ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 47, nº 1 (1987), p. 85-111
<http://www.numdam.org/item?id=AIHPA_1987__47_1_85_0>

Ann. Inst. Henri Poincaré,
Vol. 47, n° 1, 1987, p. 85-111. *Physique théorique*

Rigorous numerical stability estimates for the existence of KAM tori in a forced pendulum

by

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ABSTRACT. — We study KAM estimates for a one dimensional, time dependent hamiltonian system corresponding to a forced pendulum. We study the stability of a KAM torus with rotation number equal to the golden section as a function of a parameter ε. We first use the Birkhoff algorithm in order to reduce the perturbation to high order (up to eighth), then we apply four different methods, comparing our results with numerical experiments ([*7*], [*21*]). We find that the KAM torus is rigorously stable for $|\varepsilon| < \varepsilon^*$, with ε^* smaller than the experimental value by a factor ~ 40 (the previously known rigorous estimate was lower by a factor $\sim 10^6$.

RÉSUMÉ. — On étudie les estimations de type KAM pour un système hamiltonien unidimensionnel dépendant du temps correspondant à un pendule forcé. On étudie la stabilité d'un tore KAM dont le nombre de rotation est égal au nombre d'or en fonction d'un paramètre ε. On utilise d'abord un algorithme de Birkhoff pour réduire la perturbation à un ordre élevé (jusqu'à huit), on utilise ensuite quatre méthodes différentes et on compare les résultats avec ceux d'expériences numériques ([*7*], [*21*]). On trouve que le tore KAM est rigoureusement stable pour $|\varepsilon| < \varepsilon^*$, avec ε^* plus petit que la valeur expérimentale par un facteur ~ 40 (l'estimation rigoureuse connue précédemment était inférieure par un facteur $\sim 10^6$).

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(**) Research partially supported by CNR-GNFM.

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§ 1. INTRODUCTION

Rigorous stability estimates on the quasi periodic solutions of almost integrable hamiltonian systems with given (diophantine) rotation numbers appeared to be still quite far from « reality », if reality is defined either by the computer experiments or by the requirement that the results should have practical relevance [*1*] [*2*] [*4*] [*5*] [*6*].

This has led to the widespread belief that the KAM-theory, which is essentially the only general tool available for such estimates, does not have a practical interest.

The problem with KAM-theory is its great generality: in fact it provides simple stability theorems under minimal assumptions (e. g. non isochrony, or twist, property of the unperturbed system and regularity, say analyticity, of the perturbing function); a closer analysis of the proofs of the KAM stability theorems shows that in fact the estimates met in the proofs are not really bad (i. e. not too pessimistic) given the generality under which they hold).

This remark can be used to implement a different use of the KAM-theory which may lead to (much more) realistic stability estimates, and is therefore closer to practical usefulness.

The idea is to make explicit use of the peculiarities of the actual hamiltonian that one is using, which is always very special, by performing a few changes of variables which bring the system much closer to an integrable one, near the selected quasi periodic motions, at the expense of making the perturbing function very complicated.

It is to this new, equivalent, hamiltonian system that one shall apply the stability estimates of KAM-theory: such estimates will no longer be « too bad » because they are now applied to a rather complicated hamiltonian (hence « close to a general hamiltonian ») and one can hope that the net result would be a not unreasonably small bound on the stability threshold of the selected quasi periodic motion.

In this paper, pursuing a program initiated in [*5*] (and applied also to the Siegel problem in [*14*] [*15*]), using the KAM-estimates given in [*5*] [*6*] [*16*], we shall study a one dimensional non autonomous hamiltonian system describing a forced pendulum

$$H(A, \phi, t) = \frac{A^2}{2} + \varepsilon \cos \phi + \varepsilon \cos (\phi - t). \qquad (1.1)$$

Numerical as well as non rigorous theoretical results on the stability threshold $\varepsilon^*(\omega_0)$ are available in the literature; this threshold is defined as the largest ε^* such that for $\varepsilon < \varepsilon^*(\omega_0)$ the quasi periodic motions with rotation number

$$\omega_0 = \frac{1 + \sqrt{5}}{2} \qquad \text{(golden section)}$$

exists on a torus $\mathscr{T}^*(\varepsilon)$, which is obtained by continuity as ε grows from 0 to $\varepsilon^*(\omega_0)$ from the unperturbed torus $\mathscr{T}_0 = \{ A, \phi, t/A = \omega_0 \}$ in the (three dimensional) phase space of (1.1).

A non rigorous computation [*21*] suggests that

$$3.22\ \varepsilon^*(\omega_0) = 1 \tag{1.2}$$

and non rigorous theoretical results are also available ([*7*]). For simplicity of notations, we denote with ε^*_{ED} the critical value (1.2) given in [*7*].

A « naive » application of KAM-theory ([*4*]) gives instead of (1.2) the estimate $10^{20}\,\varepsilon^*(\omega_0) > 1$ (a rigorous but « useless » result, if one believes (1.2)), while a first application of the above ideas gives a result like $3.10^7\,\varepsilon^*(\omega_0) > 1$ ([*5*]).

However the latter rigorous estimates appear to be improvable because the main limitation was the excessive length of the calculations: one gets the feeling that dramatic improvements would be possible if only one was able to perform very long, straightforward, calculations. This led to the idea of improving the bounds via a computer assisted analysis where the role of the computer was to evaluate actual bounds on the value of very long, but algebraically simple, formulae arising in perturbation theory.

In this paper we report a method leading to the lower bound on $\varepsilon^*(\omega_0)$

$$7.93 \cdot 10^3\ \varepsilon^*(\omega_0) > 1 \tag{1.3}$$

which is « only » about three order of magnitude away from the empirical result (1.2). The same method, applied by using the new stability statement of the KAM theory [*16*], gives a better lower bound:

$$1.30 \cdot 10^2\ \varepsilon^*(\omega_0) > 1 \tag{1.4}$$

which is now away from (1.2) by a factor 40.

The value $\varepsilon^*(\omega_0)$ is often called the « stochasticity threshold » and it has been closely investigated in many other systems starting with the numerical experiments of Henon-Heiles on the axisymmetric potential [*9*]:

$$V(x, y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right) \tag{1.5}$$

and successively on the restricted three body problem [*10*]: this analysis was based on perturbation theory.

More recently, new numerical methods have been developed in [*11*] [*12*] and theoretical results have been obtained (see [*7*] [*13*] [*14*] [*15*]): to some of the above problems the methods of this paper might be applicable too.

The scheme of this paper is the following. In § 2 we explain our perturbation theory method. In § 3 we give four algorithms, each of them parameterized by an integer N called the « order ». In method 1 we use the rough estimate (3.9) of the perturbation function and we obtain the final estimate on ε, making use of the general bound (3.3). In method 2 we use again (3.3), computing explicitly the new hamiltonian and perturbing functions and evaluating various constants by using the explicit formulae for the new hamiltonian and perturbation with the help of a computer. In method 3 we improve (3.3) replacing it with the set of conditions (3.12), (3.13), (3.14), (3.16) below; their discussion for a general analytic hamiltonian would lead to (3.3), but one can improve their discussion (and in this way the final result), by making use of the explicit form of the hamiltonian and perturbing functions. In method 4 we use the new proof of KAM theorem given in [*16*]. In § 4 we discuss the numerical aspects of the implementation of the algorithm (which we have been able to carry only up to « 8-th order » because of memory and time limitations of the available computer (Vax 11/780)).

§ 2. THEORY OF THE NUMERICAL ALGORITHM: COORDINATE TRANSFORMATION AND CORRESPONDING BOUNDS

We consider the hamiltonian system (1.1) and we wish to follow continuously, as ε grows, the invariant torus $\mathscr{T}(\varepsilon)$ on which a quasi periodic motion with rotation number

$$\omega_0 = \frac{1+\sqrt{5}}{2}$$

takes place and which at $\varepsilon = 0$ is the torus described by the parametric equations

$$\begin{cases} A = \omega_0 = A_0 \\ \phi = \phi' \qquad\qquad (\phi', t') \in \mathscr{T}^2 . \\ t = t' \end{cases} \tag{2.1}$$

We start by building explicitly a canonical change of variables transforming the hamiltonian (1.1) into a new hamiltonian, which has the form of an integrable part plus a perturbation of order ε^{N+1} for some fixed N.

Although we are interested in the simple case (1.1), we will formulate the method in a general form.

Let

$$H_\varepsilon(A, \phi, t) = h_0(A) + \varepsilon f_0(A, \phi, t) \tag{2.2}$$

where h_0 is real analytic on $S_{\rho_0}(A_0)$ and holomorphic on $\hat{S}_{\rho_0}(A_0)$, while f_0 is real analytic on $S_{\rho_0}(A_0) \times \mathcal{T}^2$ and holomorphic on $\hat{S}_{\rho_0}(A_0) \times C^2_{\xi_0}$ where, for $A_0 \in \mathcal{R}$:

$$\begin{aligned} S_\rho(A_0) &= \{ A/A \in \mathcal{R}, |A - A_0| < \rho \} \\ \hat{S}_\rho(A_0) &= \{ A/A \in \mathcal{C}, |A - A_0| < \rho \} \\ \mathcal{T}^2 &= \text{standard 2-dimensional torus} \\ C_\xi &= \{ z/z \in \mathcal{C}, e^{-\xi} < |z| < e^{\xi} \} \end{aligned} \tag{2.3}$$

and f_0 is regarded as a function of $\hat{S}_{\rho_0}(A_0) \times C^2_{\xi_0}$ by regarding it as the analytic continuation of its restriction to $S_{\rho_0}(A_0) \times \mathcal{T}^2$ and identifying the points $(\phi, t) \in \mathcal{T}^2$ with $\underline{z} = (z_1, z_2) = (e^{i\phi}, e^{it}) \in C^2_{\xi_0}$ (as it is natural to do).

