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Scattering theory for Hamiltonians with Stark effect

by

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ABSTRACT. — For a large class of Hamiltonians with Stark effect, existence and asymptotic completeness of the wave operators are shown. For operators $H = -\Delta + Fx_1 + U(x_1) + V$, $F > 0$, we assume that V decays as $|x_1|^{-\frac{1}{2}-\varepsilon}$ as $x_1 \to -\infty$, and that $U = W''$, W a smooth function with all derivatives bounded. Furthermore, U is periodic, or $\| U' \|_\infty < F$. The last case allows U to be a sum of periodic and almost periodic functions. More general operators than H above are considered.

RÉSUMÉ. — On démontre l'existence et la complétude asymptotique des opérateurs d'onde pour une classe étendue de Hamiltoniens avec effet Stark. On considère des opérateurs $H = -\Delta + Fx_1 + U(x_1) + V$, $F > 0$, on suppose que V décroît comme $|x_1|^{-1/2-\varepsilon}$ pour $x_1 \to -\infty$ et que $U = W''$, où W est une fonction lisse avec toutes ses dérivées bornées. De plus, U est supposée périodique ou satisfaisant $\| U' \|_\infty < F$. Le dernier cas permet à U d'être une somme de fonctions périodiques et presque périodiques. On considère aussi des opérateurs plus généraux que le H ci-dessus.

1. INTRODUCTION. STATEMENT OF RESULT

In this paper we continue our study of Stark effect Hamiltonians with a new class of potentials, which was begun in [*10*] [*11*]. We obtain an extension to dimension $n > 1$ of these results.

There has been a considerable interest recently in studying the spectrum of Stark effect Hamiltonians, in particular in connection with random potentials, see e. g. [2] [3] [4] [5] [6]. One finds that in many cases the presence of an electric field implies that the spectrum has an absolutely continuous component equal to $\mathbb{R}$. However, as shown in [6], if the potential is too singular, the situation is different. A sufficiently strong electric field is required to obtain the absolutely continuous component. Many of these results were obtained in one dimension using ordinary differential equation techniques. Thus it is of interest to study these operators in dimension $n > 1$, and to include a large class of potentials.

To describe the results we need some notation. In $\mathcal{H}_1 = L^2(\mathbb{R})$ we consider for $F > 0$

$$h_0 = -\frac{d^2}{dx_1^2} + Fx_1,$$

and for U a bounded realvalued function

$$h_1 = -\frac{d^2}{dx_1^2} + Fx_1 + U(x_1).$$

Both operators are essentially selfadjoint on the Schwartz space $\mathcal{S}(\mathbb{R})$. Let T_2 be a selfadjoint operator with domain $\mathcal{D}(T_2)$ in a separable Hilbert space $\mathcal{H}_2$. We consider on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ the selfadjoint operators determined by

$$H_0 = h_0 \otimes I_2 + I_1 \otimes T_2$$
$$K = h_1 \otimes I_2 + I_1 \otimes T_2 = H_0 + U \otimes I_2.$$

These operators are essentially selfadjoint on $\mathcal{D} = \mathcal{S}(\mathbb{R}) \otimes \mathcal{D}(T_2)$ (algebraic tensor product). We use the same notation for the closures.

Let $\mathcal{B}^k(\mathbb{R})$ denote the bounded realvalued functions on $\mathbb{R}$ with k bounded derivatives, and let $\mathcal{B}^\infty(\mathbb{R}) = \bigcap_{k=1}^{\infty} \mathcal{B}^k(\mathbb{R})$. The assumption on U is either

ASSUMPTION 1.1. — $U(x_1) = W''(x_1)$ for some $W \in \mathcal{B}^\infty(\mathbb{R})$, and furthermore

$$\| U' \|_{L^\infty(\mathbb{R})} < F. \tag{1.1}$$

or

ASSUMPTION 1.2. — $U(x_1) \in C^\infty(\mathbb{R})$ is realvalued and periodic with period ξ. Furthermore,

$$\int_0^\xi U(x_1)dx_1 = 0.$$

We note that Assumption 1.2 implies $U = W''$ for some $W \in \mathcal{B}^\infty(\mathbb{R})$, see [10]. Under these assumptions on U the electric field dominates in the propagation properties of e^{-itK}. This is the central part of our results. Propagation estimates are obtained using the extensions of the Mourre method given in [9] [12] with a conjugate operator

$$A = (-p_1) \otimes I_2, \qquad \text{where } p_1 = -i\frac{d}{dx_1}.$$

The key step is to prove the Mourre estimate (2.2) for K and A. We can then use the results from [12] to obtain a scattering theory for K and $H = K + V$, with V short-range with respect to A. We use the following assumption expressed in terms of H and K.

