha_{6,1}^{-1} \sum_{m \neq m_1} \Theta_m^{(1)}(\psi_1) u_m \zeta_m \right\|_1^2 + \lambda c_2^{(+)} |\zeta'_{m_1}|^2 \right\} dx,$$

$$c_2^{(+)} = c_1^{(+)} \min_{\chi\in[0,2\pi]} \alpha_{7,1}^{-2},$$

$$a_{4,2}^{(+)}[\zeta;\varepsilon,\tau] = \int_{-\tau}^{\tau} \Big\{ (\langle \alpha_{5,1}u_{m_1}, u_{m_1}\rangle_1 - \| w_{m_1}\|_2^2 + \varepsilon)|\zeta_{m_1}|^2 - 2\,\mathrm{Re}\,\overline{\zeta}_{m_1}(\langle \theta_{m_1}u_{m_1} + v, w_{m_1}\rangle_2 x\zeta'_{m_1} + \sum_{m\neq m_1} \theta_m \langle u_m, w_{m_1}\rangle_2 \zeta'_m) \Big\} dx,$$

$$a_{4,3}^{(+)}[\zeta;\tau,c] = c \sum_{m\neq m_1} \int_{-\tau}^{\tau} |\zeta_m|^2 dx, \qquad c > 0,$$

$$a_4^{(+)}(\lambda,\varepsilon,\tau,c) = a_{4,1}^{(+)}(\lambda,\varepsilon,\tau) - a_{4,2}^{(+)}(\varepsilon,\tau) - a_{4,3}^{(+)}(\tau,c). \tag{7.3}$$

The domain $D[a_4^{(+)}]$ is the closure of finite sequences $\zeta = \{\zeta_m\}_{m\in\mathbb{Z}}$, $\zeta_m \in C_0^\infty(-\tau,\tau)$, in respect to the norm generated by the QF

$$\tilde{a}_4^{(+)}[\zeta;\tau] = \sum_{m\in\mathbb{Z}} \int_{-\tau}^{\tau} (|\zeta'_m|^2 + m^2|\zeta_m|^2)dx. \tag{7.4}$$

Next we set

$$b_4^{(+)}[\zeta;\tau] = \sum_{m\in\mathbb{Z}} \int_{-\tau}^{\tau} (|\zeta_m|^2 + |\zeta'_m/\Theta_m^{(2)}(\psi_1)|^2)dx \tag{7.5}$$

and define $D[b_4^{(+)}]$ as the closure of $D[a_4^{(+)}]$ in the $b_4^{(+)}$-norm.

Identify $\zeta \in D[a_3^{(+)}]$ with the sequence of its Fourier components $\{\zeta_m\}$, $m \in \mathbb{Z}$, and apply the estimates (4.6)-(4.8) and (7.1)-(7.2) in order to verify that the estimate

$$a_3^{(+)}[\zeta;\lambda,\varepsilon',\tau] \geqq a_4^{(+)}[\zeta;\lambda,\varepsilon'',\tau,c] \tag{7.6}$$

holds for each $\varepsilon' \in (0,1)$, $\varepsilon'' \in (\varepsilon',1)$ which satisfy the inequality $\varepsilon'\psi^{-1/2} < \varepsilon''$, $\forall\psi \in \bar{I}_1$, and some independent of λ constant $c > 0$, if τ is sufficiently small. Besides, the QFs $b_3^{(+)}(\tau)$ and $b_4^{(+)}$ are equivalent. Consequently we have

$$N_0(a_3^{(+)}(\lambda,\varepsilon',\tau)/b_3^{(+)}(\tau)) \leqq N_0(a_4^{(+)}(\lambda;\varepsilon'',\tau,c)/b_4^{(+)}(\tau)). \tag{7.7}$$

Applying corollary 3.3 and lemma 3.6 we obtain the inequality

$$N_0(a_4^{(+)}(\lambda,\varepsilon'',\tau,c)/b_4^{(+)})(\tau)) \leqq n_{1-\mu}(a_{4,2}^{(+)}(\varepsilon'',\tau)/a_{4,1}^{(+)}(\lambda,\varepsilon'',\tau)) + n_\mu(a_{4,3}^{(+)}(\tau,c)/a_{4,1}^{(+)}(\lambda,\varepsilon'',\tau)), \qquad \forall\mu \in (0,1). \tag{7.8}$$

It is easy to check that the second term at the right hand side of (7.8) is bounded for each $\mu > 0$, uniformly with respect to $\lambda \geqslant 0$.

Put

$$\begin{aligned} a_5^{(+)}(\lambda,\varepsilon,\tau) &= a_4^{(+)}(\lambda,\varepsilon,\tau,c) + a_{4,3}^{(+)}(\tau,c), \qquad D[a_5^{(+)}] = D[a_4^{(+)}], \\ b_5^{(+)}(\tau) &= b_4^{(-)}(\tau), \qquad D[b_5^{(+)}] = D[b_4^{(+)}]. \end{aligned} \tag{7.9}$$

Fix $\varepsilon' \in (0,1)$, $\varepsilon \in (\varepsilon',1)$ and choose $\mu \in (0,1)$ so that $(1-\mu)(1-\varepsilon') > 1-\varepsilon$. Then (7.7)-(7.8) entail

$$N_0(a_3^{(+)}(\lambda,\varepsilon',\tau)/b_3^{(+)}(\tau)) \leqq N_0(a_5^{(+)}(\lambda,\varepsilon,\tau)/b_5^{(+)}(\tau)) + 0(1), \lambda \downarrow 0. \tag{7.10}$$

7.3. The next step is to estimate from below the QF $a_5^{(+)}(\lambda, \varepsilon, \tau)$ by some QF which depends only on ζ_{m_1} but not ζ_m, $m \neq m_1$. Of course, we can fix ζ_{m_1} and try to minimize $a_5^{(+)}[\zeta]$ with respect to ζ_m, $m \neq m_1$, but it would be difficult to solve explicitly the arising infinite sequence of Euler-Lagrange equations. That is why we subject the QFs $a_5^{(+)}$ and $b_5^{(+)}$ to some further transformations.