The parameters ρ_0, ξ_0 measure the regularity of h_0 and f_0: their size will be measured by suitably using the norms

$$\begin{aligned} |g|_\rho &= \sup |g(A)| \qquad \text{taking sup on } \hat{S}_\rho(A_0) \\ |g|_{\rho,\xi} &= \sup |g(A, z)| \qquad \text{taking sup on } \hat{S}_\rho(A_0) \times C^2_\xi \end{aligned} \tag{2.4}$$

(see below), where g is holomorphic in $\hat{S}_\rho(A_0)$ and $\hat{S}_\rho(A_0) \times C^2_\xi$, respectively. Let in fact:

$$\omega_0(A) = \frac{\partial h_0}{\partial A}(A), \qquad M_0(A) = \frac{\partial \omega_0}{\partial A}(A) \tag{2.5}$$

and let E_0, η_0, ε_0, be three numbers such that

$$\begin{aligned} E_0 &\geq \left| \frac{\partial h_0}{\partial A} \right|_{\rho_0}, \qquad \eta_0 \geq |M_0^{-1}|_{\rho_0} \\ \varepsilon_0 &\geq \left(\left| \frac{\partial f_0}{\partial A} \right|_{\rho_0,\xi_0} + \frac{1}{\rho_0} \left| \frac{\partial f_0}{\partial \phi} \right|_{\rho_0,\xi_0} \right) |\varepsilon| \end{aligned} \tag{2.6}$$

The physical interpretation of E_0^{-1}, ε_0^{-1} is that of time scales characteristic of the unperturbed motion or of the perturbation respectively, while η_0 measures the non isochrony of the unperturbed system (i. e. the strength of the twist it generates in phase space as A changes).

With the above notations the hamiltonian (1.1) takes the form (remembering our notation $z_1 = e^{i\phi}$, $z_2 = e^{it}$):

$$H(A, z_1, z_2) = \frac{A^2}{2} + \frac{\varepsilon}{2}\left(z_1 + \frac{1}{z_1}\right) + \frac{\varepsilon}{2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right) \tag{2.7}$$

and comparing (2.6), (2.7) we see that we can regard it as holomorphic in $\hat{S}_{\rho_0} \times C^2_{\xi_0}$ with *any* pair ρ_0, ξ_0 of positive numbers, and correspondingly, given ρ_0, ξ_0, one can take

$$E_0 = |A_0| + \rho_0, \qquad \varepsilon_0 = (\text{ch}\, \xi_0 + \text{ch}\, 2\xi_0)\,|\varepsilon| \tag{2.8}$$

where $A_0 = (1 + \sqrt{5})/2$.

We keep the freedom of choosing ρ_0, ξ_0 with the intention of trying to choose eventually values which optimize the final estimate of the threshold $\varepsilon^*(\omega_0)$. Other parameters will arise and will be treated in the same way.

The golden section that we have chosen as rotation number has very convenient arithmetic (« diophantine ») properties; namely for all integers p, $q \in \mathscr{Z}$ (see [*8*]):

$$\left| \frac{1+\sqrt{5}}{2} - \frac{p}{q} \right|^{-1} \leq Cq^2 \tag{2.9}$$

$$C = (3 + \sqrt{5})/2\,.$$

In terms of the above conventions we define a canonical change of coordinates « removing the perturbation to order higher than N »: we use here the standard perturbation theory (a method sometimes called « Birkhoff method »), and construct the canonical map via a generating function of the form:

$$\Psi_N(A', \phi, t) = \sum_{k=1}^{N} \varepsilon^k \Phi^{(k)}(A', \phi, t) \tag{2.10}$$

where

$$\Phi^{(k)}(A', \phi, t) = \sum_{(n,m)\neq \underline{0}} \phi^{(k)}_{nm}(A') e^{i(n\phi + mt)} \tag{2.11}$$

$$\phi^{(k)}_{00} \equiv 0$$

and the coefficients $\phi^{(k)}_{nm}$ must be determined so that in the new canonical variables (A', ϕ', t) the hamiltonian (2.7) takes the form:

$$H_N(A', \phi', t) = h_N(A', \varepsilon) + \varepsilon^{N+1} f_N(A', \phi', t, \varepsilon)\,. \tag{2.12}$$

The change of coordinates $\mathscr{C}_N(A', \phi', t) = (A, \phi, t)$ is related to ψ_N by the well known relations (see for instance [*1*]):

$$\begin{aligned} A &= A' + \frac{\partial \Psi_N}{\partial \phi}(A', \phi, t) \\ \phi' &= \phi + \frac{\partial \Psi_N}{\partial A'}(A', \phi, t) \\ t' &= t \end{aligned} \tag{2.13}$$

or in its complex form:

$$\begin{aligned} A &= A' + iz_1 \frac{\partial \Psi_N}{\partial z_1}(A', z_1, z_2) \\ z_1' &= z_1 \exp\left(i \frac{\partial \Psi_N}{\partial A'}(A', z_1, z_2) \right) \\ z_2' &= z_2 . \end{aligned} \tag{2.14}$$

Of course we must not only deal with the problem of determining the $\phi_{nm}^{(k)}$-coefficients, but also with that of transforming the (2.14) into a map $\mathscr{C}_N(A', z_1', z_2') = (A, z_1, z_2)$. The latter problem is a somewhat involved implicit function problem which contains also the problem of determining the domain of definition of $\mathscr{C}_N$, so far unspecified.

The determination of $\phi_{nm}^{(k)}$ is a purely algebraic task and eventually it will be computerized: one just inserts (2.14) in (2.7) obtaining

$$\frac{1}{2}\left(A' + \frac{\partial \Psi_N}{\partial \phi}(A', \phi, t) \right)^2 + \frac{\varepsilon}{2}\left(z_1 + \frac{1}{z_1} \right) + \frac{\varepsilon}{2}\left(\frac{z_1}{z_2} + \frac{z_2}{z_1} \right) \tag{2.15}$$

and one imposes that (2.15) plus $\frac{\partial \Psi_N}{\partial t}(A', \phi, t)$ (which by the general theory of the, time dependent, canonical maps is the new hamiltonian) has the form (2.12), i. e. the new hamiltonian is $\underline{z}$-independent up to order N (included) in ε. One easily finds:

$$\phi_{nm}^{(1)}(A') = \begin{cases} -i/2(nA'+m) & \text{if } |n|=1 \text{ and } |n+m|=0{,}1 \text{ otherwise} \\ 0 & \end{cases}$$

$$\phi_{nm}^{(k)}(A') = \frac{1}{2i(nA'+m)} \sum_{r+s=k} \sum_{\substack{n'+n''=n \\ m'+m''=m}} n'n'' \phi_{n'm'}^{(r)}(A') \phi_{n''m''}^{(s)}(A') \tag{2.16}$$

The new hamiltonian (2.12) will then be in the (A', z)-variables:

$$\begin{aligned} h_N(A', \varepsilon) &= \frac{A'^2}{2} + \frac{1}{2} \sum_{2}^{N}{}_{k} \varepsilon^k \sum_{r+s=k} \sum_{nm} n^2 \phi_{nm}^{(r)}(A') \phi_{-n-m}^{(s)}(A') \\ f_N(A', \phi, t) &= \sum_{N+1}^{2N}{}_{k} \varepsilon^{k-N-1} \sum_{\substack{r+s=k \\ r,s \le N}} \frac{\partial \Phi^{(r)}}{\partial \phi}(A', \phi, t) \frac{\partial \Phi^{(s)}}{\partial \phi}(A', \phi, t) \end{aligned} \tag{2.17}$$

where $\Phi^{(r)}(A', \phi, t) = \sum_{nm} \phi_{nm}^{(r)}(A') z_1^m z_2^n$ and in (2.17) one has to think that ϕ is a function $\phi(A', \phi', t, \varepsilon)$ obtained by inverting the second term of (2.13) or (2.14) at fixed A', t, ε.

The inversion problem is in fact a very important (and somewhat delicate) aspect of the theory.