ASSUMPTION 1.3. — Let H be a selfadjoint operator on $\mathscr{H}$ such that $(H + i)^{-1} - (K + i)^{-1}$ is compact, and such that for some integer $k \geq 0$ and real number $\mu > 1$ the operator

$$(H + i)^{-k}((H + i)^{-1} - (K + i)^{-1})(K + i)^{-k}(A^2 + 1)^{\mu/2} \tag{1.2}$$

extends to a bounded operator on $\mathscr{H}$.

An explicit assumption on V is also used, when T_2 is bounded below.

ASSUMPTION 1.4. — Let V be a symmetric operator with $\mathscr{D}(V) \supset \mathscr{D}(H_0)$, such that $V(H_0 + i)^{-1}$ is compact, and such that for some $\delta > \frac{1}{2}$ the operator

$$(H_0 + i)^{-1}V(1 + x_1^2)^{\delta/2}\chi_{(-\infty,0)}(x_1)$$

extends to a bounded operator on $\mathscr{H}$.

The main result of this paper is

THEOREM 1.5. — Let U satisfy either Assumption 1.1 or Assumption 1.2. Assume either

i) H satisfies Assumption 1.3

or

ii) T_2 is bounded below, and $H = K + V$ with V satisfying Assumption 1.4.

Then the wave operators

$$W_{\pm}(H, H_0) = \underset{t \to \pm\infty}{s\text{-lim}}\, e^{itH}e^{-itH_0}$$

exist and are asymptotically complete.

Furthermore, the singular continuous spectrum of H is empty, and the point spectrum of H is discrete in $\mathbb{R}$.

We shall outline the proof: Let us first note that under either assumption on U the wave operators $W_{\pm}(K, H_0)$ exist and are unitary. In particular K has purely absolutely continuous spectrum equal to $\mathbb{R}$. This follows from

$W_{\pm}(K, H_0) = W_{\pm}(h_1, h_0) \otimes I_2$ and the results in [10]. Using the chain rule for wave operators (see [13])

$$W_{\pm}(H, H_0) = W_{\pm}(H, K)W_{\pm}(K, H_0).$$

we see that the problem is reduced to proving existence and completeness of $W_{\pm}(H, K)$. This result is obtained from the abstract scattering theory developed in [12], the propagation estimates for e^{-itK} proved in section 2, and the fact the assumptions on H or V are precisely those required in order to apply the results in [12]. An extension of this result is given in Theorem 3.6. In Section 3 examples of potentials satisfying our conditions are given.

The conditions on V can be made explicit, if we specify T_2. Taking $\mathscr{H}_2 = L^2(\mathbb{R}^{n-1}, dx')$ and $T_2 = -\Delta_{x'}$, we get from Assumption 1.4 the usual conditions on V, roughly $V(x_1, x') = 0(|x_1|^{-\frac{1}{2}-\varepsilon})$ as $x_1 \to -\infty$, $V(x_1, x') = o(|x_1|)$ as $x_1 \to +\infty$, and $V(x_1, x') \to 0$ as $|x'| \to \infty$. Thus the previous results [1] [2] [7] [15] [17] [18] [19] [21] are generalized by allowing the inclusion of U. The existence and completeness of $W_{\pm}(H, K)$ for U periodic was obtained in [3], using resolvent estimates and stationary Kato-Kuroda scattering theory.

2. PROPAGATION ESTIMATES

The proofs of existence and completeness of wave operators given here depend on propagation estimates for K relative to the operator A. We use the extension of the Mourre method given in [9] [12] to obtain these estimates. The key result to be established is the Mourre estimate for K and A, see (2.1) and (2.2).