Set

$$b_6^{(+)}[\zeta\,;\tau] = \int_{-\tau}^{\tau} \left\{ \left\| \theta_{m_1}(x\zeta_{m_1})' u_{m_1} + \sum_{m \neq m_1} \theta_m \zeta_m' u_m \right\|_2^2 + |\zeta_{m_1}'|^2 \right\} dx ,$$

define $D[b_6^{(+)}]$ as the closure of $D[a_5^{(+)}]$ in the $b_6^{(+)}$-norm and apply corollary 3.3 in order to verify the equality

$$N_0(a_5^{(+)}(\lambda, \varepsilon, \tau)/b_5^{(+)}(\tau)) = N_0(a_5^{(+)}(\lambda, \varepsilon, \tau)/b_6^{(+)}(\tau)) . \tag{7.11}$$

Next we introduce the QF

$$a_6^{(+)}[\zeta\,;\lambda, \varepsilon, \tau] = a_5^{(+)}[\zeta\,;\lambda, \varepsilon, \tau] - (1-\varepsilon) \int_{-\tau}^{\tau} \left\| \alpha_{6,1}^{-1} \sum_{m \neq m_1} \Theta_m^{(1)}(\psi_1) u_m \zeta_m \right\|_1^2 dx \tag{7.12}$$

with a domain $D[a_6^{(+)}] = D[b_6^{(+)}]$. Obviously for each $\varepsilon < 1$ we have

$$N_0(a_5^{(+)}(\lambda, \varepsilon, \tau)/b_6^{(+)}(\tau)) \leqq N_0(a_6^{(+)}(\lambda, \varepsilon, \tau)/b_6^{(+)}(\tau)) , \tag{7.13}$$

Now we construct an operator $\mathscr{W}_2$ which is an automorphism in $D[a_6^{(+)}]$. At first we set $\varkappa_{s,l} = \langle \alpha_{7,1} u_s, u_l \rangle_1$, $s, l \in \mathbb{Z}$, then introduce the sequence $M_{m_1} \in l_2$ with components $\mu_{s,m_1} = K\overline{\varkappa_{s,m_1}}$, where $K = (2\pi\beta_{0,1})^{1/2} \| \alpha_{7,1} \|_1^{-1}$, and choose an arbitrary orthogonal basis $M_m = \{ \mu_{s,m} \}$, $s \in \mathbb{Z}$, $m \neq m_1$, in $l_2 \ominus \{ M_{m_1} \}$; thus the sequences M_m, $m \in \mathbb{Z}$, form an orthogonal basis in l_2.

Let $\vec{f} = \{ f_l \}_{l \in \mathbb{Z}} \in D[a_6^{(+)}]$. Set

$$v_{l,m} = \sum_{s \in \mathbb{Z}} \varkappa_{s,l} \mu_{s,m} ,$$

$$(\mathscr{W}_2 \vec{f})_m = \theta_m^{-1} \sum_{l \in \mathbb{Z}} v_{l,m} f_l , \qquad m \in \mathbb{Z} . \tag{7.14}$$

$\mathscr{W}_2$ acts in $D[a_6^{(+)}]$ as an algebraic operator with constant coefficients; since $\mathscr{W}_2$ is an automorphism in l_2, it can be extended to an automorphism in $D[a_6^{(+)}]$.

Having set $\theta = \theta_{m_1}$, $w = w_{m_1}$, $u = u_{m_1}$, $\omega_m = \sum_{l \neq m_1} v_{l,m} u_l$, $m \in \mathbb{Z}$;

$$\tilde{\mathscr{Y}}_1 = \tilde{\mathscr{Y}}_1(\varepsilon) = (1-\varepsilon)(1+\theta^{-2}\mathrm{K}^{-2}\| v \|_2^2),$$
$$\tilde{\mathscr{Y}}_2 = \tilde{\mathscr{Y}}_2(\varepsilon) = \theta^{-2}\mathrm{K}^{-2}\{ \langle v+\theta\mathrm{K}^2 u, w\rangle_2 - (1-\varepsilon)(\langle v,u\rangle_2 + \theta^2\mathrm{K}^2)\},$$
$$\tilde{\mathscr{Y}}_3 = \tilde{\mathscr{Y}}_3(\varepsilon) = (1-\varepsilon)\mathrm{K}^{-2}(\| u\|_2^2 - 2\mathrm{K}^2) + 2\theta^{-1}\mathrm{K}^{-1}\operatorname{Re}\langle \omega_{m_1}, w\rangle_2 - \theta^{-2}\mathrm{K}^{-2}(\langle \alpha_{5,1}u,u\rangle_1 - \| w\|_2^2 + \varepsilon);$$
$$\mathrm{Y}_1^{(m)} = \mathrm{Y}_1^{(m)}(\varepsilon) = (1-\varepsilon)\theta^{-1}\mathrm{K}^{-1}\langle v, \omega_m\rangle_2, \qquad m \neq m_1,$$
$$\mathrm{Y}_2^{(m)} = \mathrm{Y}_2^{(m)}(\varepsilon) = \theta^{-1}\mathrm{K}^{-1}\langle w-(1-\varepsilon)\theta u, \omega_m\rangle_2, \qquad m \neq m_1;$$
$$\mathrm{C}_+ = \theta^{-1}\mathrm{K}^{-1}c_+^{(2)},$$

we find that

$$b_6^{(+)}[\mathscr{W}_2\vec{f};\tau] = \int_{-\tau}^{\tau}\Big\{(x^2+\theta^{-2}\mathrm{K}^{-2})\,|f'_{m_1}|^2 + \sum_{m\neq m_1}|f'_m|^2\Big\}dx \qquad (7.15)$$

and

$$a_6^{(+)}[\mathscr{W}_2\vec{f};\lambda,\varepsilon,\tau] = \int_{-\tau}^{\tau}((\tilde{\mathscr{Y}}_1 x^2 + \lambda\mathrm{C}_+)\,|f'_{m_1}|^2 + 2x\operatorname{Re}\tilde{\mathscr{Y}}_2 f'_{m_1}\bar{f}_{m_1} + \tilde{\mathscr{Y}}_3\,|f_{m_1}|^2 + \sum_{m\neq m_1}\big[(1-\varepsilon)\,|f'_m|^2 + 2\operatorname{Re}\big(x\mathrm{Y}_1^{(m)}f'_{m_1} + \mathrm{Y}_2^{(m)}f_{m_1}\big)\bar{f}'_m\big]\Big\}dx. \qquad (7.16)$$