We shall invert (2.14) in the form:

$$\begin{aligned} A &= A' + \Xi(A', z_1', z_2') \\ z_1 &= z_1' \exp(i\Delta(A', z_1', z_2')) \\ z_2 &= z_2' \end{aligned} \tag{2.18}$$

with Ξ, Δ defined and holomorphic on a domain of the form

$$(A', z_1', z_2') \in \hat{S}_{\rho_1} \times C^2_{\xi_1}$$

for suitably chosen ρ_1, ξ_1, with $\rho_1 < \rho_0$, $\xi_1 < \xi_0$ and for $|\varepsilon| < \varepsilon_0$: furthermore for $|\varepsilon| \varepsilon_0$, Ξ and Δ are also holomorphic in ε and

$$\begin{aligned} \Xi(A', z_1', z_2') &= \frac{\partial \Psi_N}{\partial \phi}(A', z_1' e^{i\Delta}, z_2') \\ \Delta(A', z_1', z_2') &= -\frac{\partial \Psi_N}{\partial A'}(A', z_1' e^{i\Delta}, z_2') \end{aligned} \tag{2.19}$$

In ref. [1] (§ 5.11 and appendix G) a general theory of the inversion problem leading from (2.14) to (2.18) is presented: it is shown that if $\rho_1 = \rho_0 e^{-\tau}$, $\xi_1 = \xi_0 - \delta$ for $\tau, \delta > 0$ then Ξ, Δ exist and are holomorphic in $(A', z_1', z_2', \varepsilon) \in \hat{S}_{\rho_1}(A_0) \times C^2_{\xi_1} \times \hat{S}_{\varepsilon_0}(0)$ provided, for $|\varepsilon| < \varepsilon_0$, one has:

$$\begin{aligned} \gamma_1(\tau) \left| \frac{\partial \Psi_N}{\partial \phi} \right|_{\rho_1, \xi_1} (\rho_0 \tau)^{-1} &< 1 \\ \gamma_2(\xi, \delta) \left| \frac{\partial \Psi_N}{\partial A'} \right|_{\rho_1, \xi_1} \delta^{-1} &< 1 \end{aligned} \tag{2.20}$$

where γ_1, γ_2 are numerical constants (in the given reference they are respectively, in our notations, $2^8 e^\tau$ and 2^8: however a careful examination of the proof easily leads to the much smaller values, if $\xi \le 1$, used in [5] [6] [14], which are the values we shall in fact use here:

$$\left. \gamma_1(\tau) = e^\tau \tau^{-1} \quad \text{and} \quad \gamma_2(\xi, \delta) = \frac{\pi}{2}(e^{2\xi} + e^\delta) e^{\delta/\pi} \right).$$

The new hamiltonian H_N, holomorphic in $\hat{S}_{\rho_1}(A_0) \times C^2_{\xi_1}$, describes the motion canonically in this domain: the parameters δ, τ will be eventually chosen to optimize the final estimate.

We now look for a point A_N in $\hat{S}_{\rho_1}(A_0)$ such that

$$\omega_0 = \frac{\partial h_N}{\partial A}(A_N, \varepsilon). \tag{2.21}$$

This is another implicit function problem: we rewrite (2.21), using (2.17), as

$$A_0 = A_N + \frac{1}{2}\sum_{2}^{N}{}_{k}\, \varepsilon^k \sum_{r+s=k}\sum_{mn} n^2 \frac{\partial}{\partial A}(\phi^{(r)}_{nm}(A_N)\phi^{(s)}_{-n-m}(A_N)) = A_N + n^{\varepsilon}_{N}(A_N) \tag{2.22}$$

and we apply the implicit function theorem (sect. 1.1 of [*1*], proposition 19) to deduce that if

$$\gamma_3 \sup |\, n^{\varepsilon}_{N}(A)\,| < \rho_1 \qquad \text{taking sup on } \hat{S}_{\rho_1/2}(A_0) \tag{2.23}$$

where γ_3 is a suitable constant ($= 2^8$ in the quoted reference; however a remark like the one following (2.20) applies here as well and leads to $\gamma_3 = 2$ as a possible choice for γ_3), then A_N exists and verifies

$$|\, A_N - A_0\,| < \frac{\rho_1}{2}. \tag{2.24}$$

We shall now restrict further the domain of definition of h_N, f_N to be $\hat{S}_{\rho_N}(A_N) \times C^2_{\xi_N}$ with $\rho_N = \rho_1 e^{-\tau}/2$, $\xi_N = \xi_1 - \delta$ « centering » it around A_N, which will appear to be a convenient choice.

Numerical analysis of data and results, see § 4, will lead to the following « good » choices of the parameters ρ_N, ξ_N (in methods 1, 2, 3):

$N+1$	ρ_N	ξ_N
2	.1	1
3	.1	1
4	.1	1
5	.1	1
6	.1	1
7	.1	1
8	.45	1

For methods 1, 2, 3, we choose ρ_N, ξ_N in a large set of values, making trials on the find estimate on ε; the result is that the « optimal » values of ρ_N, ξ_N for each method (when $N \le 6$) differ slightly from $\rho_N = 0.1$, $\xi_N = 1$. For this reason and in order to make a comparison among methods 1, 2, 3, we whoose the same values $\rho_N = 0.1$, $\xi_N = 1$.

The decrease of ρ_N from $\rho_N = .1$ to $\rho_N = .045$ is due to the presence of new small denominators, which appear at $N = 7$.

The choice of ρ_N, ξ_N in method 4 is different from the others. The values ρ_7, ξ_7 are chosen so that the final estimate on ε is empirically optimal (i. e. among the various choices tried), taking into account small denominators and the conditions (2.20), (2.23), which impose a further decrease of ρ_N.

The function $e^{i\Delta}$ determining the canonical map exists, if $|\,\varepsilon\,| < \bar{\varepsilon}$

where $\bar{\varepsilon}$ is so small that (2.20), (2.23) hold with the above choices of ρ_N, ξ_N, and is holomorphic in $(A', z_1', z_2', \varepsilon) \in \hat{S}_{\rho_1}(A_0) \times C^2_{\xi_1} \times \hat{S}_{\varepsilon}(0)$.

Therefore it can be expanded in powers of ε and leads to the expression of z_1 in terms of $A', z_1', z_2', \varepsilon$ as

$$z_1 = z_1'\left(1 + \sum_{k=1}^{N} \varepsilon^k \Theta_k(A', z_1', z_2') + R^*_{N+1}\right). \tag{2.25}$$

The remainder R^*_{N+1} is the part of order greater equal to $N+1$ in ε of the function (exp $(i\Delta) - 1$ and, being holomorphic for $|\varepsilon| < \bar{\varepsilon}$, can be bounded (by Cauchy's theorem):

$$|R^*_{N+1}| \leq \left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{N+1} e^{|\Delta|_{\rho_1,\xi_1}} |\Delta|_{\rho_1,\xi_1} \tag{2.26}$$

while the coefficients Θ_k can be easily determined as follows.

Set $g_k = \dfrac{\partial \Phi^{(k)}}{\partial A}$ and write the second equality of (2.14), using (2.10) and expanding $\exp(-ig_k\varepsilon^k)$ in powers, as

$$\begin{aligned} z_1 &= z_1' \prod_{k=1}^{N}\left(\sum_{j=0}^{\left[\frac{N}{k}\right]} \frac{(-i)^j}{j!}\varepsilon^{kj} g_k(A', z_1, z_2)^j + \sum_{j=\left[\frac{N}{k}\right]+1}^{\infty} \frac{(-i)^j}{j!}\varepsilon^{kj} g_k(A', z_1, z_2)\right) \\ &= z_1'\left(1 + \sum_{p=1}^{N} \varepsilon^p H_p + R_{N+1}\right) \end{aligned} \tag{2.27}$$

with

$$H_p = \sum_{\sum_{1}^{p} k j_k = p} \prod_{k=1}^{p} \prod_{j_k=1}^{p} \frac{(-g_k)^{j_k}}{j!} \tag{2.28}$$

and H_p is a function of $A', z_1, z_2 - z_2'$.

Substituting (2.25) in (2.27) one obtains

$$\begin{aligned} z_1 &= z_1'\left(1 + \sum_{p=1}^{N} \varepsilon^p H_p\left(z_1' + z_1' \sum_{k=1}^{N} \varepsilon^k \Theta_k + R^*_{N+1}\right) + R_{N+1}\right) \\ &= z_1'\left(1 + \sum_{p=1}^{N} \varepsilon^p \sum_{q=0}^{N} \frac{1}{q!}\frac{\partial^q H_p}{\partial z_1^q}(z_1') z_1'^q \left(\sum_{k=1}^{N} \varepsilon^k \Theta_k\right)^q + \tilde{R}_{N+1}\right) \\ &= z_1'\left(1 + \sum_{p=1}^{N} \sum_{q=0}^{N} \sum_{k_i=0}^{N}{}^{*} \frac{\partial^q H_p}{\partial z_1^q}(z_1') z_1'^q \frac{\Theta_1^{k_1}}{k_1!} \cdots \frac{\Theta_N^{k_N}}{k_N!} \varepsilon^{p+\Sigma_i i k_i} + R_{N+1}\right) \end{aligned} \tag{2.29}$$

(* means: $p + k_1 + \ldots + \mathrm{N}k_\mathrm{N} \leq \mathrm{N}$ and $k_1 + \ldots + k_\mathrm{N} = q$) where the arguments A', z'_2, have been omitted in H_p and A', z'_1, z'_2 have been omitted in Θ_j.

Comparing (2.29) and (2.25) we see that

$$\Theta_\mathrm{N} = \sum_{p=1}^{\mathrm{N}} \sum_{q=0}^{\mathrm{N}} \sum_{k_i=0}^{\mathrm{N}}{}^{*} \frac{\partial^q \mathrm{H}_p}{\partial z_1^q} (z'_1)(z'_1)^q \frac{\Theta_1^{k_1}}{k_1\,!} \cdots \frac{\Theta_\mathrm{N}^{k_\mathrm{N}}}{k^\mathrm{N}\,!} \tag{2.30}$$

(* means: $p + k_1 + \ldots + \mathrm{N}k_\mathrm{N} = \mathrm{N}$ and $k_1 + \ldots + k_\mathrm{N} = q$) which determines Θ_N inductively because the restrictions on the sums and $p \geq 1$ imply $k_\mathrm{N} \equiv 0$.

The (2.14), (2.17), (2.25), (2.30) and the bound (2.26) complete the description of the change of variables.