We start by proving that K and A satisfy the conditions in [12; Definition 2.1], i. e. that A is conjugate to K, and K is ∞-smooth with respect to A. To simplify reference to [12], we follow the numbering of properties of operators used there. E_K denotes the spectral measure of K. In this section we set $F = 1$ to simplify notation.

LEMMA 2.1. — Let U satisfy Assumption 1.1. Then K and A have the following properties:

a) $\mathscr{D}(A) \cap \mathscr{D}(K)$ is dense in $\mathscr{D}(K)$ with the graph norm.

b) $e^{i\theta A}$ maps $\mathscr{D}(K)$ into $\mathscr{D}(K)$, and $\sup_{|\theta| \leq 1} \| K e^{i\theta A} f \| < \infty$ for each $f \in \mathscr{D}(K)$.

c_{∞}) The form $i[K, A]$, defined on $\mathscr{D}(A) \cap \mathscr{D}(K)$, is bounded from below and closable. The selfadjoint operator associated with its closure is denoted iB_1. We have $\mathscr{D}(B_1) \supset \mathscr{D}(K)$. For $j = 2, 3, \ldots$ we have that the form $i[iB_{j-1}, A]$, defined on $\mathscr{D}(A) \cap \mathscr{D}(K)$, is bounded from below and closable. The associated selfadjoint operator is denoted iB_j. We have $\mathscr{D}(B_j) \supset \mathscr{D}(K)$.

e) There exists $\alpha > 0$ such that for any interval $J \subset \mathbb{R}$ we have

$$E_K(J) iB_1 E_K(J) \geqq \alpha E_K(J) . \tag{2.1}$$

Proof. — We noted above that K is essentially selfadjoint on

$$\mathscr{D} = \mathscr{S}(\mathbb{R}) \otimes \mathscr{D}(T_2), \qquad \text{and} \qquad \mathscr{D} \subseteq \mathscr{D}(A) \cap \mathscr{D}(K) .$$

Thus *a*) holds. By definition $e^{i\theta A} = e^{-i\theta p_1} \otimes I_2$, and therefore

$$e^{-i\theta A} K e^{i\theta A} = \theta + H_0 + U(x_1 + \theta) \otimes I_2$$

on $\mathscr{D}$. From this relation we get *b*). Computing as a quadratic form on $\mathscr{D} \times \mathscr{D}$ we find

$$i[K, A] = I + U'(x_1) \otimes I_2$$

U' is by assumption bounded, and thus $iB_1 = I + U' \otimes I_2$ is a bounded selfadjoint operator, so $\mathscr{D}(B_1) \supset \mathscr{D}(K)$ holds trivially. By assumption $U \in \mathscr{B}^\infty(\mathbb{R})$, so we get

$$iB_j = U^{(j)} \otimes I_2, \qquad j = 2, 3, \ldots,$$

and all the properties in c_∞) have been verified. Condition (1.1) implies

$$iB_1 \geqq \alpha I$$

with $\alpha = 1 - \| U' \|_{L^\infty} > 0$, and thus (2.1) holds. □

LEMMA 2.2. — Let U satisfy Assumption 1.2. Then K satisfies *a*), *b*), and c_∞) of Lemma 2.1. Furthermore, for any $E \in \mathbb{R}$ there exist $\alpha > 0$ and $\delta > 0$ such that with $J = (E - \delta, E + \delta)$ we have

$$E_K(J) iB_1 E_K(J) \geqq \alpha E_K(J) . \tag{2.2}$$

Proof. — We have $U \in \mathscr{B}^\infty(\mathbb{R})$ by assumption. This was the only property used in Lemma 2.1 in proving *a*), *b*), and c_∞), so these results also hold here. To prove (2.2), fix $E \in \mathbb{R}$. Write $J_\delta = (E - \delta, E + \delta)$. It suffices to prove that (operator norm in $\mathscr{H}$)

$$\| E_K(J_\delta)(U' \otimes I_2) E_K(J_\delta) \| \rightarrow 0$$

as $\delta \downarrow 0$. We recall the representation (see [*20*])

$$E_K(J_\delta) = \int_{-\infty}^{\infty} E_{h_1}(J_\delta - \lambda) \otimes I_2 d(I_1 \otimes E_{T_2}(\lambda))$$

where the integral is weakly convergent. Thus it suffices to prove

$$\lim_{\delta \downarrow 0} (\sup_{\lambda \in \mathbb{R}} \| E_{h_1}(J_\delta - \lambda) U' E_{h_1}(J_\delta - \lambda) \|) = 0 .$$