Checking (7.15)-(7.16), note that $(\mathscr{W}_2 f)_{m_1} = \theta^{-1}\mathrm{K}^{-1}f_{m_1}$ and

$$\omega_{m_1} = (\mathrm{K}\alpha_{7,1}^2 - \mathrm{K}^{-1})u, \qquad \omega_m = \alpha_{7,1}\sum_{s\in\mathbb{Z}}\mu_{s,m}u_s, \qquad m\neq m_1.$$

Obviously the estimate

$$a_6^{(+)}[\mathscr{W}_2\vec{f};\lambda,\varepsilon,\tau] \geqslant a_7^{(+)}[f_{m_1};\lambda,\varepsilon,\tau] \equiv \int_{-\tau}^{\tau}\{(\mathscr{Y}_1 x^2 + \lambda\mathrm{C}_+)\,|f'_{m_1}|^2 + 2x\operatorname{Re}\mathscr{Y}_2 f'_{m_1}\bar{f}_{m_1} + \mathscr{Y}_3\,|f_{m_1}|^2\}\,dx \qquad (7.17)$$

holds with

$$\mathscr{Y}_1 = \tilde{\mathscr{Y}}_1(\varepsilon) - (1-\varepsilon)^{-1}\sum_{m\neq m_1}|\mathrm{Y}_1^{(m)}(\varepsilon)|^2,$$
$$\mathscr{Y}_2 = \tilde{\mathscr{Y}}_2(\varepsilon) - (1-\varepsilon)^{-1}\sum_{m\neq m_1}\mathrm{Y}_1^{(m)}(\varepsilon)\overline{\mathrm{Y}_2^{(m)}(\varepsilon)}, \qquad (7.18)$$
$$\mathscr{Y}_3 = \tilde{\mathscr{Y}}_3(\varepsilon) - (1-\varepsilon)^{-1}\sum_{m\neq m_1}|\mathrm{Y}_2^{(m)}(\varepsilon)|^2.$$

Since the QF $a_7^{(+)}[\vec{f}]$ depends only on f_{m_1}, the estimates (7.16)-(7.18) imply that the inequality

$$N_0(a_6^{(+)}(\lambda, \varepsilon, \tau)/b_6^{(+)}(\tau)) \leqq N_0(a_7^{(+)}(\lambda, \varepsilon, \tau)/\overset{\circ}{H}{}^1(-\tau, \tau)) \qquad (7.19)$$

holds for each $\lambda > 0$, $\varepsilon \in (0, 1)$.

It is clear that the quantities $\mathcal{Y}_j(\varepsilon)$, $j = 1, 2, 3$, are continuous with respect to $\varepsilon \in [0, 1)$. Note that $\langle v, u \rangle_1 = 0$ and $\langle w, u \rangle_1 = \langle \alpha_{4,1}u, u \rangle_1$. Then tedious but straightforward calculations yield

$$\mathcal{Y}_1(0) = 1, \qquad \mathcal{Y}_2(0) = -1 + \theta^{-1} \langle \alpha_{4,1}u, u \rangle_1,$$
$$\mathcal{Y}_3(0) = -1 + \theta^{-2} \langle \alpha_{4,1}u, u \rangle_1^2 - \theta^{-2}K^{-2} \langle \alpha_{5,1}u, u \rangle_1 .$$

Take into account that $\mathcal{Y}_2(0)$ is real and $f_{m_1}(-\tau) = f_{m_1}(\tau) = 0$ in order to verify the estimate

$$2 \operatorname{Re} \mathcal{Y}_2(\varepsilon) \int_{-\tau}^{\tau} x f'_{m_1} \bar{f}_{m_1} dx$$
$$\geqq - \int_{-\tau}^{\tau} \{ \mathcal{Y}_2(0) | f_{m_1} |^2 + | \mathcal{Y}_2(\varepsilon) - \mathcal{Y}_2(0) |^2 (x^2 | f'_{m_1} |^2 + | f_{m_1} |^2) \} dx .$$

Fixing arbitrary $\varepsilon \in (0, 1)$ and chosing $\varepsilon' > 0$ small enough, we obtain

$$a^{(+)}[f_{m_1}; \lambda, \varepsilon', \tau] \geqq (1 - \varepsilon)(\mathcal{S}_2^{(2)}[f_{m_1}; \lambda, \tau, C_+] - y \mathcal{S}_2^{(2)}[f_{m_1}; \tau]) \quad (7.20)$$

where

$$y = y(\varepsilon) = (1 - \varepsilon)^{-1}(\mathcal{Y}_2(0) - \mathcal{Y}_3(0) + \varepsilon) \qquad (7.21)$$

and the QFs $\mathcal{S}_j^{(2)}, j = 1, 2$, are defined just below (3.10)-(3.11).

REMARK 7.1. — If we had assumed $\psi_1 = \Psi$ then parallel arguments would entail an analogous to (7.20) estimate in which the QFs $\mathcal{S}_l^{(2)}$ were substituted for $\mathcal{S}_l^{(1)}$, $l = 1, 2$.

Applying (7.20) and lemma 3.8 we get the estimate

$$N_0(a_7^{(+)}(\lambda, \varepsilon, \tau)/\overset{\circ}{H}{}^1(-\tau, \tau)) \leqq \Phi(y(\varepsilon)) | \log \lambda | + 0(1), \qquad \lambda \downarrow 0, \quad (7.22)$$

where $\Phi(y)$ is defined in (3.4). It is clear that $\Phi(y(\varepsilon))$ is continuous with respect to $\varepsilon \in [0, 1)$ and not difficult to check that

$$\Phi(y(0)) = (-\Xi(\psi_1))_+^{1/2}/4\pi^3 \,|\, \beta_1'(\psi_1) \,| = \mathcal{C}_k$$

(see (2.3)).

It follows from the estimates obtained in sections 5-7 (see (5.20), (6.6), (6.7), (7.10), (7.11), (7.13), (7.19) and (7.22)) that (2.6) holds under hypothesis *i*) of theorem 2.1 and (2.3) holds under hypothesis *ii*) of this theorem.

8. ESTIMATION OF $\mathcal{N}_\lambda^{(k)}$ FROM BELOW

i) Reduction to a scalar operator and localization with respect to ψ.