§ 3. THEORY OF THE NUMERICAL ALGORITHM: THE BOUNDS ON THE NEW HAMILTONIAN AND THE KAM THEOREM

Having under control the coordinate transformation, we must analyze the properties of the hamiltonian describing the motion in the new variables $\left(\text{i. e. (2.15) plus } \frac{\partial \Psi_\mathrm{N}}{\partial t}\right)$:

$$\mathrm{H}_\mathrm{N}(\mathrm{A}', z'_1, z'_2, \varepsilon) = h_\mathrm{N}(\mathrm{A}', \varepsilon) + \varepsilon^{\mathrm{N}+1} f_\mathrm{N}(\mathrm{A}', z'_1, z'_2, \varepsilon) \tag{3.1}$$

(see (2.17)), regarded as defined in $\hat{\mathrm{S}}_{\rho_\mathrm{N}}(\mathrm{A}_\mathrm{N}) \times \mathrm{C}^2_{\xi_\mathrm{N}}$.

We plan to use the KAM stability theorem which allows us to infer the stability of the motion which would take place on the torus $\mathrm{A}' = \mathrm{A}_\mathrm{N}$, if in (3.1) one disregarded the term $\varepsilon^{\mathrm{N}+1} f_\mathrm{N}$. We introduce the characteristic parameters E_N, η_N, ε_N of the hamiltonian (3.1):

$$\mathrm{E}_\mathrm{N} = \left|\frac{\partial h_\mathrm{N}}{\partial \mathrm{A}'}\right|_{\rho_\mathrm{N}}, \qquad \eta_\mathrm{N} = \left\|\frac{\partial^2 h_\mathrm{N}^{-1}}{\partial \mathrm{A}'^2}\right\|_{\rho_\mathrm{N}},$$

$$\varepsilon_\mathrm{N} = \varepsilon^{\mathrm{N}+1}\left(\left|\frac{\partial f_\mathrm{N}}{\partial \mathrm{A}'}\right|_{\rho_\mathrm{N}, \xi_\mathrm{N}} + \frac{1}{\rho_\mathrm{N}}\left|\frac{\partial f_\mathrm{N}}{\partial \phi'}\right|_{\rho_\mathrm{N}, \xi_\mathrm{N}}\right) \tag{3.2}$$

The result that we use in methods 1,2 is a general bound of the form:

$$\Gamma \mathrm{C} \varepsilon_\mathrm{N} (\mathrm{C}\mathrm{E}_\mathrm{N})^\alpha (\eta_\mathrm{N} \rho_\mathrm{N}^{-1} \mathrm{E}_\mathrm{N})^\beta \xi_\mathrm{N}^{-\gamma} < 1 \tag{3.3}$$

(see [6]) where $\xi_\mathrm{N} < 1$, $\Gamma = 1.43 \cdot 10^{17}$, $\alpha = 2.07$, $\beta = 2$, $\gamma = 10.36$ and C is related to the arithmetic properties of the rotation number ω_0 (i. e.

$C = (3 + \sqrt{5})/2$, see (2.9) in our case). Eq. (3.3) is very general and holds for any h_N, f_N holomorphic in the domain $\hat{S}_{\rho_N}(A_N) \times C^2_{\xi_N}$.

The estimate (3.3) can be used for our purposes as soon as we find estimates for ε_N, E_N, ρ_N, ξ_N, η_N.

Such estimates for $N = 0$ are easily deduced from (2.8) and lead to a stability theorem (note that $\eta_0 = 1$) of the form: if

$$2.26 \cdot 10^{22} \, | \varepsilon | < 1 \tag{3.4}$$

then the motion with rotation number ω_0 is stable.

We propose to apply the following alternative improvements to obtain eventually the stability result:

$$1.30 \cdot 10^{2} \, | \varepsilon | < 1 \,. \tag{3.5}$$

We follow four different methods providing successive improvements on the final estimate on ε and we always compare it with the experimental value ε^*_{ED} (see (1.2)). Dividing the procedure leading to this estimate in a first part, in which we compute the estimates of the renormalized hamiltonian and perturbing functions h_N, f_N (namely E_N, η_N, ε_N) and a second part, which is the discussion of the conditions under which the KAM stability holds, we have the following possibilities.

Method 1: application of (3.9) in order to obtain ε_N (without computer assistance), numerical evaluation of E_N, η_N and application of KAM generalized condition (3.3).

Method 2: improvement by numerical evaluation of the value ε_N and application of (3.3).

Method 3: numerical evaluation of ε_N, E_N, η_N; the final result on ε_N is obtained by discussing the set of conditions (3.12), (3.13), (3.14), (3.16) below.

Method 4: same ε_N, E_N, η_N as in method 3, with the new KAM conditions derived in [*16*].

We explain each method, reporting every time a table for the final estimates on ε. The first column refers to the order $N+1$ of the perturbation (see (3.1)). The second column is the final estimate on ε, while $\varepsilon^*_{ED}/\varepsilon$ is the comparison with the bound (1.2). ρ_N, ξ_N are the parameters chosen each time. The last column « ε_{init} » indicates the initial hypothesis we make on ε: $| \varepsilon | < \varepsilon_{init}$: this is necessary because our bounds are derived by supposing *a priori* that $| \varepsilon |$ is not too large, i. e. not larger than a prefixed ε_{init}. After having applied KAM theorem, we check if the final ε is less than ε_{init} (if not we start again with a greater ε_{init}. Then, in order to minimize the difference between ε and ε_{init}, we change ε_{init} and we reapply KAM theorem.

METHOD 1. — Just apply (3.3) to (3.1) evaluating carefully E_N, η_N; this requires the use of a computer not only for the actual calculation of E_N,

η_N, but for the optimization of the final result on ε in terms of the choice of the original parameters.

The results obtained in this rather simple way are summarized, for each N, in the following table (remember that $\varepsilon^*_{ED} = .31$ see (1.2)):

N+1	ε	$\varepsilon^*_{ED}/\varepsilon$	ρ_N	ξ_N	ε_{init}
3	$4.92 \cdot 10^{-12}$	$6.3 \cdot 10^{10}$	.1	1	10^{-5}
4	$5.12 \cdot 10^{-10}$	$6.05 \cdot 10^{8}$	.1	1	10^{-5}
5	$8.30 \cdot 10^{-9}$	$3.73 \cdot 10^{7}$	.1	1	$5 \cdot 10^{-5}$
6	$5.32 \cdot 10^{-8}$	$5.82 \cdot 10^{6}$	.1	1	10^{-4}
7	$1.75 \cdot 10^{-7}$	$1.77 \cdot 10^{6}$	.1	1	10^{-4}
8	$2.20 \cdot 10^{-7}$	$1.40 \cdot 10^{6}$	.045	1	$5 \cdot 10^{-4}$

To apply this method we had to find also an estimate for ε_N and f_N. For this purpose we simply observe that $\varepsilon^{N+1} f_N$ is the part of order greater or equal to N+1 of

$$\frac{1}{2}(A^2 - A_0^2) + \frac{\varepsilon}{2}\left(z_1 + \frac{1}{z_1}\right) + \frac{\varepsilon}{2}\left(\frac{z_1}{z^2} + \frac{z_2}{z_1}\right) + \frac{\partial \Psi_N}{\partial t}(A', z_1, z_2) \tag{3.6}$$

once A, z_1, z_2 are expressed in terms of A', z'_1, z'_2 via (2.14).

If $\bar{\varepsilon}$ is a value such that for $|\varepsilon| < \bar{\varepsilon}$ the map (2.14) is well defined and holomorphic in A', z'_1, z'_2, ε (the conditions determining $\bar{\varepsilon}$ in terms of ρ_0, ξ_0 are the (2.20) to which (2.23) should also be added because we want A_N to be well defined), then

$$|\varepsilon^{N+1} f_N| \leq \left(\frac{|\varepsilon|}{\bar{\varepsilon}}\right)^{N+1} M \quad \text{in} \quad \hat{S}_{\rho_1}(A_0) \times C^2_{\xi_1} \tag{3.7}$$

where M is the maximum of (3.6) as A, z_1, z_2, ε vary in the image via (2.14) of $\hat{S}_{\rho_1}(A_0) \times C^2_{\xi_1} \times \hat{S}_{\bar{\varepsilon}}(0)$. The maximum is bounded by the maximum in the initial domain:

$$M = \rho_0 |A_0| + \rho_0^2/2 + \bar{\varepsilon}(\operatorname{ch} \xi_0 + \operatorname{ch} 2\xi_0) + \rho_0 \tau/\gamma_1 \tag{3.8}$$

having bounded $\dfrac{\partial \Psi_N}{\partial t}$ via (2.20).

Then in the smaller domain $\hat{S}_{\rho_N}(A_N) \times C^2_{\xi_N}$ we bound the derivatives of f_N by a « dimensional estimate », i. e. by Cauchy's theorem and (3.8):

$$\varepsilon^{N+1}\left|\frac{\partial f_N}{\partial A}\right| + \frac{\varepsilon^{N+1}}{\rho_N}\left(\left|\frac{\partial f_N}{\partial \phi}\right| + \left|\frac{\partial f_N}{\partial t}\right|\right) \leq \left(\frac{|\varepsilon|}{\bar{\varepsilon}}\right)^{N+1} M \cdot Q \tag{3.9}$$

(where $Q = 2[\rho_1(1 - e^{-\tau})]^{-1} + 2e^{\tau}[\rho_1(1 - e^{-\delta})]^{-1}$).

After evaluating numerically E_N, η_N (see § 4 for the numerical method and the error control methods used) we substitute them in (3.3) and then optimize numerically (in a large set of tentative values) over the allowed choices of ρ_0, ξ_0, τ, δ (i. e. subject to the conditions (2.20), (2.23)). The results are in the above table.

METHOD 2. — The main advantage of method 1 is that the numerical analysis is rather limited: in particular the coefficients (2.30) do not need to be evaluated.