(Here the norm is operator norm in $\mathscr{H}_1$.) First we replace the spectral projection with a smooth cutoff function. Choose $\varphi \in C_0^\infty(\mathbb{R})$, $0 \leqq \varphi \leqq 1$, $\varphi(\mu) = 1$ for $|\mu| \leqq 1$, $\varphi(\mu) = 0$ for $|\mu| \geqq 2$. Define $\varphi_\delta(\mu) = \varphi((\mu - E)/\delta)$. Note that $e^{-i\lambda p_1}$ is the translation operator in $\mathscr{H}_1$. We write

$$U_\lambda(x_1) = e^{-i\lambda p_1}U(x_1)e^{i\lambda p_1} = U(x_1 - \lambda).$$

We have $e^{-i\lambda p_1}\mathscr{D}(h_1) \subseteq \mathscr{D}(h_1)$,

$$e^{-i\lambda p_1}h_1 e^{i\lambda p_1} = h_0 + U_\lambda - \lambda$$

and thus

$$e^{-i\lambda p_1}\varphi_\delta(h_1+\lambda)U'\varphi_\delta(h_1+\lambda)e^{i\lambda p_1} = \varphi_\delta(h_0+U_\lambda)U'_\lambda\varphi_\delta(h_0+U_\lambda). \quad (2.3)$$

Let $\Phi_\delta(\lambda) = \|\varphi_\delta(h_1 + \lambda)U'\varphi_\delta(h_1 + \lambda)\|$. Since U is periodic with period ξ, we get $\Phi_\delta(\lambda + \xi) = \Phi_\delta(\lambda)$ from (2.3). Thus it suffices to consider $\lambda \in [0, \xi]$. Since φ_δ is smooth, $\Phi_\delta(\lambda)$ is continuous in λ. Consider a fixed $\lambda \in [0, \xi]$. We write $\tilde{\varphi}_\delta(\mu) = (\mu + i)\varphi_\delta(\mu)$ and $\tilde{\tilde{\varphi}}_\delta(\mu) \doteq (\mu + i)^2\varphi_\delta(\mu)$. We have

$$\varphi_\delta(h_1 + \lambda)U'\varphi_\delta(h_1 + \lambda) = (\tilde{\tilde{\varphi}}_\delta(h_1 + \lambda)(h_1 + \lambda + i)^{-1}(p_1 + i)^{-1} \\ (p_1 + i)(h_1 + \lambda + i)^{-1}U'(h_1 + \lambda + i)^{-1}\tilde{\varphi}_\delta(h_1 + \lambda).$$

It is proved in [*10*] that

$$(p_1 + i)(h_1 + \lambda + i)^{-1}U'(h_1 + \lambda + i)^{-1}$$

extends to a bounded operator on $\mathscr{H}_1$, and that $(h_1+\lambda+i)^{-1}(p_1+i)^{-1}$ is compact. Since h_1 has purely absolutely continuous spectrum, $s\text{-}\lim_{\delta\downarrow 0} \tilde{\varphi}_\delta(h_1 + \lambda) = 0$, and there exists $\delta=\delta(\lambda)$ such that $\Phi_\delta(\lambda) < \frac{1}{2}$. Now we use the periodicity and the compactness of $[0, \xi]$ to conclude that there exists $\delta > 0$ such that for all $\lambda \in \mathbb{R}$ $\Phi_\delta(\lambda) \leqq \frac{1}{2}$. Thus (2.2) follows with $\alpha = \frac{1}{2}$, since $E_{h_1}(J_\delta-\lambda) \doteq E_{h_1}(J_\delta-\lambda)\varphi_\delta(h_1+\lambda)$. □

REMARK 2.3. — The difficulty in proving the estimate (2.2) comes from the fact that $U' \otimes I_2$ is not compact relative to K, except in the trivial case $\mathscr{H}_2 = \mathbb{C}^n$. Since h_1 is not semibounded, the technique developed in [*16*] to handle similar problems, encountered in the context of many-body Schrödinger operators, cannot be used here. Thus it seems that an additional hypothesis is needed; in our case either smallness or periodicity.