8.1. Let $\{\psi_l\}$ be the set of 1-degenerate values on $[0, \Psi]$. Fix $\tau > 0$ so that for each l the set $[\psi_l - \tau, \psi_l + \tau] \cap [0, \Psi]$ contains no 1-degenerate values except ψ_l and no 2-degenerate values. Now fix arbitrary $\delta > 0$ and assume $\lambda > 0$ small enough so that $\Lambda \equiv (\lambda/\delta)^{1/2} < \tau$. Put

$$\mathrm{I}_l^{(-)} = (\psi_l - \tau, \psi_l - \Lambda), \qquad \mathrm{I}_l^{(+)} = (\psi_l + \Lambda, \psi_l + \tau),$$
$$\Pi_l^{(\pm)} = \mathrm{I}_l^{(\pm)} \times (0, 2\pi),$$

(if $\psi_l = \Psi$, we define only $\mathrm{I}_l^{(-)}$ and $\Pi_l^{(-)}$. Introduce the QFs $a_1^{(-)}(\lambda, \varepsilon, \mathrm{I}_l^{(\pm)})$ and $b_1^{(-)}(\mathrm{I}_l^{(\pm)})$ which are analogous respectively to the QFs $a_1^{(+)}(\lambda, \varepsilon)$ with $\varepsilon \leqq 0$ and $b_1^{(+)}$ (see (5.10) and (5.11)) except that the domain of integration Π is replaced by $\Pi_l^{(\pm)}$ and $c_1^{(+)}$ is substituted for $c_1^{(-)} = \max_{(\psi,\chi)\in\Pi} \rho\mathscr{B}^{-4}$. Set

$$\mathrm{D}[a_1^{(-)}(\mathrm{I}_l^{(\pm)})] = \{ \zeta \in \mathrm{H}^1(\Pi_l^{(\pm)}) : \zeta_{\restriction\{\psi_l \pm \Lambda\} \cup \{\psi_l \pm \tau\}} = 0 \}$$

and define $\mathrm{D}[b_1^{(-)}(\mathrm{I}_l^{(\pm)})]$ as the closure of $\mathrm{D}[a_1^{(-)}(\mathrm{I}_l^{(\pm)})]$ in the $b_1^{(-)}(\mathrm{I}_l^{(\pm)})$-norm.

Since $\mathrm{I}_l^{(\pm)}$ contains no 1-degenerate values, $\zeta \in \mathrm{D}[a_1^{(-)}(\mathrm{I}_l^{(\pm)})]$ implies

$$\mathrm{V}_0\zeta \equiv (\mathscr{B}^{-2}\alpha_3 - \partial_\psi\mathscr{B}^{-2})\zeta - (i\alpha_0^{-1}\partial_\chi\mathscr{B}^{-2})\mathscr{F}_2^{-1}\mathscr{D}\zeta \in \mathrm{D}(\mathscr{F}_1^{-1}).$$

Put

$$a_8[\zeta; \mathrm{I}_l^{(\pm)}] = \int_{\Pi_l^{(+)}} \mathscr{B}^4 \,|\, \mathscr{F}_1^{-1}\mathrm{V}_0\zeta \,|^2 \alpha_0 d\Pi,$$
$$a_2^{(-)}(\lambda, \varepsilon, \mathrm{I}_l^{(\pm)}) = a_1^{(-)}(\lambda, \varepsilon, \mathrm{I}_l^{(\pm)}) + \lambda c_1^{(-)} a_8(\mathrm{I}_l^{(\pm)}), \qquad \mathrm{D}[a_2^{(-)}] = \mathrm{D}[a_1^{(-)}],$$
$$b_2^{(-)}(\mathrm{I}_l^{(\pm)}) = b_1^{(-)}(\mathrm{I}_l^{(\pm)}) + a_8(\mathrm{I}_l^{(\pm)}), \qquad \mathrm{D}[b_2^{(-)}] = \mathrm{D}[b_1^{(-)}].$$

8.2. Fix l and choose an arbitrary $\zeta \in \mathrm{D}[a_2^{(-)}(\mathrm{I}_l^{(+)})]$ (or respectively $\zeta \in \mathrm{D}[a_2^{(-)}(\mathrm{I}_l^{(-)})]$. Set

$$\vec{\eta}(\psi, \chi) = (\eta_1, \eta_2, \eta_3) = \begin{cases} (\zeta, -\mathscr{F}_2^{-1}\mathscr{D}\zeta, \mathscr{B}^2\mathscr{F}_1^{-1}\mathrm{V}_0\zeta), & (\psi, \chi) \in \Pi_l^{(\pm)} \\ 0, (\psi, \chi) \in \Pi \backslash \Pi_l^{(\pm)} \end{cases} \tag{8.1}$$

Then we have $\vec{\eta} \in \mathrm{D}[a_0]$ and

$$a_0[\vec{\eta}] + \lambda b_0[\vec{\eta}] \leqq a_2^{(-)}[\zeta; \lambda, 0, \mathrm{I}_l^{(\pm)}]. \tag{8.2}$$

Note that if the QF $a_0 + \lambda b_0$ is evaluated with $\vec{\eta}$ in the form of (8.1), then the only term containing η_3 (see (5.22)) is estimated from avove by $\lambda c_1^{(-)} a_8[\zeta]$. Besides we have $\tilde{b}_0[\vec{\eta}] = b_2^{(-)}[\zeta; \mathrm{I}_l^{(\pm)}]$. Then (8.2), lemma 3.1 and corollary 3.2 imply the estimate

$$\mathcal{N}_\lambda^{(k)} \geqq \sum_l \{ \mathrm{N}_0(a_2^{(-)}(\lambda, 0, \mathrm{I}_l^{(+)})/b_2^{(-)}(\mathrm{I}_l^{(+)})) + \mathrm{N}_0(a_2^{(-)}(\lambda, 0, \mathrm{I}_l^{-})/b_2^{(-)}(\mathrm{I}_l^{(-)})) \}. \tag{8.3}$$

Without any loss of generality we assume that there exists a single 1-degenerate value $\psi_1 \in (0, \Psi)$. Moreover, a conspicuous change of the variable allows us to perform the analysis of $N_0(a_2^{(-)}(\lambda, 0, I_1^{(-)})/b_2^{(-)}(I_1^{(-)}))$ parallely with the analysis of

$$N_0(a_2^{(-)}(\lambda, 0, I_1^{(+)})/b_2^{(-)}(I_1^{(+)})). \tag{8.4}$$

For this reason we restrict our attention to the estimation from below of the quantity (8.4) and omit $I_1^{(+)}$ in the notations.

9. ESTIMATION OF $\mathcal{N}_\lambda^{(k)}$ FROM BELOW

ii) Localization with respect to the Fourier component.