One can rightly suspect, however, that the bound (3.9) is a rather poor one: in fact, one can improve it by computing the coefficients (2.30) and expressing in terms of them the term of order N+1 in ε of $\varepsilon^{N+1} f_N$ explicitly, where f_N is the perturbation (2.17), computed in terms of the new variables (A', z'_1, z'_2) via (2.25). As we explain in § 4, we memorize the generating function in a matrix with relatively large dimension. From the recursive construction (2.16) of $\Phi^{(k)}(A', \phi, t)$, one can easily see that the generating function is a sum of products of the form:

$$\prod_{i=1}^{k+1} \frac{d_i}{(a_i A' + b_i)^{c_i}} \cdot e^{i(n\phi + mt)} \qquad a_i, b_i, c_i, d_i \in \mathbb{Z} \tag{3.10}$$

(m, n refers to the particular Fourier component $\phi_{nm}^{(k)}$ we are computing) and every row of the matrix in which we store $\Phi^{(k)}(A', \phi, t)$ (up to order N included) contains a product of this form. Formula (2.17) for the construction of h_N, f_N involves the products $\phi_{nm}^{(r)} \phi_{-n-m}^{(s)}$ or $\frac{\partial \Phi^{(r)}}{\partial \phi} \cdot \frac{\partial \Phi^{(s)}}{\partial \phi}$ which are sums of terms like:

$$\prod_{i=1}^{r+1} \frac{d_i^{(r)}}{(a_i^{(r)} A' + b_i^{(r)})^{c_i^{(r)}}} \prod_{i=1}^{s+1} \frac{d_i^{(s)}}{(a_i^{(s)} A' + b_i^{(s)})^{c_i^{(s)}}} e^{i((n^{(s)} + n^{(r)})\phi + (m^{(s)} + m^{(r)})t)}$$

and in order to obtain the estimate of h_N, f_N we bound it as:

$$\prod_{i=1}^{r+1} \frac{| d_i^{(r)} |}{| a_i^{(r)} A' + b_i^{(r)} |^{c_i^{(r)}}} \prod_{i=1}^{s+1} \frac{| d_i^{(s)} |}{| a_i^{(s)} A' + b_i^{(s)} |^{c_i^{(s)}}} e^{(|n^{(s)} + n^{(r)}| + |m^{(s)} + m^{(r)}|)\xi_N}$$

(see § 4 for the analysis of the numerical method and error control).

Using the estimate of the new hamiltonian h_N and the perturbation f_N, the final estimate on ε is obtained through (3.3).

This leads to significant numerical improvements as the table shows

N+1	ε	$\varepsilon_{ED}^*/\varepsilon$	ρ_N	ξ_N	ε_{init}
3	$2.78\cdot10^{-9}$	$1.11\cdot10^{8}$	.1	1	10^{-5}
4	$2.02\cdot10^{-7}$	$1.53\cdot10^{6}$	.1	1	10^{-5}
5	$2.23\cdot10^{-6}$	$1.39\cdot10^{5}$	.1	1	$5\cdot10^{-5}$
6	$1.17\cdot10^{-5}$	$2.64\cdot10^{4}$	.1	1	10^{-4}
7	$3.78\cdot10^{-5}$	$8.20\cdot10^{3}$	.1	1	10^{-4}
8	$9.21\cdot10^{-5}$	$3.36\cdot10^{3}$	.045	1	$5\cdot10^{-4}$

Before examining the next method, we report here a table in which we compare the values of $\| f \|_{\rho,\xi}$ at each order. The first column refers to Method 1, while the second is the explicit computer assisted (but rigorous) estimate of f (Method 2 and, as we shall see, Methods 3, 4):

N+1	Methode 1	Methode 2
3	$1.47 \cdot 10^{9}$	$8.71 \cdot 10^{2}$
4	$2.93 \cdot 10^{12}$	$8.05 \cdot 10^{3}$
5	$5.83 \cdot 10^{15}$	$1.69 \cdot 10^{5}$
6	$1.16 \cdot 10^{19}$	$2.74 \cdot 10^{6}$
7	$2.31 \cdot 10^{22}$	$5.17 \cdot 10^{7}$
8	$2.01 \cdot 10^{27}$	$1.36 \cdot 10^{8}$

METHOD 3. — The (3.3) is obtained under general assumptions, namely without making use of the special form of the functions $h_1, \ldots, h_N, f_1, \ldots, f_N$.

In this method we consider conditions (3.12), (3.13), (3.14), (3.16) below to be discussed separately. Their discussion for a general analytic hamiltonian leads to a condition equivalent to (3.3).

But using the special form of $h_1, \ldots, h_N, f_1, \ldots, f_N$, the discussion of these conditions gives better bounds of apparently more complicated structure, but trivial to evaluate numerically.

The explicit computation of $E_N, \eta_N, \varepsilon_N$ (used instead of (3.9) as in method 1) provides better results in comparison with method 1, while the discussion of conditions (3.12), (3.13), (3.14), (3.16) gives an improvement of the condition (3.3) used in method 2.

We refer to [*1*] for the general KAM theory and to [*4*] [*5*] [*6*] for the necessary few modifications to adapt it to non autonomous one-dimensional hamiltonian systems.

Let us consider the hamiltonian (3.1) with associated parameters ω_N, $E_N, \eta_N, \varepsilon_N, \rho_N, \xi_N$, as defined in (2.5), (2.6). The KAM theorem leads to a procedure of reduction of the hamiltonian (1.1) to higher order in ε (see [*1*]), which is valid under some conditions (namely (3.12), (3.13), (3.14), (3.16)) and requires the definition of iterated parameters related to the hamiltonian (3.1) of order N at the j-th step (namely after having reduced the perturbation to order ε^{2^j}):

$$\begin{aligned} \xi_{j+1} &= \xi_j - 2\delta_j \\ \rho_{j+1} &= \rho_j(4CE_jG_j^2)^{-1} \\ \eta_{j+1} &= \eta_j\rho_j(\rho_j - \varepsilon_j\eta_j)^{-1} \\ E_{j+1} &= E_j + \varepsilon_j \\ \varepsilon_{j+1} &= B(\xi_N, \delta_0)CE_j^2\delta_j^{-9}(C\varepsilon_j)^2 (\log (C\varepsilon_j)^{-1})^2 \end{aligned} \tag{3.11}$$

(we mean ξ_j, ρ_j, η_j, E_j, ε_j as j-th iterated of the initial parameters $\xi(0) = \xi_N$,

$\rho(0) = \rho_N, \dots ; \{ \delta_j \}$ is the « analyticity loss » sequence related to ξ_N), where (see [6]):

$$B(\xi_N, \delta_0) = 200\pi(e^{2\xi_N + 2\delta_0/3} + e^{5\delta_0/3}).$$

But in this method, we use a more complicated expression for B, obtained through the explicit form of h_N and f_N; its simplified expression (which is also an upper bound for it) is:

$$B(\xi_N, \delta_0) = 2.74 + 57e^{4\delta_0} + 11.2e^{\xi_N} + 183.35e^{\xi_N + 4\delta_0} + \\ + .5e^{2\xi_N + 5\delta_0}\delta_0^{-1} + (.571 + .912e^{2\xi_N})e^{4\delta_0}\delta_0^{-2} + .36e^{\xi_N + 6\delta_0}\delta_0^{-3}.$$

We obtained (3.11) by applying the iterated definitions of parameters (2.5), (2.6) on the new hamiltonian H_N and making use of dimensional estimates (see [1] [2] [4] [6]).

We list now the conditions appearing in the KAM theorem (see [1] [4] [5] [6] for their derivation), having in mind the definitions of iterated parameters (3.11):

$$4\varepsilon_j\eta_j^2\rho_j^{-2}E_j < 1 \\ 16CE_jC\varepsilon_j\delta_j^{-4}(\log(C\varepsilon_j)^{-1})^4 < 1 \tag{3.12}$$

(insuring the existence of a quasi periodic motion with pulsation ω_0 for the renormalized hamiltonian)

$$\Gamma_1 C\varepsilon_j CE_j\delta_j^{-5} < 1 \tag{3.13}$$

(for the inversion on the circles)

$$\Gamma_2 C\varepsilon_j CE_j\delta_j^{-5}(\log(C\varepsilon_j)^{-1})^2 < 1 \tag{3.14}$$

(for the inversion on the annuli), where:

$$\Gamma_1(\xi_N, \delta_0) = \gamma_1(\xi_N, \delta_0) = \frac{\pi}{2}(e^{2\xi_N} + e^{\delta_0})e^{\delta_0/\pi}$$

$$\Gamma_2(\tau_N) = \gamma_2(\tau_N) = e^{\tau_N}\tau_N$$

(see [6] for their derivation) and τ_N is a positive constant depending on ρ_N. Note that Γ_1 and Γ_2 depend on the choice of parameters ρ_N, ξ_N, associated to h_N and f_N.

In order to insure the convergence of our method we must discuss for every $j = 0, 1, \dots$ conditions (3.12), (3.13), (3.14) by means of iterated parameters to which we have added a « fast convergence » hypothesis, that is:

$$(C\varepsilon_0)^{2^j} \leq C\varepsilon_j \leq (C\varepsilon_0)^{(2-\lambda)^j} \tag{3.15}$$

where λ has to be chosen belonging to (0, 1).