PROPOSITION 2.4. — Let U satisfy Assumption 1.1 or Assumption 1.2. Let $g \in C_0^\infty(\mathbb{R})$. Then for $0 < s' < s$ we have

$$\|(1 + A^2)^{-s/2}e^{-itK}g(K)(1 + A^2)^{-s/2}\| \leqq c(1 + |t|)^{-s'}$$

for all $t \in \mathbb{R}$, and

$$\| (1 + A^2)^{-s/2} e^{-itK} g(K) P_A^{\pm} \| \leqq c(1 + |t|)^{-s'}$$

for all t, $\pm t > 0$.

Proof. — In Lemmas 2.1 and 2.2 we have verified that K and A satisfy the assumptions in the theory developed in [*12*]. Thus the results follow from those in [*9*] [*12*]. Note that the assumption of semiboundedness is not needed here. One can verify [*9*; Lemma 3.8] directly. One needs to show that $[g(K), A]$ and higher commutators all extend to bounded operators on $\mathscr{H}$. This follows by straightforward commutator computations, using the representation ($\hat{g}$ denotes the Fourier transform)

$$g(K) = \int_{-\infty}^{\infty} \hat{g}(\sigma) e^{i\sigma K} d\sigma$$

and the fact that $[K, A]$ and higher commutators $[[K, A], A], \ldots$, all extend to bounded operators on $\mathscr{H}$, as verified in the proof of Lemma 2.1.

□

To make explicit the conditions on perturbations of K in the applications we need to prove propagation estimates, where the localization $(1+A^2)^{-s/2}$ is replaced by a localization in x_1; cf. the discussion in [*9*].

Let $\chi_+ \in C^\infty(\mathbb{R})$, $0 \leq \chi_+ \leq 1$, $\chi_+(x_1) = 1$ for $x_1 \geq 2$, $\chi_+(x_1) = 0$ for $x_1 \leq 1$, and let $\chi_-(x_1) = 1 - \chi_+(x_1)$. Define

$$\rho(x_1) = \chi_+(x_1) + \chi_-(x_1)(1 + x_1^2)^{-1/2}. \tag{2.4}$$

In order to prove the result we need to assume T_2 bounded below, a condition which is verified in all applications; see section 3.

LEMMA 2.5. — Assume that T_2 is bounded below. For δ, $0 \leq \delta \leq 1$, the operator

$$(1 + A^2)^\delta (K + i)^{-1} \rho(x_1)^\delta$$

extends to a bounded operator on $\mathscr{H}$.

Proof. — The result is obtained by interpolation. It suffices to show that

$$A^2(K + i)^{-1} \rho(x_1)$$

extends to a bounded operator on $\mathscr{H}$. The details are given in Appendix A. □

PROPOSITION 2.6. — Let U satisfy Assumption 1.1 or Assumption 1.2. Assume that T_2 is bounded below. Let $g \in C_0^\infty(\mathbb{R})$. For $0 < s' < s \leq 1$ we have

$$\| \rho(x_1)^s e^{-itK} g(K) \rho(x_1)^s \| \leq c(1 + |t|)^{-2s'}$$

for all $t \in \mathbb{R}$, and

$$\| \rho(x_1)^s e^{-itK} g(K) P_A^{\pm} \| \leq c(1 + |t|)^{-2s'}$$

for all t, $\pm t > 0$.

Proof. — The results follow from Proposition 2.4 and Lemma 2.5. Note that the restriction $s \leq 1$ comes from Lemma 2.5. □

3. EXAMPLES AND REMARKS

In this section we give some examples of operators T_2 and potentials U and V satisfying our assumptions. We also give a few remarks on our results.

EXAMPLE 3.1. — A large class of potentials satisfying Assumption 1.1 can be obtained by the following construction: Let U be a realvalued function which can be represented as

$$U(x_1) = \int_{-\infty}^{\infty} e^{i\omega x_1} d\mu(\omega),$$

where μ is a Borel measure satisfying

$$\int_{-\infty}^{\infty} \omega^{-2} d |\mu|(\omega) < \infty$$

and

$$\int_{-\infty}^{\infty} \omega^{2l} d |\mu|(\omega) < \infty$$

for all $l = 0, 1, 2, \ldots$. Additionally one needs the condition $\| U' \|_{L^\infty} < F$. A large class of almost-periodic functions is obtained if one takes μ to be a sum of point measures.