9.1. Recall the notations of section 7 and expand $\zeta \in D[a_2^{(-)}]$ into a Fourier series $\zeta = \sum_m \zeta_m u_m$. If we apply the explicit formula (4.5) for the resolvent of $\mathcal{F}_1$ and take into account that the operator $x\mathcal{F}_1^{-1} \equiv (\psi - \psi_1)\mathcal{F}_1^{-1}$ is bounded on $L_2(\Pi, \alpha_0)$ and $\int_0^{2\pi} \partial_\chi \mathcal{B}^{-2} d\chi = 0$, we find that the estimate

$$c_1^{(-)}\lambda b_2^{(-)}[\zeta] \leqq c\delta \int_\Lambda^\tau \left\{ x^2 \,|\, \zeta'_{m_1} |^2 + |\, \zeta_{m_1} |^2 + \sum_{m \neq m_1} (|\, \zeta'_m |^2 + |\, \zeta_m |^2) \right\} d\Pi \tag{9.1}$$

holds with an independent of λ and δ constant c.

Introduce the QFs $a_3^{(-)}(\Lambda, \tau, \varepsilon, c)$, $a_{3,j}^{(-)}$, $j = 1, 2, 3$, and $b_3^{(-)}(\Lambda, \tau)$ which are analogous respectively to $a_4^{(+)}(\lambda, \tau, \varepsilon, c)$, $a_{4,j}^{(+)}$, $j = 1, 2, 3$, with $\lambda = 0$, $\varepsilon \leqq 0$, $c < 0$, and $b_4^{(+)}(\tau)$ (see (7.3) and (7.5)) except that the interval of integration $(-\tau, \tau)$ is substituted for (Λ, τ). (The domains $D[a_3^{(-)}]$ and $D[b_3^{(-)}]$ are defined by analogy with $D[a_4^{(+)}]$ and $D[b_4^{(+)}]$.) The estimates (4.6)-(4.7), (7.1)-(7.2) and (9.1) imply that the inequality

$$a_2^{(-)}[\zeta : \lambda, 0] \leqq a_3^{(-)}[\zeta ; \Lambda, \tau, \varepsilon, c], \qquad c < 0 .$$

holds for each $\varepsilon < 0$ provided that δ is small enough (cf. (7.6)). Besides, the QFs $b_2^{(-)}$ and $b_3^{(-)}(\Lambda, \tau)$ are equivalent. Then, by analogy with (7.7), we have

$$N_0(a_2^{(-)}(\lambda, 0)/b_2^{(-)}) \geqslant N_0(a_3^{(-)}(\Lambda, \tau, \varepsilon, c)/b_3^{(-)}(\Lambda, \tau)), \qquad \forall \varepsilon < 0 . \tag{9.2}$$

Put

$$a_4^{(-)}(\Lambda, \tau, \varepsilon) = a_3^{(-)}(\Lambda, \tau, \varepsilon, c) + a_{3,3}^{(-)}(\Lambda, \tau, c),$$
$$b_4^{(-)}(\Lambda, \tau) = b_3^{(-)}(\Lambda, \tau), \qquad D[a_4^{(-)}] = D[a_3^{(-)}], \qquad D[b_4^{(-)}] = D[b_3^{(-)}]$$

(cf. (7.9)). By analogy with (7.10) we show that (9.2) entails the estimate

$$N_0(a_2^{(-)}(\lambda, 0)/b_2^{(-)}) \geqq N_0(a_4^{(-)}(\Lambda, \tau, \varepsilon)/b_4^{(-)}(\Lambda, \tau)) + O(1), \quad \lambda \downarrow 0, \quad \forall \varepsilon < 0, \quad (9.3)$$

9.2. Introduce the functional sequences $f = \{ f_m \}$ and $\{\zeta_m\}$:

$$f_{m_1} = f \in \overset{\circ}{H}{}^1(\Lambda, \tau), \qquad \int_\Lambda^\tau f\,dx = 0,$$

$$f_m(x) = (1-\varepsilon)^{-1}\left\{ -x Y_1^{(m)} f(x) + (Y_1^{(m)} - Y_2^{(m)}) \int_\Lambda^x f(t)\,dt \right\}, \quad m \neq m_1, \quad \varepsilon < 0, \quad (9.4)$$

$$\zeta_m = (\mathscr{W}_2 \vec{f})_m, \quad m \in \mathbb{Z}, \quad (9.5)$$

where $\mathscr{W}_2$ is the operator defined by (7.14). Denote by $a_5^{(-)}[f; \Lambda, \tau, \varepsilon]$ the QF $a_4^{(-)}[\zeta; \Lambda, \tau, \varepsilon]$ evaluated with $\zeta(\varepsilon)$ in the form of (9.4)-(9.5); the domain $D[a_5^{(-)}]$ is defined by (9.4). Then we have $a_5^{(-)} = a_{5,1}^{(-)} + a_{5,2}^{(-)}$, where the QFs $a_{5,j}^{(-)}[f; \Lambda, \tau, \varepsilon]$, $j = 1, 2$, coincide respectively with the QFs

$$(2-j)a_4^{(-)}[\zeta; \Lambda, \tau, \varepsilon] + (-1)^{2-j}(1-\varepsilon)\int_\Lambda^\tau \left\| a_{6,1}^{-1} \sum_{m \neq m_1} \Theta_m^{(1)}(\psi_1) u_m \zeta_m \right\|_1^2 dx$$

with $\zeta(\varepsilon)$ in the form of (9.4)-(9.5). Further, set

$$b_5^{(-)}[f; \Lambda, \tau] = \int_\Lambda^\tau x^2 |f'|^2 dx, \qquad f \in D[a_5^{(-)}].$$

Note that the QF $b_4^{(-)}[\zeta; \Lambda, \tau]$ evaluated with ζ in the form of (9.4)-(9.5) is equivalent to the QF $b_5^{(-)}[f; \Lambda, \tau]$. Then, by lemma 3.1 and corollaries 3.2 and 3.5, we obtain the estimate

$$N_0(a_4^{(-)}(\Lambda, \tau, \varepsilon)/b_4^{(-)}(\Lambda, \tau)) \geqq N_0(a_5^{(-)}(\Lambda, \tau, \varepsilon)/b_5^{(-)}(\Lambda, \tau))$$
$$\geqq N_0(a_{5,1}^{(-)}(\Lambda, \tau, 2\varepsilon)/b_5^{(-)}(\Lambda, \tau)) - n_{-\varepsilon}(a_{5,2}^{(-)}(\Lambda, \tau)/b_5^{(-)}(\Lambda, \tau)), \qquad \forall \varepsilon < 0. \quad (9.6)$$