Conditions (3.12), (3.13), (3.14) are implemented by

$$B(\xi_N, \delta_0)(CE_j)^2(\log(C\varepsilon_j)^{-1})^2\delta_j^{-9}(C\varepsilon_j)^{\lambda} < 1. \tag{3.16}$$

Using (3.11), we obtain also:

$$\rho_j \geq \rho_0 \Big/ \left[(4CE_0)^j \, (\log\,(C\varepsilon_0)^{-1})^{2j} \prod_{0}^{j-1}{}_n \, 2^{2n}\delta_n^{-2} \right]. \qquad (3.17)$$

We report in the following table the values of B, ξ_N, $\{ \delta_n \}$ for methods 2, 3. Note that the values of $\{ \delta_n \}$ and B are different in method 3 for N=1−6 and N=7, because they depend on ρ_N, which decreases at N=7.

	Method 2	Method 3 (N=1−6)	Method 3 (N=7)
B :	6.10^3	1560	2020
ξ_N :	≤ 1	1	1
δ_0 :	$\xi_N/5$	.2	.2265
δ_1 :	$\xi_N/5$	.2	.1
δ_2 :	$\xi_N/15$	.06667	.066
δ_3 :	$\xi_N/40$	.025	.06375
δ_4 :	$33\xi_N/8000$	.004125	.026875
....	.....	.....	.....
δ_n :	$33\xi_N/(500\cdot 2^n)$	$33/(8000\cdot 2^n)$	$43/(3200\cdot 2^n)$

We find that for $j \geq 5$ the condition insuring convergence (namely, the validity of (3.12), (3.13), (3.14), (3.16) for every $j \geq j' = 5$), is obtained giving the bound $E_N < 2E_0$, using (3.15), (3.17) and choosing $\{ \delta_n \}$ as shown in the table. Using (3.11), (3.15), (3.17), we discuss conditions (3.12), (3.13), (3.14), (3.16) by using the computer for $j = 1, \ldots, j'$ with suitable j' and with a suitable choice of $\{ \delta_n \}_{n \geq 5}$ as explained in the table. Then we make some more general choices of the parameters and we compute the same conditions with the assumption $j > j'$ to insure convergence. This condition is of the same order of magnitude of the stronger statement obtained by discussing each of (3.12), (3.13), (3.14), (3.16) for $j=0, 1, 2, 3, 4$: this is the reason why we pick $j' = 5$ (see [*6*] for the method of discussion of the conditions).

We provide here the results on the final estimates on $\varepsilon(\varepsilon^*_{ED} = .31$, see (1.2)):

N+1	ε	$\varepsilon^*_{ED}/\varepsilon$	ρ_N	ξ_N	ε_{init}
3	$4.47\cdot 10^{-9}$	$6.93\cdot 10^7$	.1	1	10^{-5}
4	$2.90\cdot 10^{-7}$	$1.06\cdot 10^6$	.1	1	10^{-5}
5	$2.97\cdot 10^{-6}$	$1.04\cdot 10^5$	.1	1	$5\cdot 10^{-5}$
6	$1.49\cdot 10^{-5}$	$2.08\cdot 10^4$	.1	1	$5\cdot 10^{-5}$
7	$4.65\cdot 10^{-5}$	$6.66\cdot 10^3$	.1	1	$6\cdot 10^{-5}$
8	$1.26\cdot 10^{-4}$	$2.46\cdot 10^3$	.045	1	$1.5\cdot 10^{-4}$

METHOD 4. — The best result (1.4) we have obtained, by applying again the procedure of method 3, is due to the recent estimate for a general one-dimensional non autonomous hamiltonian system, found in [*16*] (to which we refer for more details).

Like in method 3, we report the conditions to be satisfied. We do not perform here the optimization of constants appearing in (3.23)-(3.28), although it has been considered in the numerical work.

Considering the hamiltonian:

$$H_j(A, \phi, t) = h_j(A) + \varepsilon^{2^j} f_j(A, \phi, t) \tag{3.18}$$

we introduce another set of parameters:

$$\begin{aligned} &F_j = \| f_j \|_{\rho_j, \xi_j} \\ &B_j = \| h_j'' \|_{\rho_j, \xi_j} \\ &\eta_j = \| h_j''^{-1} \|_{\rho_j, \xi_j} \\ &\lambda_j' \geq [1 - (\rho_{j+1}/\rho_j)]^{-1} \\ &\lambda_j \geq [1 - (\rho_{j+1} + \varepsilon^{2^j} k_j^{(1)} C F_j/\rho_j)]^{-1} \\ &Q_j = (\alpha k_j^{(7)} \varepsilon^{2^j} C^2 F_j B_j)^{-1} \\ &P_j \geq \varepsilon^{2^j} C^2 F_j B_j \end{aligned} \tag{3.19}$$

where

$$k_j^{(1)} \equiv 4e^{2\delta_j}(\beta_j^3 + 2\beta_j^4), \qquad \beta_j \equiv \frac{e^{\delta_j}}{1 - e^{-\delta_j}},$$
$$k_j^{(7)} \equiv (k_j^{(1)})^2/(4\beta_j \delta_j^{-1})$$

and $\alpha > 0$ is to be chosen.

We denote with δ_j the (first) « analyticity loss » due to the control of the derivatives of the generating function $\Phi_j(A', \phi)$ and with δ_j (the (second) « analyticity loss » for the inversion problem related to the canonical transformation:

$$A = A' + \varepsilon^{2^j} \frac{\partial \Phi_j}{\partial \phi}$$
$$\phi' = \phi + \varepsilon^{2^j} \frac{\partial \Phi_j}{\partial \phi}$$
$$t' = t.$$

We shall choose

$$\delta_j = \frac{\delta}{2^{j+1}}, \qquad j \geq 0 \tag{3.21}$$

where $\delta > 0$ is to be fixed.

We introduce also the cut-off parameter of the regularized perturbation (corresponding to the N_j of [*1*], p. 502):

$$G_j \equiv \delta_j^{-1} [\log Q_j + 2 \log (k_j^{(6)} + \log Q_j)] \tag{3.22}$$

where

$$k_j^{(6)} \equiv (\beta_j + 1)\delta_j.$$

After having renormalized the hamiltonian at the j-th step by means of the canonical change of variables (3.20), we define the iterated parameters: B_{j+1}, λ_j, λ'_j, P_{j+1}, G_j, η_{j+1}, δ_j, in terms of the parameters at the $(j-1)$-th step.

We can conclude (see [16], where the result is derived in detail) that if ε satisfies the following conditions (3.23)-(3.28), then the invariant torus is shown to exist:

$$\begin{aligned} &4\,|\,\varepsilon\,|\,CF_0\eta_0B_0G_0^2\rho_0^{-1} < 1 \qquad j=0 \\ &32\,P_jB_j\eta_j(G_{j-1}G_j)^2 \le 1 \qquad j \ge 1 \end{aligned} \tag{3.23}$$

(for the control of the domains $\hat{S}_{\rho_{j+1}}(A_{j+1}^0) \subset \hat{S}_{\rho_j}(A_j^0)$),

$$\begin{aligned} &(2CB_0\rho_0^{-1}G_0^2)^{-1} + |\,\varepsilon\,|\,(2F_0\eta_0\rho_0^{-2} + k_0^{(1)}CF_0\rho_0^{-1}) < 1 \qquad j=0 \\ &2(G_{j-1}/G_j)^2 + P_j(32\eta_jB_jG_{j-1}^4 + 4k_j^{(1)}G_{j-1}^2) \le 1 \qquad j \ge 1 \end{aligned} \tag{3.24}$$

(so that the map (3.20) is well defined, being $A = A' + \varepsilon^{2^j}\dfrac{\partial \Phi_j}{\partial \phi} \in \hat{S}_{\rho_j}(A_j^0)$),

$$\begin{aligned} &|\,\varepsilon\,|\,(k_0^{(4)}CF_0\rho_0^{-1} + k_0^{(5)}C^2F_0B_0) < 1 \qquad j=0 \\ &P_j(4k_j^{(4)}G_{j-1}^2 + k_j^{(5)}) \le 1 \qquad j \ge 1 \end{aligned} \tag{3.25}$$

(for the implicit function problem related to (3.20),

$$16\,P_je^{-k_j^{(6)}}\alpha k_j^{(7)} \le 1 \tag{3.26}$$

(for the definition of the new perturbation function f_{j+1}),

$$\begin{aligned} &|\,\varepsilon\,|\,\lambda_0^2F_0\eta_0\rho_0^{-2} < 1 \qquad j=0 \\ &P_j(4\lambda_j)^2\eta_jB_jG_{j-1}^4 \le 1 \qquad j \ge 1 \end{aligned} \tag{3.27}$$

(for the definition of the iterated parameters),

$$k_0^{(3)}P_0 + |\,\varepsilon\,|\,k_0^{(2)}CF_0\rho_0^{-1} + \sum_{1}^{j}{}_n\, P_n(4k_n^{(2)}G_{n-1}^2 + k_n^{(3)}) \le 1 - \delta \qquad j \ge 1 \tag{3.28}$$

(for the control of the analyticity losses δ_j, $\dot{\delta}_j$), where:

$$\begin{aligned} k_j^{(2)} &\equiv 4\lambda'_j\beta_j^3e^{2\delta_j} \\ k_j^{(3)} &\equiv 8(\beta_j^3 + 6\beta_j^4 + 6\beta_j^5)e^{2\delta_j} \\ k_j^{(4)} &\equiv 4\lambda'_j(\beta_j^3 + 2\beta_j^4)e^{2\delta_j} \\ k_j^{(5)} &\equiv 8(\beta_j^3 + 14\beta_j^4 + 36\beta_j^5 + 24\beta_j^6)e^{2\delta_j}. \end{aligned}$$

But in practical problems, we can not control (3.23)-(3.28) for every $j \ge 0$; for this reason we fix an integer j_0: we control (3.23)-(3.28) when $j \le j_0$ and then we impose a stronger condition which guarantees (3.23)(-(3.28) for every $j > j_0$.