REMARK 3.2. — In our Theorem 1.5 the operator T_2 can be any selfadjoint operator on a Hilbert space. Due to the tensor product structure, the spectrum of T_2 can be quite arbitrary. It is clear that to prove Lemma 2.5, one needs to have T_2 bounded below. A counterexample is provided by $\mathcal{H}_2 = L^2(\mathbb{R}, dx_2)$ and $T_2 = -\dfrac{d^2}{dx_2^2} + Fx_2$, $F \neq 0$. Typically we take $T_2 = -\Delta_{x'}$, on $L^2(\mathbb{R}^{n-1}; dx')$, and identify H_0 with $-\Delta + x_1$ on $L^2(\mathbb{R}^n)$. Thus we get results on

$$H = -\Delta + x_1 + U(x_1) + V.$$

When $U \equiv 0$, our results reduce to those previously obtained, at least for V a multiplication operator, see [*1*] [*2*] [*7*] [*15*] [*18*] [*21*]. We give an example (cf. [*21*]).

EXAMPLE 3.3. — For the case $T_2 = -\Delta_{x'}$ on $L^2(\mathbb{R}^{n-1}; dx')$, a sufficient condition on V to satisfy Assumption 1.4 is given by the following Stummel-type condition: The functions χ_+ and χ_- are those used in defining ρ, see (2.4).

$$V(x) = \{(1 + x_1^2)^{-\delta/2}\chi_-(x_1) + (1 + x_1^2)^{1/2}\chi_+(x_1)\}(V_1(x) + V_2(x))$$

with $\delta > \frac{1}{2}$,

$$V_1 \in L^\infty(\mathbb{R}^n), \qquad \lim_{|x|\to\infty} V_1(x) = 0$$

and $V_2 \in L^2_{loc}(\mathbb{R}^n)$ such that for some ν, $0 < \nu < 4$,

$$\lim_{|x|\to\infty}\left((1 + x_1^2)\int_{|x-y|\leq 1} |V_2(y)|^2 |x - y|^{\nu-n} dy\right) = 0 .$$

We emphasize that our Assumption 1.4 does not require V to be a multiplication operator. It is easy to give examples of finite rank operators which satisfy the assumption.

EXAMPLE 3.4. — We can also take

$$T_2 = -\Delta_{x'} + Z \qquad \text{on} \qquad \mathcal{H}_2 = L^2(\mathbb{R}^{n-1}; dx'),$$

where Z is a suitable perturbation of $-\Delta_{x'}$. If Z is a usual short range perturbation (see e. g. [*15*]), then the wave operators

$$W_\pm(-\Delta_{x'} + Z, -\Delta_{x'})$$

exist and are complete. Thus using the chain rule for wave operators, we get existence and completeness of the wave operators between $-\Delta + x_1$ and $-\Delta + x_1 + U(x_1) + Z(x') + V(x_1, x')$ on $L^2(\mathbb{R}^n)$ (with an obvious notation). Here U satisfies Assumption 1.1 or 1.2 and V Assumption 1.4.

REMARK 3.5. — There is a different version of Theorem 1.5, where one requires less smoothness on U, but faster decay in the x_1-variable of the potential V. We state the following result. The detailed proof is quite long, so it will only be sketched here.

THEOREM 3.6. — Let U satisfy either

i) $\quad U = W'', \qquad W \in \mathcal{B}^5(\mathbb{R}), \qquad \| U' \|_{L^\infty(\mathbb{R})} < F,$

or

ii) $\quad U \in \mathcal{B}^3(\mathbb{R}), \qquad$ U periodic with period ξ, $\qquad \int_0^\xi U(x_1)dx_1 = 0 .$

Let V satisfy Assumption 1.4 with $\delta > 1$ in (1.2). Let $H = K + V$, and let T_2 be bounded below. Then the wave operators $W_\pm(H, H_0)$ exist and are asymptotically complete. Furthermore, the singular continuous spectrum of H is empty, and the point spectrum of H is discrete in $\mathbb{R}$.