It is easy to check that the QFs ratio $a_{5,2}^{(-)}(\Lambda, \tau)/b_5^{(-)}(\Lambda, \tau)$, $\Lambda > 0$, generates a compact operator. Besides we have

$$n_\mu(a_{5,2}^{(-)}(\Lambda, \tau)/b_5^{(-)}(\Lambda, \tau)) = O(1), \qquad \Lambda \downarrow 0, \qquad \forall \mu > 0. \quad (9.7)$$

9.3. Put

$$a_6^{(-)}[f; \Lambda, \tau, \varepsilon] = \gamma_1[f; \Lambda, \tau] - y(\varepsilon)\gamma_2[f; \Lambda, \tau], \qquad f \in \overset{\circ}{H}{}^1(\Lambda, \tau).$$

(the QFs γ_j, $j = 1, 2$, are introduced in (3.2)-(3.3) and $y(\varepsilon)$ is defined by (7.21)). Fix an arbitrary $\varepsilon < 0$ and verify that the inequality

$$a_5^{(-)}[f; \Lambda, \tau, \varepsilon'] \leqq (1-\varepsilon)a_6^{(-)}[f; \Lambda, \tau, \varepsilon], \qquad f \in D[a_5^{(-)}],$$

holds if $\varepsilon' < 0$ and $|\varepsilon'|$ is sufficiently small. Then corollaries 3.2 and 3.4 imply the estimate

$$N_0(a_{5,1}^{(-)}(\Lambda, \tau, \varepsilon')/b_5^{(-)})) \geqq N_0(\gamma_1(\Lambda, \tau) - y(\varepsilon)\gamma_2(\Lambda, \tau)/\gamma_1(\Lambda, \tau)) - 1. \quad (9.8)$$

Recalling that $\Lambda = (\lambda/\delta)^{1/2}$ and applying lemma 3.7, we find that the estimates (9.6)-(9.8) entail

$$N_0(a_4^{(-)}(\Lambda, \tau, \varepsilon'')/b_4^{(-)}(\Lambda, \tau)) \geqq \frac{1}{2}\Phi(y(\varepsilon))\,|\log\lambda| + O(1), \qquad \lambda \downarrow 0, \quad (9.9)$$

where $\varepsilon < 0$ is arbitrary, $\varepsilon'' < 0$ and $|\varepsilon''|$ is small enough.

The estimates obtained in sections 8-9 (see (8.3), (9.3) and (9.9)) entail the inequality (2.7).

10. THE CASE OF COMPLETELY DEGENERATE VALUES

10.1. In this section we discuss briefly the complementary arguments in the proof of theorem 2.1 which are needed in the case when there exist completely degenerate values $\psi \in (0, \Psi)$; without any loss of generality we assume that there is a unique 1-degenerate value $\psi_1 \in (0, \Psi)$ which is completely degenerate. Denote $k\beta_2(\psi_1) = -m_2$.

Fix some $\tau_1 > 0$ such that $I \equiv (\psi_1 - \tau_1, \psi_1 + \tau_1) \subset\subset (0, \Psi)$ and the set $\bar{I}\backslash\{\psi_1\}$ contains no 2-degenerate values. Choose an arbitrary number h which satisfies the inequalities

$$0 < h < (\beta_0(\psi))^{-1}(1 - (k\beta_2(\psi) + m_2)), \qquad \forall\psi \in \bar{I}.$$

Set $p(\psi) = hp_0(\psi)$, where $p_0(\psi) \in [0, \Psi]$ is an arbitrary function such that $\operatorname{supp} p_0 \subset I$ and $0 \leqq p_0(\psi) < p_0(\psi_1) = 1$ for each $\psi \in \bar{I}$, $\psi \neq \psi_1$. Put $\hat{\beta}_2 = \beta_2 + \beta_0 k^{-1} p$. Note that $k\hat{\beta}_2(\psi_1) \bar{\in} \mathbb{Z}$. If $\psi \in \bar{I}$, then $k\hat{\beta}_2(\psi)$ takes no integer values except may be $-m_2$; besides $k\beta_2(\psi)$ may take the value $-m_2$ at no more than two points on $\bar{I}$.

Define the self-adjoint in the HS $\mathscr{L}(\psi)$ operator $\hat{F}_2(\psi) = F_2(\psi) + p(\psi)$ on the domain $D(\hat{F}_2(\psi)) = D(F_2(\psi))$, $\psi \in [0, \Psi]$. The eigenvalues $\hat{\Theta}_m^{(2)}(\psi)$ and resolvent of $\hat{F}_2(\psi)$ are defined respectively by the relations (4.1) and (4.5), if we replace in them β_2 by $\hat{\beta}_2$ and α_2 by $\hat{\alpha}_2 = \alpha_2 + \alpha_0 p$. Obviously $\hat{F}_2(\psi)$ commutes with $F_1(\psi)$ and its eigenfunctions are $U_m(\psi)$ (see (4.2)). Also the estimates (4.6)-(4.7) hold if we substitute in them $F_2(\psi)$ for $\hat{F}_2(\psi)$ and $F(\psi)$ for $\hat{F}(\psi) = F_1(\psi)F_2^{-1}(\psi)$.

Introduce the self-adjoint in $L_2(\tilde{\Pi}, \alpha_0\psi)$ operator $\mathscr{F}_2$ as in (4.10) replacing $F_2(\psi)$ by $\hat{F}_2(\psi)$. In what follows we denote by $\hat{a}_j^{(+)}$, $\hat{a}_{j,i}^{(\pm)}$ and $\hat{b}_j^{(\pm)}$, $j \geqslant 1$, respectively the QFs $a_j^{(\pm)}$, $a_{j,i}^{(\pm)}$ and $b_j^{(\pm)}$ which were defined in sections 5-9 substituting in them $\mathscr{F}_2$ for $\hat{\mathscr{F}}_2$ and $\mathscr{F}$ for $\hat{\mathscr{F}} \equiv \mathscr{F}_1\hat{\mathscr{F}}_2^{-1}$.