To this end, we consider again conditions (3.23)-(3.28) with (3.24) substituted by:

$$\gamma\left(\frac{1}{2} + 4P_j k_j^{(1)} G_{j-1}^2\right) \leq 1 \qquad j \geq j_0 \tag{3.29}$$

$$32\gamma' P_j \eta_j B_j G_{j-1}^4 \leq 1 \qquad j \geq j_0 \tag{3.30}$$

and (3.27) substituted by:

$$\frac{l}{l-1} P_j (4\lambda)^2 B_j \eta_j G_{j-1}^4 \leq 1 \tag{3.32}$$

with $\gamma, \gamma' \geq 2$ to be chosen and

$$\lambda \equiv \gamma/(\gamma - 1)$$

$$l \equiv (1 - P_{j_0}(4\lambda_{j_0})^2 B_{j_0} \eta_{j_0} G_{j_0-1}^4)^{-1}$$

$$g \equiv 2 - 1/l$$

We denote with:

$$\psi_j^* \equiv \prod_{1}^{j}{}_k (k_{j_0+k-1}^{(1)})^{1/2^k}$$

$$\psi_j \equiv \prod_{1}^{j}{}_k (\sigma'_k + \tau'_k \chi_{k-1}^2)^{1/2^k}$$

where

$$\chi_{j+1} \equiv \log\left[(a k_{j+j_0}^{(8)})^{1/2^j} P_{j_0} \psi_j^*\right]^{-1} + \\ + 2^{1-j}\left\{\log\left(k_{j+j_0}^{(6)} + 2^j \log\left[(a k_{j+j_0}^{(7)})^{1/2^j} P_{j_0} \psi_j^*\right]^{-1}\right)\right\}$$

$$\sigma'_k \equiv g(1+\alpha)(k_{j_0+k-1}^{(1)})^2$$

$$\tau'_k \equiv \begin{cases} \dfrac{g\lambda k_{j_0}^{(1)}}{(1-e^{-\delta_{j_0}})^2}\,\delta_{j_0-1}^{-2} & k = 1 \\[2ex] \dfrac{4g\lambda k_{j_0+k-1}^{(1)}}{(1-e^{-\delta_{j_0}})^2}\,\delta_{j_0+k-2} 4^{k-2} & k > 1 \end{cases}$$

If

$$\psi \equiv \prod_{1}^{\infty}{}_k (\sigma'_k + \tau'_k \chi_{k-1}^2)^{1/2}$$

being:

$$\psi_j^* \nearrow \psi^* \equiv \prod_{1}^{\infty}{}_k (k_{j_0+k-1}^{(1)})^{1/2^k}$$

$$\psi_j \nearrow \psi \equiv \prod_{1}^{\infty}{}_k (\sigma'_k + \tau'_k \chi_{k-1}^2)^{1/2^k}$$

$$\chi_{j+1} \searrow \chi \equiv \log(P_{j_0} \psi^*)^{-1}.$$

one can conclude (see [16]) that conditions (3.23), (3.29), (3.30), (3.24), (3.25), (3.31), (3.27), (3.23) are bounded by an expression of the form: $a2^j P_j G_j L_j N_j^b$, where $a = a(j_0) > 0$, $b \in \mathcal{N}$ and one can control these conditions as long as

$$P_{j_0} \psi < 1. \tag{3.32}$$

Once the new condition (3.32) is satisfied, if ε is small enough, say

$$\varepsilon < \varepsilon^*(j_0)$$

the torus $\mathcal{T}(\varepsilon)$ is an invariant one (for further details see [16]).

By applying the general theory to (1.1), we obtain, for $N = 0$:

$$1.98\ 10^9 \varepsilon < 1$$

(compare it with (3.4)!).

This new estimate allows us to obtain substantial improvements of the previous results, as is shown in the following table:

$N+1$	ε	$\varepsilon^*_{ED}/\varepsilon$	ρ_N	ξ_N	ε_{init}
3	$1.26 \cdot 10^{-4}$	2460.31	.1	1	.0005
4	$5.32 \cdot 10^{-4}$	582.70	.1	1	.002
5	$1.58 \cdot 10^{-3}$	196.20	.1	1	.003
6	$2.77 \cdot 10^{-3}$	111.91	.1	1	.0045
7	$4.22 \cdot 10^{-3}$	73.45	.094	1	.006
8	$7.70 \cdot 10^{-3}$	40.25	.0192	.8746	$7.704 \cdot 10^{-3}$

Note that the last value of ε is away from the numerical value ε^*_{ED} (see (1.2)) only for a factor 40.25

§ 4. NUMERICAL SCHEME

Formulae (2.17) for the construction of the new hamiltonian system are suitable for an iterative numerical computation.

Each formula depends upon the expression of the generating function $\Phi^{(k)}$, that we indicate as

$$\Phi^{(k)}(A', \phi, t) = \sum_{nm} \phi^{(k)}_{nm}(A') e^{i(n\phi + mt)}. \tag{4.1}$$

We want a good estimate of the generating and perturbing functions; to this end, we try to store completely the expression:

$$\Psi_N(A', \phi, t) = \sum_{1}^{N} {}_k\, \varepsilon^k \Phi^{(k)}(A', \phi, t)$$

in a bidimensional matrix with relatively big size, that we shall indicate with « A ».

The Fourier components $\phi^{(k)}_{nm}(A')$ are, as we can deduce from (2.16), sums of products of the form (3.10), that we rewrite in the slightly different form (we use this form in order to economize memory):

$$\frac{d_1}{a_1 A'^{c_1}} \prod_{2}^{N+1}{}_i \frac{1}{(a_i A' + b_i)^{c_i}} e^{i(n\phi + mt)}, \qquad a_i, b_i, c_i, \ d_1 \in Z \tag{4.2}$$

where a_i, b_i, c_i, d_1 depend upon N, k, n, m.

It is necessary to memorize, in each row of the matrix, the order k of the generating function we are computing, the couple of integers (n, m), the numbers a_i, b_i, c_i, d_1, which characterize every term of $\phi^{(k)}_{(n,m)}$.

For example, at $N = 1$:

$$\Phi^{(1)}(A', \phi, t) = \frac{e^{i\phi}}{2iA'} - \frac{e^{-i\phi}}{2iA'} + \frac{e^{i(\phi - t)}}{2i(A'-1)} - \frac{e^{-i(\phi - t)}}{2i(A'-1)}.$$

At this order we have four terms corresponding to the Fourier components $(1,0)$, $(-1,0)$, $(1,-1)$, $(-1,1)$ and we memorize them in two rows (note that each term appears two times with opposite signs, so that we have to store only half of these terms):

k	n	m	a	d	c	a	b	c
1	1	0	2	1	1	0	0	0
1	1	−1	2	1	0	1	−1	1

With this procedure it is possible to have more rows corresponding to the same Fourier component (n, m). In order to obtain $\phi^{(k)}_{nm}$ we have to sum over every row with the same (n, m).

Using this method, we are able to obtain a fast estimate of the generating function and of the relevant quantities.

We need some device for storing data, using them for purely algebraic operations. Once the available memory is exhausted (this happens when $N=6$), the progress continues through the recursive construction of the generating function in this way: after the computation of each row, related to a not memorized order, we pursue with every operation connected to this row. Then we delete the row in order to make more space.

Of course, this device makes the program extremely slow. For this reason and for the impossibility of memorizing large quantities of data, we are not able to compute higher orders of the generating function (and, in any event, it is clear that the results will not improve forever with the order).

Going back to the description of our numerical method, we observe that the generating function, its derivatives and the perturbation are exactly computed, namely without numerical error (in fact, we represent these functions through the set of integers a_i, b_i, c_i, d_1). At this stage of memorization of the generating function by the matrix « A », every entry is an integer number and we avoid approximation errors (that would damage the efficacy of the method), because a_i, b_i, c_i, d_1 in (4.2) are the result (less then the maximum integer allowed by the computer which is 2147483647 as stated in [17], p. 2-5) of sums and products among integers. The estimate of the generating function and of every other related expressions is:

$$\| \Phi^{(k)}(A', \phi, t) \| = \sum_{nm} | \phi^{(k)}_{nm}(A') | \, e^{(|n|+|m|)\xi_N} . \tag{4.3}$$

We adopt this estimate because numerical experiments show that cancellations are not relevant.

There is empirical evidence that every order has a leading term of the form (compare with [4]):

$$C_N \frac{e^{2N\xi_N}}{(A_0 - \rho_N - 1)^{2(N-1)}} \qquad C_N \text{ constant}, \quad C_N \simeq 2^{-2N} . \tag{4.4}$$

We deal with the remainder by using the iterative formula (2.25), which depends only upon $\Phi^{(k)}$.

Looking at the expression of the Fourier components $\phi^{(k)}_{nm}(A')$, the least value of $\Phi^{(k)}$ is reached when $| a_i A' + b_i |$ is minimum. Through (4.3), we compute the estimates of the terms $\phi^{(k)}_{nm}(A')$, which are sums of products of the form (4.2)):

$$\frac{d_1}{a_1 A'^{c_1}} \prod_{2}^{N+1}{}_i \frac{1}{| a_i A' + b_i |^{c_i}} . \tag{4.5}$$

Once obtained these values, we compute estimates of derivatives through Cauchy's estimates.