Proof. — We shall only outline the proof. The new assumptions on U imply that (in the terminology of [*9*] [*12*]) A is conjugate to K at every $E \in \mathbb{R}$, and that K is 2-smooth w. r. t. A. Thus the results follow from [*12*, Theorem 4.4]. The requirement $\delta > 1$ comes from the fact that now the estimates in Proposition 2.6 hold for $2s' > 1$, provided we can take $s > 2$. That in turn follows from $\delta > 1$ and an extension of Lemma 2.5. □

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APPENDIX A

In this appendix we prove Lemma 2.5. We assume T_2 bounded below. By adding a constant we can assume $T_2 = Q^2$, $Q = \sqrt{T_2}$ is the positive square root. We first extend a result due to Herbst-Simon [*8*, Theorem C.1] to the present setting. We recall that χ_+ is the cutoff function used in (2.4). To simplify the notation, we identify Q^2 with $I_1 \otimes Q^2$, p_1^2 with $p_1^2 \otimes I_2 = \left(-\dfrac{d^2}{dx_1^2}\right) \otimes I_2$, etc.

LEMMA A.1. — The following operators extend to bounded operators on $\mathscr{H}$:

$$\chi_+(x_1)(p_1^2 + Q^2)(H_0 + i)^{-1} \tag{A.1}$$
$$\chi_+(x_1)x_1(H_0 + i)^{-1} \tag{A.2}$$
$$\chi_+(x_1)\sqrt{x_1}Q(H_0 + i)^{-1} \tag{A.3}$$
$$\chi_+(x_1)\sqrt{x_1}p_1(H_0 + i)^{-1} \tag{A.4}$$

Proof. — We follow [*8*]. We compute as quadratic forms on $\mathscr{S}(\mathbb{R}) \otimes C^\infty(T_2)$, where $C^\infty(T_2)$ denotes the C^∞-vectors for T_2.

$$\begin{aligned}(\chi_+(p_1^2 + Q^2 + x_1))_* (\chi_+(p_1^2 + Q^2 + x_1)) &= (p_1^2 + Q^2)\chi_+^2(p_1^2 + Q^2) + x_1^2\chi_+^2 \\ &+ x_1\chi_+(p_1^2 + Q^2) + (p_1^2 + Q^2)x_1\chi_+ = (p_1^2 + Q^2)\chi_+^2(p_1^2 + Q^2) + x_1^2\chi_+^2 \\ &+ 2(p_1x_1\chi_+^2p_1 + Qx_1\chi_+^2Q) + [p_1,[p_1,x_1\chi_+^2]].\end{aligned}$$

We have $[p_1,[p_1,x_1\chi_+^2]] = -\dfrac{d^2}{dx_1^2}(x_1\chi_+(x_1)^2) \geq -c$ for some $c > 0$. For any $f \in \mathscr{S}(\mathbb{R}) \otimes C^\infty(T_2)$ we get

$$\begin{aligned}\|\chi_+H_0f\|^2 + c\|f\|^2 \geq \|\chi_+(p_1^2 + Q^2)f\|^2 \\ + \|\chi_+x_1f\|^2 + 2\|\chi_+\sqrt{x_1}Qf\|^2 + 2\|\chi_+\sqrt{x_1}p_1f\|^2.\end{aligned}$$

We note that H_0 maps $\mathscr{S}(\mathbb{R}) \otimes C^\infty(T_2)$ into itself. Thus we can take $f = (H_0 + i)^{-1}g$, $g \in \mathscr{S}(\mathbb{R}) \otimes C^\infty(T_2)$, and the results follow. □

Proof of Lemma 2.5. — We must prove that

$$p_1^2(K + i)\rho(x_1)$$

extends to a bounded operator on $\mathscr{H}$. Below we compute in the quadratic form sense on $\mathscr{S}(\mathbb{R}) \otimes C^\infty(T_2)$. We note that all operators considered here map this set into itself. We continue to use the simplified notation $K = p_1^2 + x_1 + Q^2 + U$, etc.