10.2. Define the operator $\hat{\mathscr{G}} = \hat{\mathscr{F}}_2\alpha_7^2\hat{\mathscr{F}}_2 + \mathscr{D}\alpha_6^2\mathscr{D}^*$, $D(\hat{\mathscr{G}}) = D(\mathscr{G})$, and the operator $\hat{\mathscr{W}}_1$ substituting in (5.13) $\mathscr{F}_2$ for $\hat{\mathscr{F}}_2$. As in lemma 5.2, it can be shown that the operator $\hat{\mathscr{W}}_1 : \mathrm{D}(\hat{\mathscr{W}}_1) \to \mathrm{D}[a_0]$ is an isomorphism. Set

$$\mathscr{W}_3 = \begin{pmatrix} 1 & \hat{\mathscr{G}}^{-1} p \hat{\mathscr{F}}_2^{-1} \mathscr{D} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{D}(\mathscr{W}_3) = \mathrm{D}(\hat{\mathscr{W}}_1).$$

Obviously $\mathscr{W}_3$ is an automorphism in $\mathrm{D}(\hat{\mathscr{W}}_1)$. Put

$$\mathscr{W}_4 = \hat{\mathscr{W}}_1 \mathscr{W}_3 = \hat{\mathscr{W}}_1 + \begin{pmatrix} 0 & \mathrm{V}_1 & 0 \\ 0 & \mathrm{V}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\mathrm{V}_1 = \alpha_6^2 \mathscr{D}^* \hat{\mathscr{G}}^{-1} p \hat{\mathscr{F}}_2^{-1} \mathscr{D}$, $\mathrm{V}_2 = \alpha_7^2 \hat{\mathscr{F}}_2 \hat{\mathscr{G}}^{-1} p \hat{\mathscr{F}}_2^{-1} \mathscr{D}$. Obviously the operator $\mathscr{W}_4 : \mathrm{D}(\hat{\mathscr{W}}_1) \to \mathrm{D}[a_0]$ is an isomorphism.

10.3. When we estimate $\mathscr{N}_\lambda^{(k)}$ from above, we substitute (5.21) for the estimate

$$a_0[\mathscr{W}_4 \vec{\zeta}] + \lambda b_0[\mathscr{W}_4 \vec{\zeta}] \geqq \hat{a}_1[\zeta^{(2)}; \lambda, \varepsilon, k] - \hat{c}_1 a_9[\zeta^{(2)}] + c' \int_\Pi \alpha_0 (|\hat{\mathscr{G}} \zeta^{(1)}|^2 - c |\hat{\mathscr{G}}^{1/2} \zeta^{(1)}|^2) d\Pi$$

where

$$a_9[\zeta] = \int_\Pi \{ \alpha_6^{-2} (|\mathscr{F}_1 \mathrm{V}_1 \zeta|^2 + |\mathrm{V}_1 \zeta_2|^2) + \alpha_7^{-2} (|\mathscr{F}_1 \mathrm{V}_2 \zeta|^2 + \mathrm{V}_2 \zeta|^2) \} \alpha_0 d\Pi$$

and $\hat{c}_1$ is independent of λ and h. Hence, instead of (5.23), we get

$$\mathscr{N}_\lambda^{(k)} \leqq \mathrm{N}_0(\hat{a}_1^{(+)}(\lambda, \varepsilon, k) - \hat{c}_1 a_9 / \hat{b}_1^{(+)}(k)) + 0(1), \qquad \lambda \downarrow 0.$$

Further we construct a smooth partition of unity on I such that:

i) $p = p_1 + p_2$, $\operatorname{supp} p_1 = \bar{\mathrm{I}}_1 \equiv [\Psi_1 - \tau, \psi_1 + \tau]$, $\tau \in (0, \tau_1)$;

ii) the operators $\hat{\mathrm{F}}_2^{-1}(\psi)$ and $\hat{\mathrm{F}}(\psi)$ are bounded for each $\psi \in \bar{\mathrm{I}}_1$.

Commuting the operator factors in V_1 and V_2, we obtain the crucial estimate

$$a_9[\zeta] \leqq \hat{c}_2 \sum_{j=1,2,3} a_{9,j}[\zeta], \tag{10.1}$$

where

$$a_{9,1}[\zeta] = \int_\Pi \alpha_0 p_1^2 (|\hat{\mathscr{F}} \zeta|^2 + |\hat{\mathscr{F}}_2^{-1} \zeta|^2) d\Pi,$$

$$a_{9,2}[\zeta] = \int_\Pi \alpha_0 p_2^2 (|\hat{\mathscr{F}} \mathscr{D} \zeta|^2 + |\zeta|^2) d\Pi,$$

$a_{9,3}[\zeta]$ is a compact in $L_2(\Pi)$ QF and the constant $\hat{c}_2$ is independent of λ and h. Obviously we have

$$a_{9,2}[\zeta] \leqq h^2 \int_{\Pi} \alpha_0(|\hat{\mathscr{F}}\mathscr{D}\zeta|^2 + |\zeta|^2)d\Pi. \qquad (10.2)$$

Fix some $\varepsilon' \in (0, 1)$ and $\varepsilon \in (\varepsilon', 1)$. Apply corollaries 3.2 and 3.5 together with the estimates (10.1)-(10.2) in order to verify that the estimate

$$N_0(\hat{a}_1^{(+)}(\lambda, \varepsilon', k) - \hat{c}_1 a_9/\hat{b}_1^{(+)}(k)) \leqq N_0(\hat{a}_1^{(+)}(\lambda, \varepsilon, k) - \hat{c}_3 a_{9,1}/\hat{b}_1^{(+)}(k)) + n_\mu(a_{9,3}/L_2(\Pi))$$

holds with $\hat{c}_3 = \hat{c}_1\hat{c}_2$, if $\mu > 0$ and h are sufficiently small.