This work is carried out with a Vax 11/780 with floating point notation and in double precision (1). This means that any number is represented by a sign, a normalized positive fraction less than one and with the first decimal digit different from 0, and an exponent which gives the position of the point

(1) We have seen, with a rough estimate on numerical errors and on their propagation, that the error in computing the generating function can be bounded by $1.5 \cdot 10^{N-15}$; therefore the final results are exact up to the (15-N)-th decimal significative digit. Using simple precision, the final results would be meaningless for $N \geq 5$.

(which means for example the representation of 123456.78 as .123456789. 10^6 or .001234 as $.1234 \cdot 10^{-2}$).

Moreover « double precision » means that any number is stored in eight contiguous bytes with bit 15 the sign bit, bits 7-14 an excess 128 binary exponent, i. e. represented by the binary equivalents of 0 throught 255, and bits 0-6, 16-63 a normalized 56 bit fraction. For this reason the absolute value of a number is at most of the order of 2^{128} (i. e. in the approximate range ($.29\ 10^{-38}$, $.17\ 10^{39}$)) and the precision of the computer is approximately one part in 2^{55} (i. e. typically 16 decimal digits) (see [*19*]).

We indicate in the following with « *i*-th digit » of a number, the *i*-th digit of the normalized fraction in the floating point notation.

As we noted before, we compute exactly the Fourier components of the generating function (representing them through the set of integers n, m, a_i, b_i, c_i, d_1); in this way the numerical errors (due to the precision of the machine) arise only in the estimate of the generating function and in the discussion of the final conditions of methods 2, 3, 4.

In order to obtain an upper bound on the final estimate which includes the numerical errors, we have first divided every formula of the program into the most elementary operations (precisely: sum, difference, multiplication and division), for which one obtains the result exact up to the 16-th digit, that is the 56-th bit of the normalized fraction (see [*17*]). Then, after every such operation, we have increased (or decreased where it was necessary) the result of each operation, adding (or subtracting) one to the 54-th bit of the normalized fraction; we have used the library functions LIB$INSV and LIB$FFC (see [*20*]).

For other functions, like the exponential, that we can not divide into elementary operations, we have computed their exact value with series expansion up to the 16-th digit and we have increased (or decreased) their last exact digit.

We can conclude that the final results, for the estimate of the generating function, are exact up to the 4-th digit.

After having obtained the new hamiltonian and the new perturbing function we pursue by applying KAM-theory in the general form: we discuss the set of the final conditions, repeating the same procedure above (splitting formulas, decreasing or increasing results of each operation, etc.), in order to take into account the numerical error.

We find our results at each order, by choosing the analyticity parameters ρ_N, ξ_N in a large set of tentative values. The decrease of ρ_N and ξ_N is caused by two factors: first, we have to control the conditions (2.20), (2.23), which impose a constraint on ρ_N, ξ_N; second, the presence of small denominators. At every order, the number of Fourier components increases, while the domain of holomorphy of h and f is smaller.

Finally, the results are obtained imposing $\varepsilon < \varepsilon_{\text{iniz}}$ and verifying the agreement between the final and the former hypotheses as explained in § 3.

The flow-chart given here represents the part of the program which evaluates the generating function, computed through (2.16). We denote with « A » the matrix related to the generating function (2.10), (2.11). Each row contains a term like (4.2) of the Fourier expansion (2.11).

The first element of the j-th row (A(j, 1) is the order k of the generating function $\Phi^{(k)}(A', \phi, t)$, that we are computing. The second and the third elements refer to the couple of integers (n, m) of the Fourier expansion term $\phi_{nm}^{(k)}(A')$: $A(j, 2) = n$, $A(j, 3) = m$.

The remaining elements of the j-th row are occupied by the set of numbers a_i, b_i, c_i, d_i, which appear in (4.2).

In this way, in order to obtain a Fourier component $\phi_{nm}^{(k)}(A')$, we have to sum over the rows such that:

$$A(j, 1) = k, \qquad A(j, 2) = n, \qquad A(j, 3) = m.$$

If $A(j, 1) = k$ and $A(j + 1, 1) = k + 1$, we define the array $C(k)$ so that $C(k) = j$, namely $C(k)$ takes memory of the index j of the last row related to the generating function of order k: $\Phi^{(k)}(A', \phi, t)$, and we set $C(0) = 0$.

We denote the subroutines with the following symbols:

[Coeff] = it calculates the value of a_i, b_i, c_i, d_i of the each row of « A »;

[Estim] = it estimates a single row of « A » and its derivatives;

[New] = it constructs the hamiltonian and perturbing functions h_N, f_N through recursive formulae (2.17).

For simplicity of notations, we set in the following flow-chart:

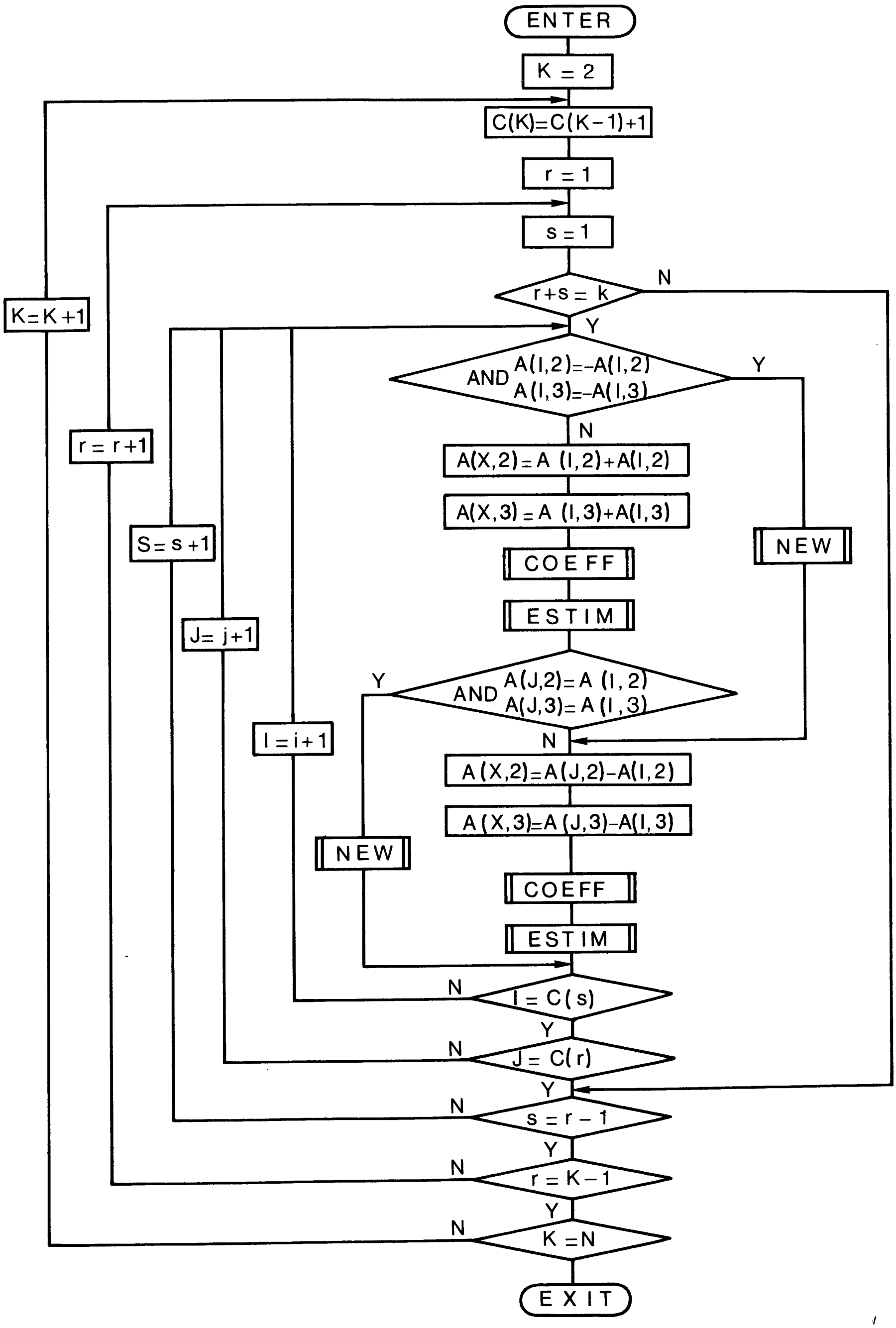
$$X = C(k), \qquad J = C(r - 1) + 1, \qquad I = C(s - 1) + 1$$

and we denote with k, r, s, the same indexes appearing in (2.16).

ACKNOWLEDGMENTS

We wish to thank Prof. G. Gallavotti, for having introduced us to this problem, for having suggested the strategy that allowed us to reach our results and for his constant precious help; Prof. G. Benfatto, who followed us throughout all our work dedicating his time and his experience; Prof. L. Galgani and A. Giorgilli for very useful discussions and advice.

We thank the Departement of Mathematics of the University of Rome « L. Sapienza » for having placed the computers at our disposal.


ENTER
K = 2
C(K)=C(K-1)+1
r = 1
s = 1
r+s = k
N
Y
K=K+1
r = r+1
S= s +1
J= j+1
I = i+1
AND A(I,2)=-A(I,2) A(I,3)=-A(I,3)
Y
N
A(X,2) = A (I,2)+A(I,2)
A(X,3) = A (I,3)+A(I,3)
COEFF
ESTIM
NEW
AND A(J,2)= A (I, 2) A(J,3)= A (I, 3)
Y
N
A (X,2)=A(J,2)-A(I,2)
A (X,3)=A (J,3)-A(I, 3)
NEW
COEFF
ESTIM
I = C(s)
J = C(r)
s = r - 1
r = K - 1
K =N
EXIT

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(Manuscrit reçu le 4 février 1987)