We first note that for any $f \in \mathscr{S}(\mathbb{R}) \otimes C^\infty(T_2)$

$$\|p_1^2(K + i)^{-1}\rho(x_1)f\| \leq \|(p_1^2 + Q^2)(K + i)^{-1}\rho(x_1)f\|.$$

Thus it suffices to prove that

$$(p_1^2 + Q^2)(K + i)^{-1}\rho(x_1)$$

extends to a bounded operator on $\mathscr{H}$. We have

$$\begin{aligned}(p_1^2 + Q^2)(K + i)^{-1}\rho &= (K - U - x_1)(K + i)^{-1}\rho \\ &= (K - U)(K + i)^{-1}\rho - x_1(K + i)^{-1}\rho.\end{aligned}$$

The first term is a bounded operator on $\mathcal{H}$. The second term is rewritten:

$$x_1(K + i)^{-1}\rho = (K + i)^{-1}x_1\rho + [x_1, (K + i)^{-1}]\rho.$$

We have, using (3.4),

$$(K + i)^{-1}x_1\rho = (K + i)^{-1}(H_0 + i)(H_0 + i)^{-1}x_1\chi_+(x_1) + (K + i)^{-1}x_1\chi_-(x_1)(1 + x_1^2)^{-\frac{1}{2}}.$$

Now (A.2) implies that both terms extend to bounded operators on $\mathcal{H}$.

$$[x_1, (K+i)^{-1}]\rho = (K + i)^{-1}[p_1^2, x_1](K + i)^{-1}\rho = -2i(K + i)^{-1}p_1(K + i)^{-1}\rho = -2i(K + i)^{-2}p_1\rho + 2i(K + i)^{-2}[p_1, x_1 + U](K + i)^{-1}\rho.$$

Since $i[p_1, x_1 + U] = 1 + U'$ is bounded, the second term is bounded on $\mathcal{H}$.

$$(K + i)^{-2}p_1\rho = (K + i)^{-2}(H_0 + i)(H_0 + i)^{-1}p_1\chi_+(x_1) + (K + i)^{-2}(H_0 + i)(H_0 + i)^{-1}p_1\chi_-(x_1)(1 + x_1^2)^{-\frac{1}{2}}.$$

The first term extends to a bounded operator on $\mathcal{H}$ by (A.4), if we note that χ_+ is supported on $[1, \infty)$, so we can write $\chi_+(x_1) = \sqrt{x_1}\cdot\chi_+(x_1).(\sqrt{x_1}^{-1}\chi_{[1,\infty)}(x_1))$. The second term is also bounded. This can be seen by computations similar to those in [*18*; Lemma 8.5]. We give the computations for the sake of completeness. We write $\tilde{\chi}_- = \chi_-(x_1)(1 + x_1^2)^{-\frac{1}{2}}$. For $f \in \mathcal{S}(\mathbb{R}) \otimes C^\infty(T_2)$ we have

$$\begin{aligned} &\| p_1\tilde{\chi}_-(H_0 + i)^{-1}f \|^2 \\ &= \langle (H_0 + i)^{-1}f, \tilde{\chi}_-p_1^2\tilde{\chi}_-(H_0 + i)^{-1}f \rangle \leq \langle (H_0 + i)^{-1}f, \tilde{\chi}_-(p_1^2+Q^2)\tilde{\chi}_-(H_0+i)^{-1}f \rangle \\ &= \langle (H_0 + i)^{-1}f, \{ \tilde{\chi}_-^2(p_1^2 + Q^2) - 2i\chi_-p_1\tilde{\chi}'_- - \tilde{\chi}_-\tilde{\chi}''_- \} (H_0 + i)^{-1}f \rangle \\ &= \langle (H_0 + i)^{-1}f, \tilde{\chi}_-^2(H_0 - x_1)(H_0 + i)^{-1}f \rangle + \langle (H_0 + i)^{-1}f, -\tilde{\chi}_-\tilde{\chi}''_-(H_0 + i)^{-1}f \rangle \\ &\quad - 2i \langle p_1\tilde{\chi}_-(H_0 + i)^{-1}f, \tilde{\chi}'_-(H_0 + i)^{-1}f \rangle \leq c \| f \|^2 + \frac{1}{2}\| p_1\tilde{\chi}_-(H_0 + i)^{-1}f \|^2. \end{aligned}$$

Thus $p_1\tilde{\chi}_-(H_0 + i)^{-1}$ extends to a bounded operator on $\mathcal{H}$. This completes the proof of Lemma 2.5.

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