Now construct the partition of the interval $(0, \Psi)$ (see (6.1)) so that $\mathscr{I}_1 = \{ I_1 \}$. By analogy with (6.2), (6.3), (6.6) and (6.7) we obtain the inequality

$$N_0(\hat{a}_1(\lambda, \varepsilon', k) - \hat{c}_3 a_{9,1}/\hat{b}_1^{(+)}(k)) \leqq N_0(\hat{a}_3^{(+)}(\lambda, \varepsilon, \tau) - \hat{c}_3 a_{9,1}/\hat{b}_3^{(+)}(\tau)), \quad \forall\varepsilon' \in (0, 1), \quad \forall\varepsilon \in (\varepsilon', 1).$$

Expand $\zeta \in D[a_3^{(+)}]$ into a Fourier series $\zeta = \sum_m \zeta_m u_m$. Then we have

$$a_{9,1}[\zeta] \leqq \hat{c}_4 \int_{-\tau}^{\tau} \left(h^2 |\zeta_{m_1}|^2 + \sum_{m \neq m_1} |\zeta_m|^2 \right) d\Pi, \qquad (10.3)$$

where $\hat{c}_4$ is independent of h. Checking (10.3) take into account the crucial relation (4.9). Consequently the estimate

$$N_0(\hat{a}_3^{(+)}(\lambda, \varepsilon', \tau) - \hat{c}_3 a_{9,1}/\hat{b}_3^{(+)}(\tau)) \leqq N_0(\hat{a}_4^{(+)}(\lambda, \varepsilon, \tau, c)/\hat{b}_4^{(+)}(\tau))$$

holds for each $\varepsilon' \in (0, 1)$, $\varepsilon \in (\varepsilon', 1)$ and some independent on λ constant $c > 0$, provided that h is sufficiently small (cf. (7.7)).

Further the quantity $N_0(\hat{a}_4^{(+)}(\lambda, \varepsilon, \tau, c)/\hat{b}_4^{(+)}(\tau))$ is handled as shown in section 7, arriving finally at an estimate which is analogous to (7.20) except that $y(\varepsilon)$ is replaced by some continuous with respect to h quantity $y(\varepsilon, h)$ such that $y(\varepsilon, 0) = y(\varepsilon)$. Since h is arbitrarily small we again come to (2.6) or respectively to (2.3).

10.4. The arguments in the estimation of $\mathscr{N}_\lambda^{(k)}$ from below are quite similar and we omit the details. Note only that the formulae (8.1) must be substituted for

$$\vec{\eta}(\psi, \chi) = (\eta_1, \eta_2, \eta_3) = \begin{cases} (\zeta + V_1\zeta, \ -\hat{\mathscr{F}}_2^{-1}\mathscr{D}\zeta + V_2\zeta, \ \mathscr{B}^2\mathscr{F}_1^{-1}V_0\zeta), (\psi, \chi) \in \Pi_1^{(\pm)} \\ 0, (\psi, \chi) \in \Pi/\Pi_1^{(\pm)} \end{cases}$$

Then instead of (8.3) we get the estimate

$$a_0[\vec{\eta}] + \lambda b_0[\vec{\eta}] \leqq \hat{a}_2^{(-)}[\zeta; \lambda, \varepsilon, I_1^{(\pm)}] + \hat{c}_5 a_9[\zeta].$$

where $\varepsilon < 0$ is arbitrary and $\hat{c}_5 > 0$ is independent on λ and h.

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REFERENCES

[1] G. Bateman, *MHD Instabilities*. MIT, Cambridge, Massachusetts, 1978.
[2] C. Mercier, H. Luc, *The MHD Approach to the Problem of Plasma Confinement in Closed Magnetic Configurations*. EUR 5127e, Luxembourg, 1974.
[3] J. P. Freidberg, Ideal magnetohydrodynamic theory of magnetic fusion systems. *Rev. Mod. Phys.*, t. **54**, no. 3, 1982, p. 801-902.
[4] C. Mercier, Un critère nécessaire de stabilité hydrodynamique pour un plasma en symétrie de révolution. *Nucl. Fusion*, t. **1**, 1961, p. 47-53.
[5] J. L. Greene, J. L. Johnson, Interchange instabilities in ideal hydromagnetic theory. *Plasma Phys.*, t. **10**, 1968, p. 729-745.
[6] D. Lortz, Sufficient stability criteria for general MHD-equilibria. *Nucl. Fusion*, t. **13**, 1973, p. 817-819.
[7] D. Lortz, J. Nührenberg, Comparison between necessary and sufficient criteria for axially symmetric equilibria. *Nucl. Fusion*, t. **13**, 1973, p. 821-827.
[8] J. P. Goedbloed, Spectrum of ideal magnetohydrodynamics of axisymmetric toroidal systems. *Phys. Fluids*, t. **18**, 1975 (10), p. 1258-1268.
[9] J. Descloux, G. Geymonat, *On the essential spectrum of an operator relative to the stability of a plasma in a toroidal geometry*. École Polytechnique Féderale de Lausanne Report, October, 1979.
[10] A. E. Lifshits, The continuous spectrum in some problems of mathematical physics. *Dokl. Akad. Nauk SSSR*, vol. **277**, no. 5, 1984, p. 1105-1108 (in Russian).
[11] E. Hamieri, On the essential spectrum of ideal magnetohydrodynamics. *Comm. Pure Appl. Math.*, t. **38**, 1985, p. 43-66.
[12] Y. P. Pao, Instabilities in toroidal plasmas. *Phys. Fluids*, t. **18**, 1975 (9), p. 1192-1196.
[13] R. L. Dewar, A. H. Glasser, Ballooning mode spectrum in deneral toroidal systems. *Phys. Fluids*, t. **26**, 1983 (10), p. 3038-3052.
[14] G. D. Raikov, Spectral properties of a linear magnetohydrodynamic model with axial symmetry. *Vestnik LGU*, ser. 4, 1986, vyp. 2, p. 74-76 (in Russian).
[15] M. S. Birman, M. Z. Solomjak, *Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory*. American Math. Society Translations, Series 2, 114 AMS, Providence, R. I., 1980.
[16] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I, Functional Analysis*. Academic Press, New York, 1972.

[17] M. REED, B. SIMON, *Methods of Modern Mathematical Physics, IV, Analysis of operators*. Academic Press, New York, 1978.

[18] L. D. LANDAU, E. M. LIFSHITS, *Course of Theoretical Physics, VIII, Electrodynamics of Continuous Media*. Pergamon International Library of Science, Technology, Engineering and Social Studies, Pergamon Press, Oxford-Elmsford, N. Y., 1984.

[19] Yu. M. BEREZANSKII, *Expansions in eigenfunctions of self-adjoint operators*. Translations of Mathematical Monographs, vol. **17**, AMS, Providence, R. I., 1968.

[20] O. A. LADYŽENSKAYA, N. N. URALTSEVA, *Linear and quasilinear equations of elliptic type*. Nauka, Moscow, 1964 (in Russian).

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